Basic Plasma Physics for Plasma based accelerators

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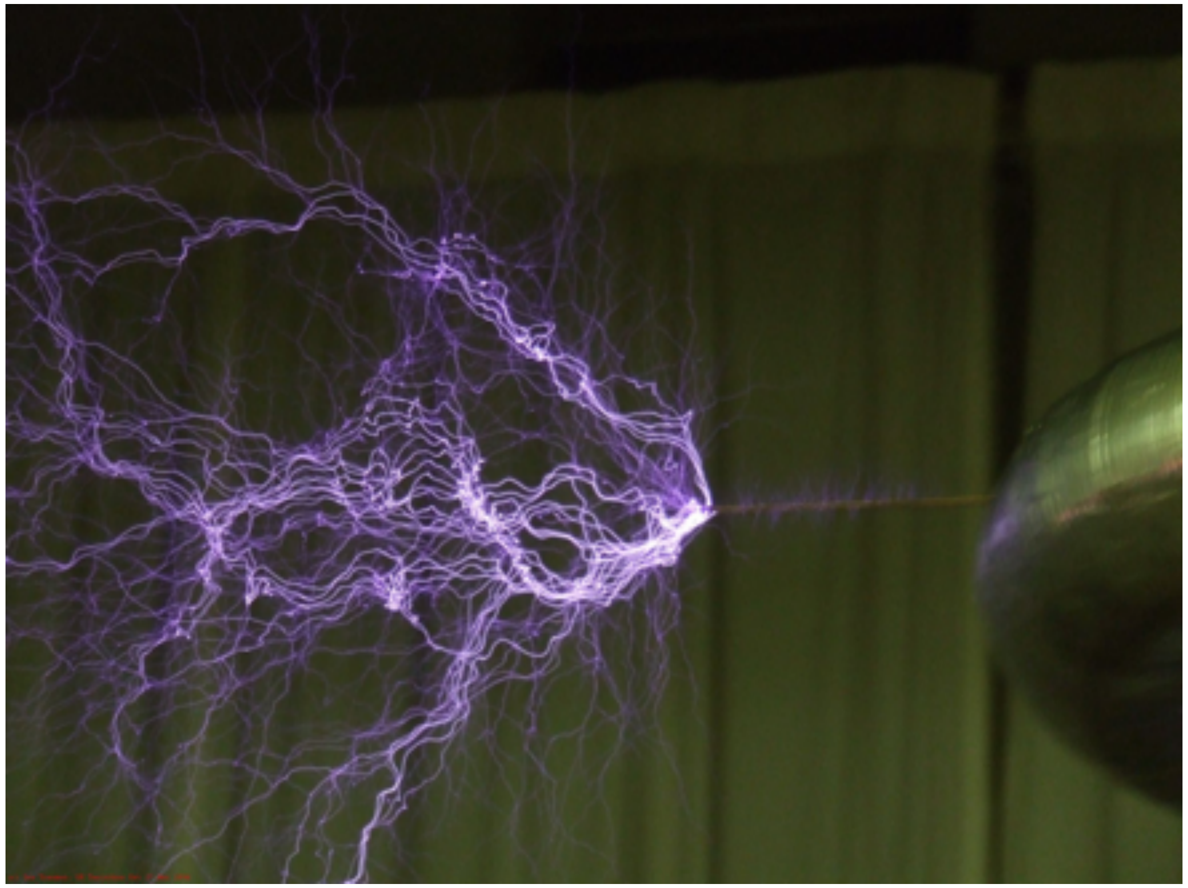
Why use plasmas in particle accelerators?

- Why are particle accelerators so big (and expensive)?
  - maximum accelerating field in RF accelerators ~100 MV/m
  - higher electric fields produce breakdown
- Use a plasma instead as breakdown already happened
  - plasmas support wave modes suitable for accelerating particles



What is in this lecture?

- This lecture introduces some key basic plasma physics concepts needed to understand plasma based particle accelerators
  - plasma frequency
  - EM waves in a plasma
    - critical density
    - ponderomotive force
  - driving plasma waves

What is a plasma?

- The "4th state" of matter
    - NB no phase change… no latent heat of plasma formation…
- ***"a quasi-neutral ionized gas that exhibits collective behaviour"***
    - interactions between plasma particles are dominated by long-range (small angle) EM forces
    - neutrality can be broken on short length and time scales… leading to large electric fields
    - Collective effects: currents, magnetic fields, waves and instabilities

