# Transverse Beam Dynamics - Tutorial

JAI lectures 2021 - Michaelmas Term

## 1 Preliminary exercices

1. Watch this Iron Man clip and discuss the main accelerator physics concepts involved either if they are properly represented or not in the movie.
2. Go through the short questions posted during lectures and try to answer them.

## 2 To think about

1. How can we measure $\beta^*$($\beta$-function at the IP) in the LHC?

We cannot measure it directly because we do not have BPMs a the IP. However using K-modulation technique, the strength of the last quadrupole before the IP is modulated. This modulation produces a measurable tune shift. The tune shift is linerly related to the $\beta$-function at the quadrupole location.

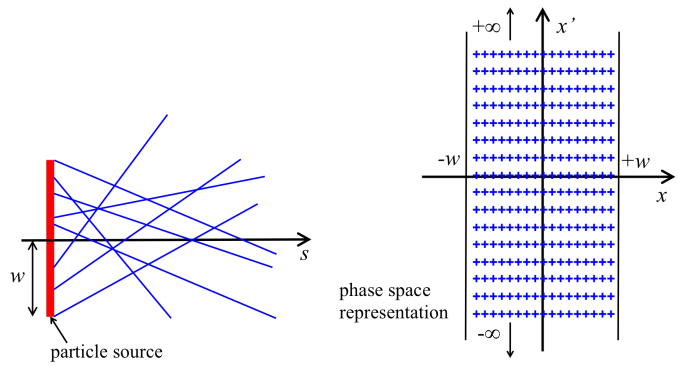
$$\Delta Q = \frac{\beta_q \Delta K}{4\pi}$$

By transporting the measured $\beta$-function at the quadrupole to the IP we can have an estimation of the $\beta$-function at that location.

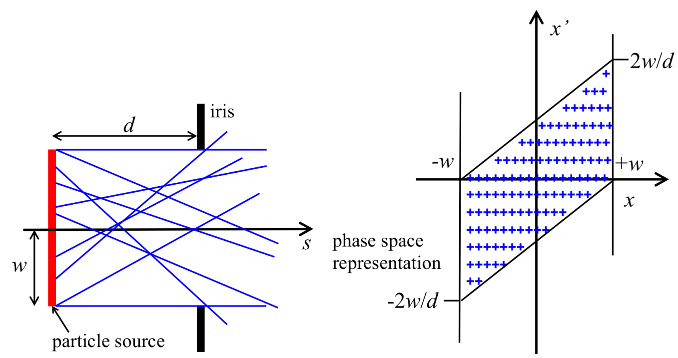
1. What are the possible effects of ground motion in the beam?
   (a) Orbit distorsion.
   (b) Emittance growth.
2. What can we do if there is a small object partially blocking the beam aperture?
   (a) Orbit bump.
   (b) Evaporate it.
   (c) Open the machine and remove it.

## 3 Exercise: Understanding the phase space concept

1. Phase Space Representation of a Particle Source:
   - Consider a source at position $s_0$ with radius $w$ emitting particles. Make a drawing of this setup in the configuration space and in the phase space. Which part of the phase space can be occupied by the emitted particles?
     **Answer.** Particles are emitted from the entire source surface $x \in [-w, +w]$ with a trajectory slope $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ , i.e. the particles can have any $x' \in \mathbb{R}$. The occupied phase space area is infinite.

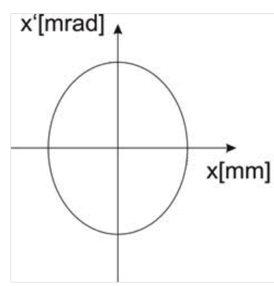



- Any real beam emerging from a source like the one above will be collimated. This can be modelled by assuming that a distance $d$ away from the source there is an iris with opening radius $R = w$. Draw this setup in the configuration space and in the phase space. Which part of the phase space is occupied by the beam, right after the collimator?
  **Answer.** Particles with angle $x' = 0$ are emitted from the entire source surface $x \in [-w, +w]$ and arrive behind the iris opening. For $x = \pm w$ there is a maximum angle $x' = \pm 2w/d$ that will still be accepted by the iris. This leads to a parallelogram in phase space. Such a beam has a specific emittance given by the occupied phase space area.

