Hamiltonian Dynamics Problem Sheet

David Kelliher

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Problem 1. Hamilton's canonical equations are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Apply the equations to the following two Hamiltonians:

$$H(x, y, p_x, p_y; t) = xp_y + yp_x^2 + p_x p_y$$

$$H(r, \theta, \phi, p_r, p_\theta, p_\phi; t) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2\theta}\right) + V(r)$$

Problem 2. A double pendulum is formed by joining one pendulum to another (Fig. 1). Consider a double pendulum with masses m_1 and m_2 attached by wires of length l_1 and l_2, respectively. Define the angles made by the two wires with the vertical as θ_1 and θ_2, respectively.

- Write down the Lagrangian, L = T - V.
- Find the canonical momenta $(p_{\theta_1}, p_{\theta_2})$.
- Bonus question - write down the Hamiltonian for the system (noting $H = \sum_{i=1}^{2} \dot{\theta}_i p_{\theta_i} - L$).

Problem 3. Show that the transformation from action angle coordinates to (q,p) as follows

$$q(I, \phi) = \sqrt{2I}\sin\phi, \; p(I, \phi) = \sqrt{2I}\cos\phi \tag{1}$$

is a canonical transformation (recall for a canonical transformation the Poisson bracket must satisfy $[q, p]_{\phi,I} = 1$)). Show that the generating function

$$F_3(p, \phi) = \frac{p^2}{2}\tan\phi \tag{2}$$

effects the same coordinate transformation as Eqn. 1.

Problem 4. The leapfrog (or *velocity Verlet*) algorithm is given by

$$v_{n+1/2} = v_n + \frac{1}{2}hF(x_n), \tag{3}$$

$$x_{n+1} = x_n + hv_{n+1/2}, \tag{4}$$

$$v_{n+1} = v_{n+1/2} + \frac{1}{2}hF(x_{n+1}). \tag{5}$$

where $x_n, v_n, F(x_n)$ are position, velocity and force at step n while h is the step length. Show that the algorithm is symplectic

Problem 5. (i) Starting with the vector potential for a multipole magnet

$$A_x = 0, \quad A_z = 0, \quad A_l = -\mathcal{R}\sum_{n=1}^{\infty}(b_n + ia_n)\frac{(x + iz)^n}{nr_0^{n-1}} \tag{6}$$

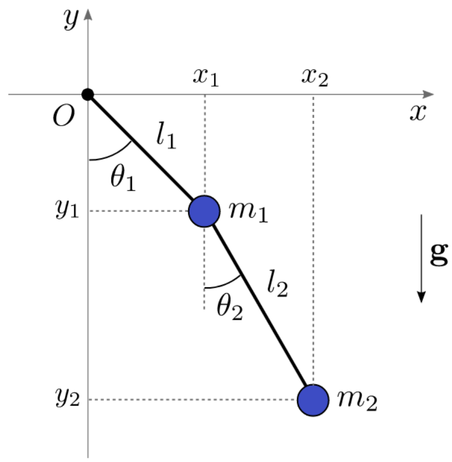

Figure 1: Double pendulum

find the Hamiltonian for a sextupole (n=3) assuming small dynamic variables. We assume a normal sextupole (i.e. the skew term $a_3 = 0$). The strength parameter b_n is given by

$$b_n = \frac{1}{(n-1)!}\frac{\delta^{n-1} B_y}{\delta x^{n-1}} \tag{7}$$

Note, one may follow the procedure used to calculate the quadrupole Hamiltonian in the lecture slides.
(ii) Given the generator f

$$f = -(L/2)\left(p_x^2 + p_z^2\right) \tag{8}$$

calculate $: f : y$ setting y to each of the transverse phase space variables (x, z, p_x, p_z) to show that f is the generator for a drift space (recall that the Lie operation $: f : y$ is equal to the Poisson bracket $[f, y]$).