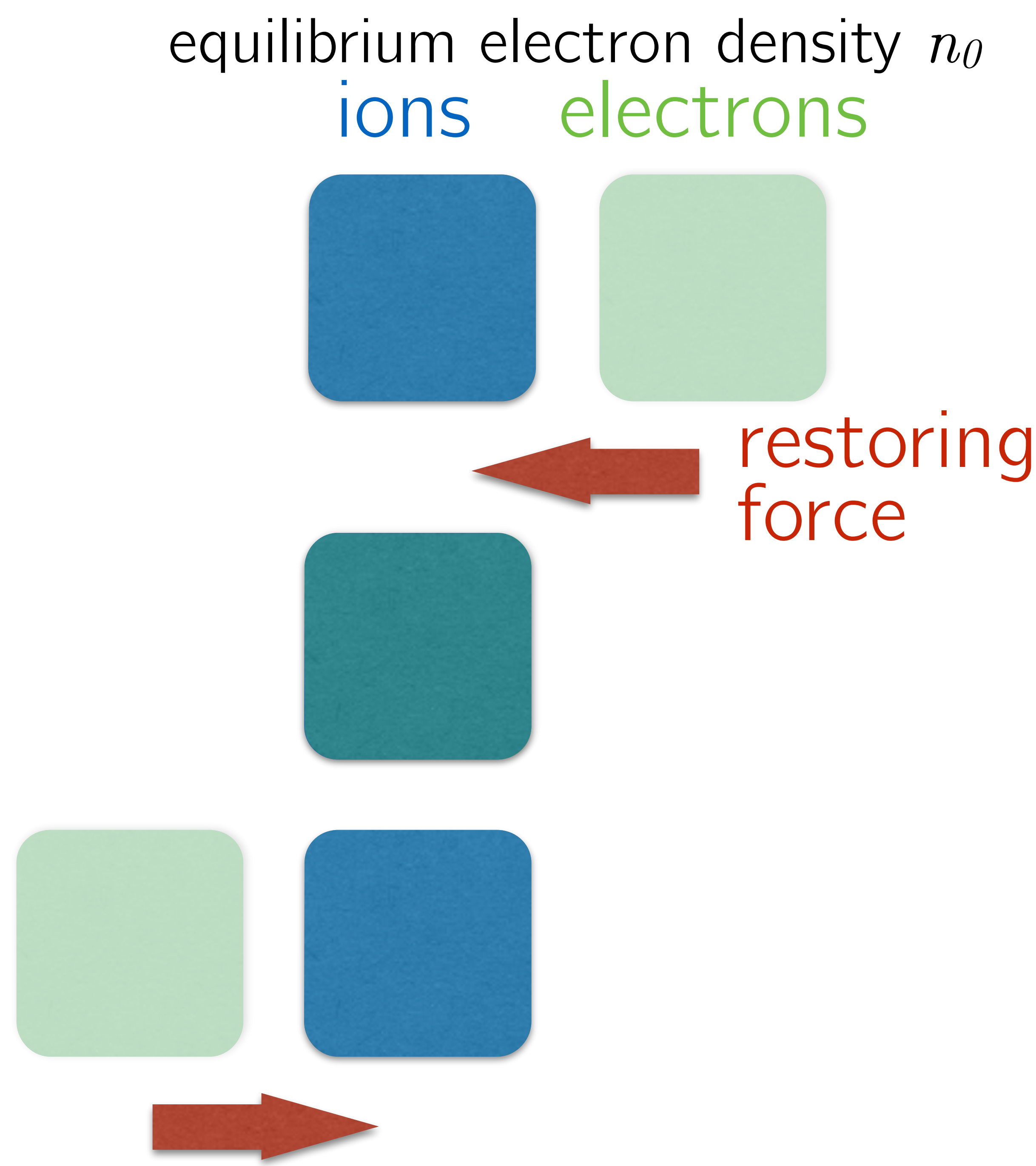
e.g. aluminium

T[K]$<$ 958	958 $<$ T[K]$<$ 2765	T[K] $>$ 23000
solid	gas	plasma
melts	*vaporises*	*ionizes*

$$-en_e + Zen_i \approx 0$$

The plasma frequency

- What happens if we displace all the electrons from a region of plasma?
  - An electric field is set up that acts to return plasma to quasi-neutrality
  - electrons overshoot original position and we get ***electron plasma oscillations***
    - oscillations at the plasma frequency ω_p



The plasma frequency

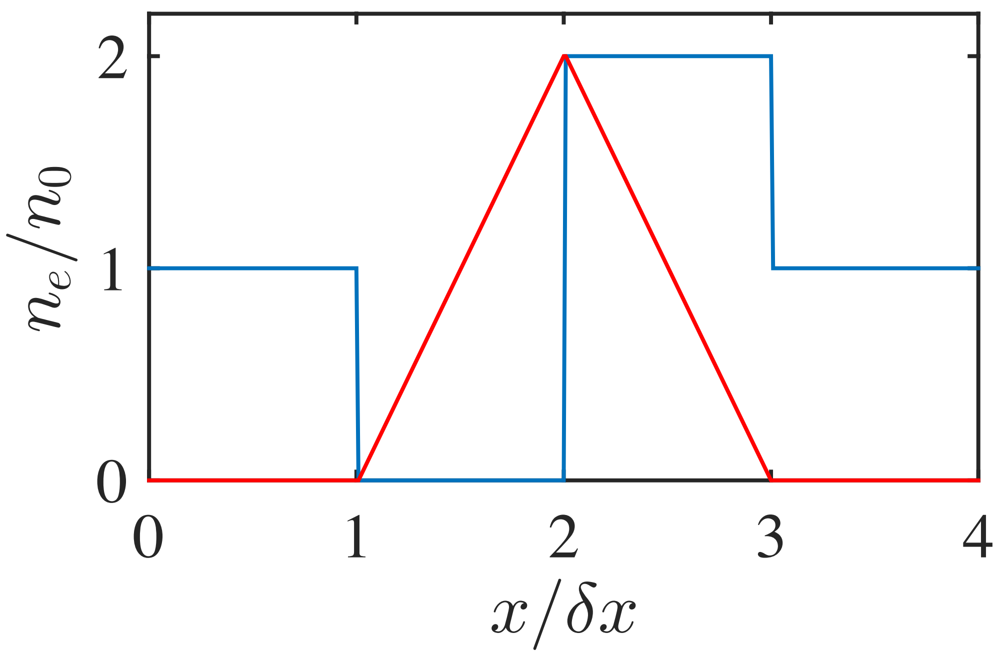
- Displace all electron in small volume by distance δx
- Find electric field from Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\frac{\mathrm{d}E_x}{\mathrm{d}x} = \frac{Zen_i - en_e}{\epsilon_0}$$

$$E_x = \int_0^{\delta x} \frac{en_0}{\epsilon_0}\mathrm{d}x$$

$$\approx \frac{en_0\delta x}{\epsilon_0}$$



The plasma frequency

- Use this field in equation of motion for an electron

$$m_e \frac{\mathrm{d}^2 \delta x}{\mathrm{d}t^2} = -eE$$

$$= -\frac{e^2 n_0 \delta x}{\epsilon_0}$$

$$\frac{\mathrm{d}^2 \delta x}{\mathrm{d}t^2} + \left(\frac{e^2 n_0}{m_e \epsilon_0}\right) \delta x = 0$$

$$\boxed{\omega_p = \left(\frac{e^2 n_0}{m_e \epsilon_0}\right)^{\frac{1}{2}}}$$

The plasma frequency

- Conventional RF accelerator operates at GHz frequencies:

$$f_p(n_e[\mathrm{cm}^{-3}]) = \omega_p/(2\pi) \approx 9000\sqrt{n_e}\ \mathrm{Hz}$$

$$f_p(n_e = 1 \times 10^{18}\mathrm{cm}^{-3}) = 9 \times 10^{13}\ \mathrm{Hz}$$

- Wavelength of conventional RF structure is

$$\lambda_{\mathrm{RF}} = \frac{c}{f} \approx 0.25\ \mathrm{m}$$

- Wavelength of plasma accelerator structure will be:

$$\lambda_p = \frac{c}{f_p}$$

$$\lambda_p(n_e = 10^{18}\ \mathrm{cm}^{-3}) \approx 33\ \mu\mathrm{m}$$

Plasma's support very strong electric fields

- Consider Electric field strength associated with a plasma density perturbation with wavenumber $k_p = \omega_p / c$

- For large amplitude plasma waves we get very large electric fields

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$k_p E_x = -e \frac{\delta n_e}{\epsilon_0}$$

$$E_x = \frac{\delta n_e}{\epsilon_0} \frac{c}{\omega_p} \times \frac{n_e}{n_e}$$

$$= \frac{\delta n_e}{n_e} \frac{m_e c \omega_p}{e}$$

$$E_x \left(n_e = 10^{18} \text{ cm}^{-3}\right) \approx 10^{11} \text{ Vm}^{-1} = 100 \text{ GVm}^{-1}$$

Electro magnetic waves in a plasma

- One way to excite a plasma wave with a phase velocity close to c is to use an intense laser pulse — **laser wakefield acceleration**
- Need to know how lasers propagate inside plasmas: the plasma dispersion relation from Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{E} = E_x e^{i(kz-\omega t)}\hat{\mathbf{x}}$$

Electro magnetic waves in a plasma

- Take time derivative of Ampere's law

$$\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- Insert Faraday's law and use vector triple product

$$-\nabla \times (\nabla \times \mathbf{E}) = \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- Look for transverse plane waves of form:

$$\mathbf{E} = E_x e^{i(kz-\omega t)}\hat{\mathbf{x}}$$

$$-k^2 E - \cancelto{0}{\nabla(\nabla \cdot \mathbf{E})} = \mu_0 \boxed{\frac{\partial j}{\partial t}} - \frac{\omega^2}{c^2} E$$

the plasma response

Electromagnetic waves in a plasma

- current density is:

$$j_x = -n_e e v_x$$

$$\frac{\partial j_x}{\partial t} = -n_e e \frac{\partial v_x}{\partial t}$$

- Acceleration of electron in x is due to wave's electric field

$$m_e \frac{\partial v_x}{\partial t} = -eE_x$$

$$\frac{\partial j_x}{\partial t} = \frac{n_e e^2}{m_e} E_x$$

- So our wave equation becomes:

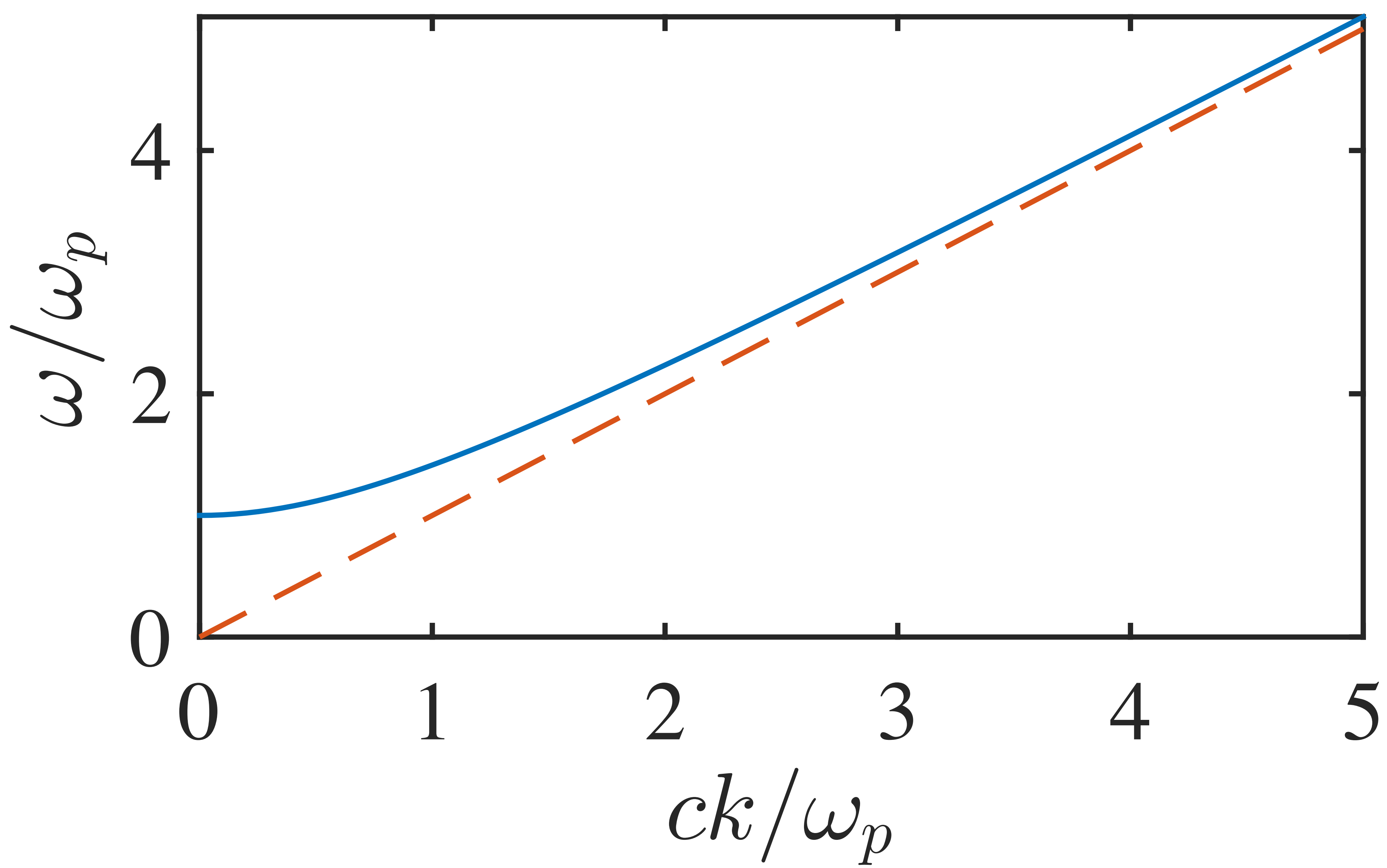
$$-k^2 E_x = \mu_0 \frac{n_e e^2}{m_e} E_x - \frac{\omega^2}{c^2} E_x$$

- Dispersion relation for EM waves in plasma:

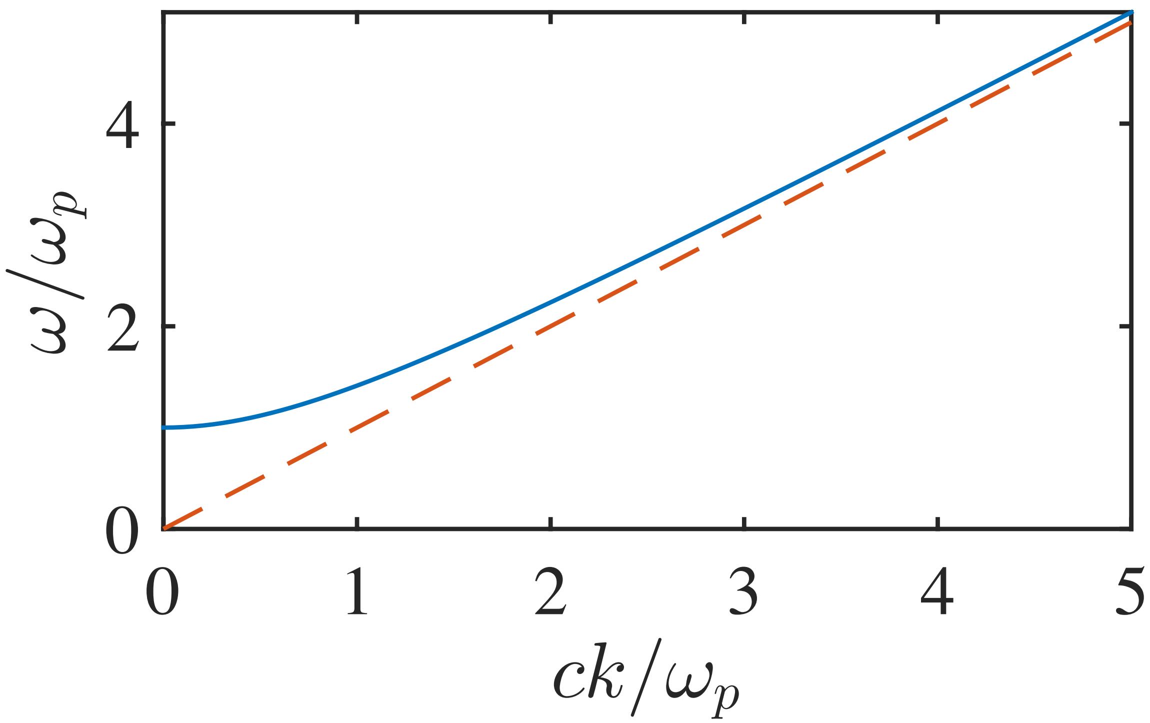
$$\boxed{\omega^2 = \omega_p^2 + c^2 k^2}$$

Electromagnetic waves in a plasma

$$\omega^2 = \omega_p^2 + c^2 k^2$$



Group and Phase velocity of EM waves in plasma



$$\omega^2 = \omega_p^2 + c^2 k^2$$

$$\omega \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}} = ck$$

$$v_p = \frac{\omega}{k} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-\frac{1}{2}}$$

$$\frac{v_p}{c} \approx 1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}$$

$$\omega^2 = \omega_p^2 + c^2 k^2$$

$$\cancel{2}\omega \frac{\mathrm{d}\omega}{\mathrm{d}k} = \cancel{2}c^2 k$$

$$v_g = c^2 \frac{k}{\omega} = \frac{c^2}{v_p}$$

$$v_g = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}$$

$$\frac{v_g}{c} \approx 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}$$

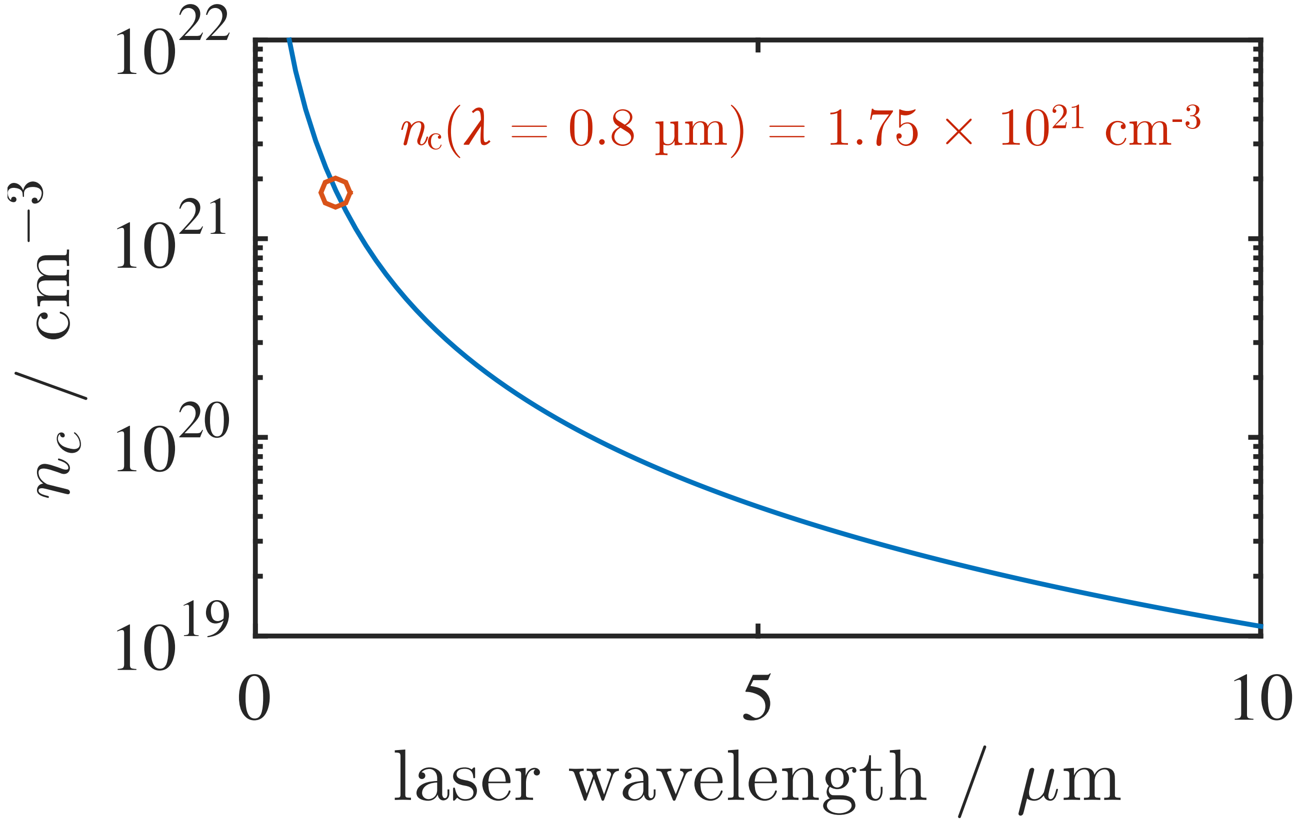
- phase velocity $v_p > c$ (!)
- group velocity $v_g < c$

Critical Density

- Dispersion relation tells us
  - if $\omega = \omega_p$; $v_g = 0$; i.e. EM wave cannot propagate
  - Using definition of ω_p we can define the critical density, n_c
  - Plasmas with density > critical are opaque
    - *laser light reflects: ion acceleration techniques*
  - Plasmas with density < critical are transparent
    - *laser light travels through plasma: wakefield acceleration*

$$\omega^2 = \frac{n_c e^2}{m_e \epsilon_0}$$

$$n_c = \frac{m_e \epsilon_0 \omega^2}{e^2}$$



Laser strength parameter

- consider motion of a single electron in a plane EM wave

$$E_x = E_0 \cos(kz - \omega t)$$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$A = \int E \; dt$$

$$A = \frac{E_0}{\omega}\sin(kz - \omega t) = A_0 \sin(kz - \omega t)$$

- Ignore the B field (v << c)
  - electron motion is a transverse oscillation (called a "quiver")
  - maximum normalised quiver momentum of electron is given by laser strength parameter, a_0

$$a_0 = \frac{eA_0}{m_e c} = \frac{eE_0}{m_e \omega c} \propto \sqrt{I\lambda^2}$$

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -eE_x$$

$$p_x = -e\int E_x \mathrm{d}t$$

$$= -\frac{eE_0}{\omega}\sin(kz - \omega t)$$

$$\frac{p_x}{m_e c} = -\frac{eE_0}{m_e \omega c}\sin(kz - \omega t)$$

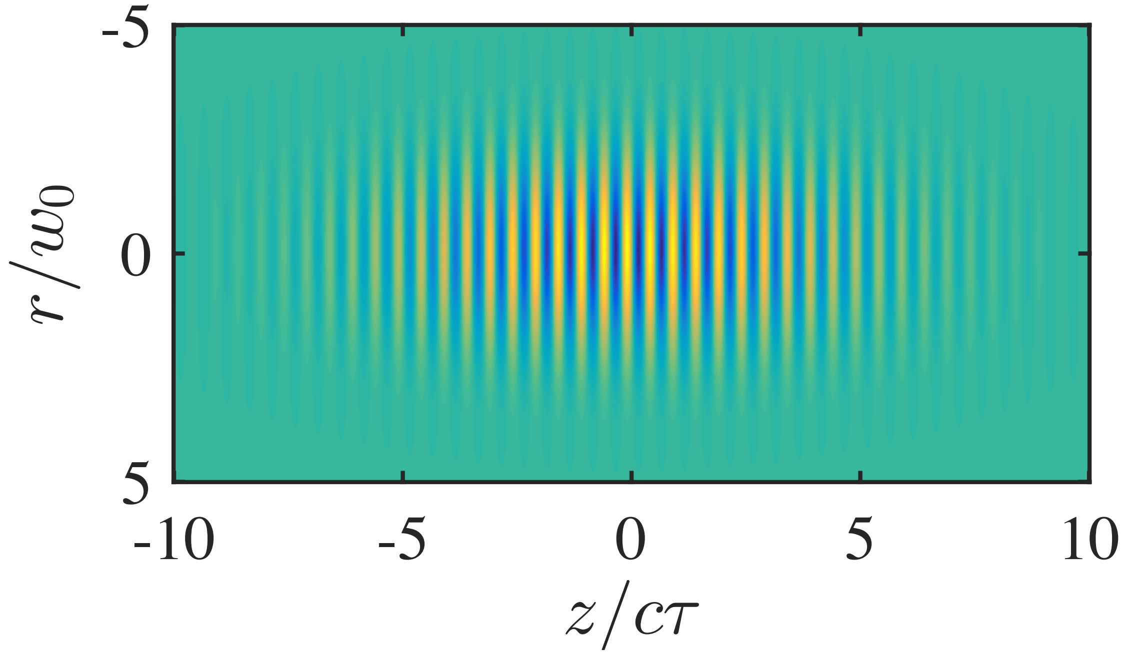
$$= -\frac{eA_0}{m_e c}\sin(kz - \omega t)$$

$$= a_0 \sin(kz - \omega t)$$

Ponderomotive force

- "real" lasers have a spatially varying intensity profile
- consider average kinetic energy of an electron at r,z
  - We can think of this position dependent energy as a potential energy
  - Gradient in this potential energy is a force — ***the ponderomotive force***
  - Ponderomotive force pushes charged particles away from regions of high laser intensity

$$
\begin{aligned}
a(r,z) &= a_0 0 \exp\left[-\frac{r^2}{w_0^2}\right] \exp\left[-\frac{(z-v_g t)^2}{c^2\tau^2}\right] \sin(kz-\omega t) \\
&= a_0(r,z)\ \sin(kz-\omega t)
\end{aligned}
$$



$$
\begin{aligned}
\langle K \rangle &= \frac{p_x^2}{2m_e} \\
&= \frac{1}{2} m_e c^2 a_0^2
\end{aligned}
$$

$$
\mathbf{F}_p = -\frac{1}{2} m_e c^2 \nabla a_0^2
$$

**this is a non-relativistic "derivation" but full result is very similar!

Ponderomotive force

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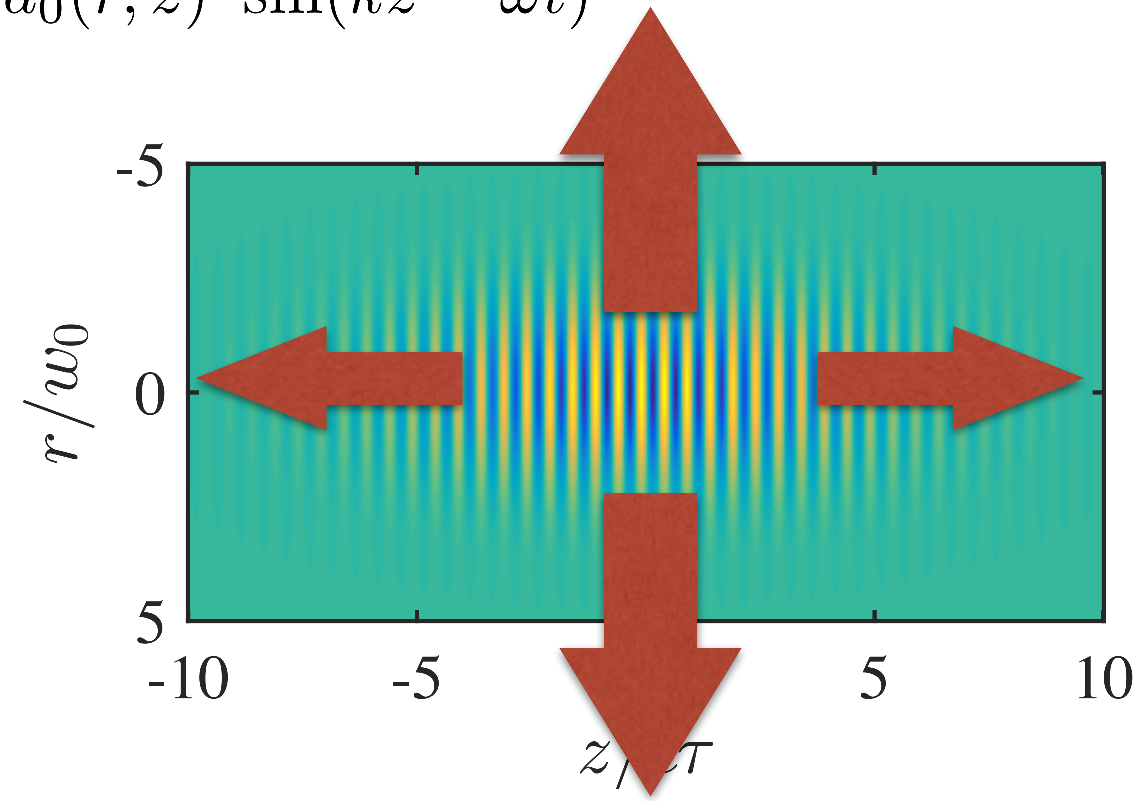


$$\langle K \rangle = \frac{p_x^2}{2m_e} = \frac{1}{2} m_e c^2 a_0^2$$

$$\mathbf{F}_p = -\frac{1}{2} m_e c^2 \nabla a_0^2$$

**this is a non-relativistic "derivation" but full result is very similar!

Driving Plasma Waves with lasers or particle beams

$$m_e\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right) = -\frac{1}{2}m_e c^2 \nabla a_{\mathrm{laser}}^2 - e\mathbf{E}_{\mathrm{beam}} - e\mathbf{E}_{\mathrm{plasma}}$$

- Treat plasma as a fluid (use momentum equation and continuity equation)
- Consider two drive terms
  - laser (through ponderomotive force)
  - charged particle beam (through electric field)
- Do in one dimension
- linearise equations (ignore products of small terms)

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e \mathbf{v}) = 0$$

$$E = \cancelto{0}{E_0} + E_1$$

$$v = \cancelto{0}{v_0} + v_1$$

$$n = n_0 + n_1$$

$$\frac{\partial n_1}{\partial t} + n_0 \frac{\partial v_1}{\partial x} = 0$$

$$m_e \frac{\partial v_1}{\partial t} = -\frac{1}{2}m_e c^2 \nabla a_{\mathrm{laser}}^2 - eE_{\mathrm{beam}} - eE_1$$

Driving Plasma Waves with lasers or particle beams

- take d/dt of continuity and d/dx of momentum equation and combine

$$\frac{\partial^2 n_1}{\partial t^2} + n_0 \frac{\partial^2 v_1}{\partial x \partial t} = 0$$

$$m_e \frac{\partial^2 v_1}{\partial t \partial x} = -\frac{1}{2} m_e c^2 \frac{\partial^2 a_{\text{laser}}^2}{\partial x^2} - e\frac{\partial E_{\text{beam}}}{\partial x} - e\frac{\partial E_1}{\partial x}$$

$$m_e \frac{\partial^2 n_1}{\partial t^2} - n_0 \frac{1}{2} m_e c^2 \frac{\partial^2 a_{\text{laser}}^2}{\partial x^2} - e n_0 \frac{\partial E_{\text{beam}}}{\partial x} - e n_0 \frac{\partial E_1}{\partial x} = 0$$

- use Gauss's law to find E field due to plasma response

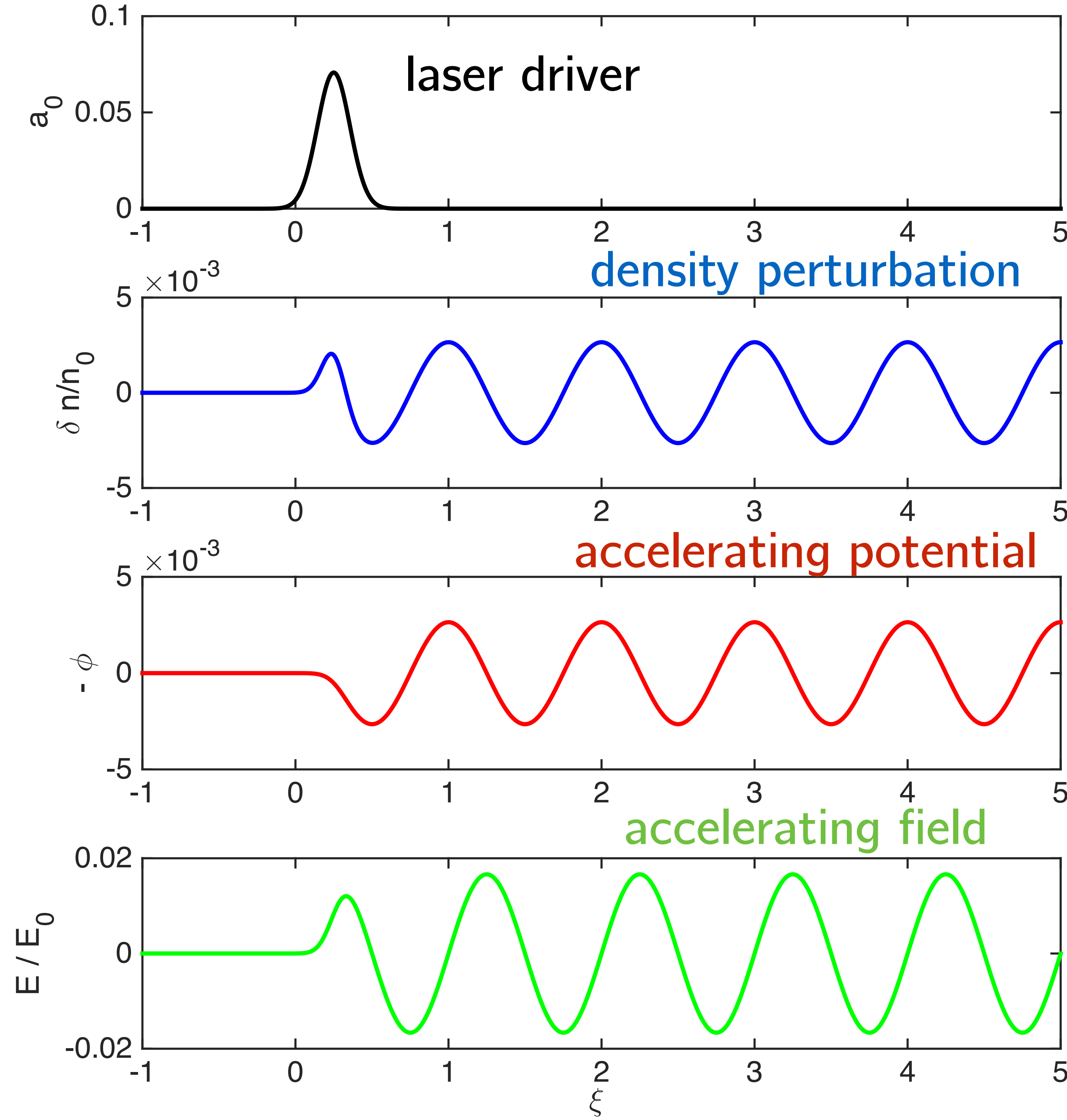
$$\begin{aligned}\frac{\partial E}{\partial x} = \frac{\rho}{\epsilon_0} &= \frac{Zen_i - en_e}{\epsilon_0} \\ &= \frac{Zen_i - e(n_0 + n_1)}{\epsilon_0} \\ &= -\frac{en_1}{\epsilon_0}\end{aligned}$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)\frac{n_1}{n_0} = -\frac{c^2}{2}\frac{\partial^2 a_{\text{laser}}^2}{\partial x^2} - \omega_p^2 \frac{n_{\text{beam}}}{n_0}$$

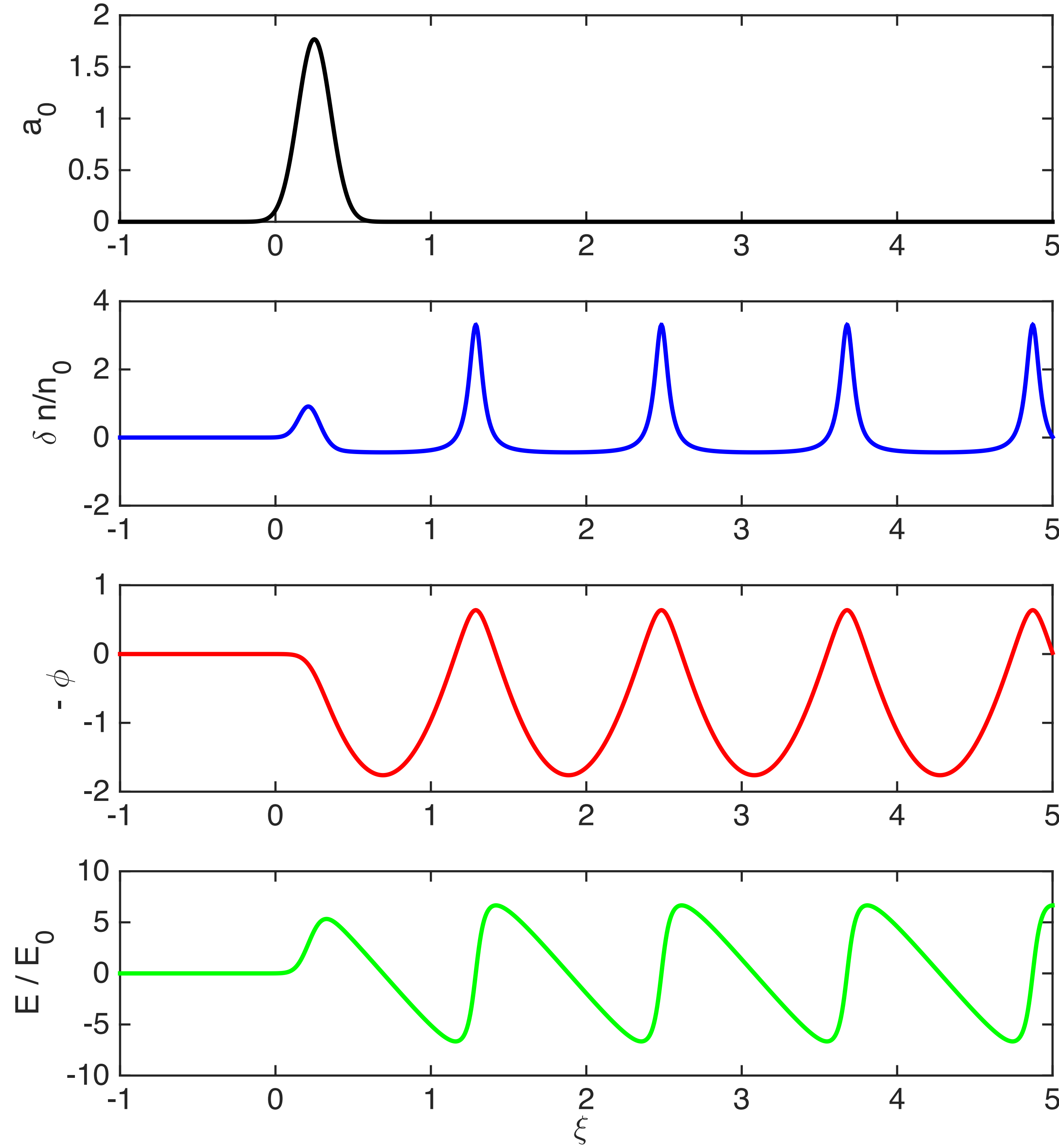
Driving Plasma Waves with lasers or particle beams

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)\frac{n_1}{n_0} = -\frac{c^2}{2}\frac{\partial^2 a_{\text{laser}}^2}{\partial x^2} - \omega_p^2\frac{n_{\text{beam}}}{n_0}$$

- Solution for this linear non-evolving driver moving at speed v is:
  - sinusoidal density pertubation
  - sinusoidal potential
  - sinusoidal electric field
  - valid if $n_1 << n_0$



Driving Plasma Waves with lasers or particle beams



- A proper non-linear treatment also possible:
  - density spikes
  - quadratic potential
  - linear electric field
  - lengthening of wavelength (a relativistic shift of ω_p)
  - occurs when $n_1 \sim n_0$

Summary - and what's next?

Introduced some key concepts you will encounter in plasma based accelerators:

- plasma oscillations (the plasma wave in a plasma accelerator)
- ponderomotive force (what drives the plasma wave)
- driving plasma waves

Next time with me we will look at:

- Key limits to plasma based accelerators
- non-linear 3D plasma waves: the blow out or bubble regime
- Radiation generation in plasma accelerators