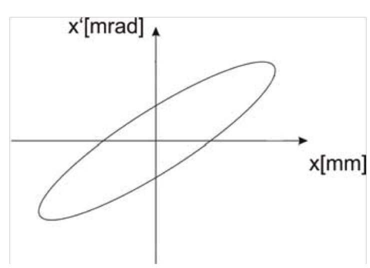



2. Sketch the emittance ellipse of a particle beam in:

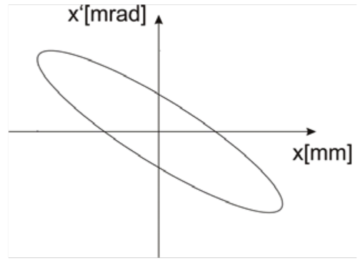
(I) horizontal $x$-$x'$ phase space at the position of a transverse waist,
**Answer.** Beam at the position of a transverse ($x$) waist




(II) when the beam is divergent, and
**Answer.** Divergent beam (positive slope):



(III) when the beam is convergent.
**Answer.** Convergent beam (negative slope):




## 4 Exercise: Stability condition

Consider a lattice composed by a single 2 meters long quadrupole, with $f = 1$ m

- Prove that if the quadrupole is defocusing, then a lattice is not stable
- Prove that if the quadrupole is focusing, then the lattice is stable

Solution:
This quadrupole is clearly thick. Therefore one should use the thick quadrupole matrices. However, I post the thin lens calculation for comparison.

- In the case of a defocusing quadrupole:

$$M_{\mathrm{QD}} = \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix}$$

which can be computed to be

$$M_{\mathrm{QD}} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

has trace $\mathrm{Tr}\,(M_{\mathrm{QD}}) = 4$, which does not fulfill the stability requirement:

$$|\mathrm{Tr}\,(M_{\mathrm{QD}})| \leq 2$$

- In the case of a focusing quadrupole:

$$M_{\mathrm{QF}} = \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which clearly satisfies the stability criterion.

## 5 Twiss functions evolution

Which of the optics parameters can be constant

1. In a drift.
2. In a quadrupole with constant strength $K$.

Justify the response.
**Hint:** The differential equation representing the evolution of the $\beta$-function reads,

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 K = 1$$

### Solution

Let's consider the two cases separately:

1. In a drift $K = 0$,

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 = 1$$

therefore, $\beta$ cannot be constant ($\beta' = \beta'' = 0$).
Taking into account that $-2\alpha = \beta'$ , if $\alpha = \text{const.}$, $\beta'' = 0$ and $\beta$ must evolve linearly with $s$.

2. In a quadrupole, $K \neq 0$
therefore $\beta$ is constant if

$$\beta = \frac{1}{\sqrt{K}}$$

In addition, if$\alpha$ is constant,

$$-\alpha^2 + \beta^2 K = 1$$

$$K = \frac{1+\alpha^2}{\beta^2}$$

## 6 Exercise: Bump and Orbit Control

Given two kickers located at the two ends of a FODO cell with phase advance 45 degrees (the two kickers are located at $L_{\text{cell}}$ distance from each other), compute the strengths of such kickers (in radians) in order to give the beam, initially at $(x_i,\, x'_i) = (0,\, 0)$, an arbitrary offset at the end of the cell while preserving its angle, $\left(x_f,\, x'_f\right) = (x_{\text{arbitrary}},\, 0)$.

### Solution

The transfer matrix of a periodic cell is:

$$M = \begin{pmatrix} \cos\varphi + \alpha\sin\psi & \beta\sin\varphi \\ -\gamma\sin\varphi & \cos\varphi - \alpha\sin\varphi \end{pmatrix}$$

Substituting the value for the phase advance we get the matrix to apply to the beam right after the first kick $k_1$:

$$\begin{pmatrix} x_f \\ x'_f \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1+\alpha & \beta \\ -\gamma & 1-\alpha \end{pmatrix}\begin{pmatrix} 0 \\ k_1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \beta k_1 \\ (1-\alpha)k_1 \end{pmatrix}$$

From this we see that to achieve an arbitrary $x_f$ we need:

$$k_1 = \frac{\sqrt{2}x_f}{\beta}$$

The second kick, $k_2$, has only to remove the final tilt:

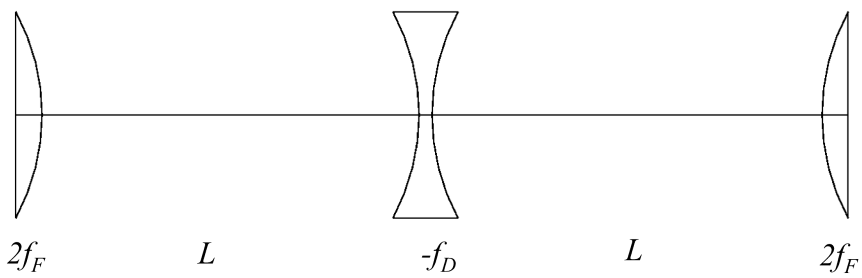
$$k_2 = -x'_f = -\frac{(1-\alpha)}{\sqrt{2}}k_1$$

Notice that one can reduce the strength of the kickers by placing them close to a focusing quadrupoles, where $\beta$ is maximum.

# 7 Exercise: Chromaticity in a FODO cell

Consider a ring made of $N_{cell}$ identical FODO cells with equally spaced quadrupoles. Assume that the two quadrupoles are both of length $l_q$, but their strengths may differ.

1. Calculate the maximum and the minimum betatron function in the FODO cell. (*Use the thin-lens approximations*)




**Answer.** First we calculate the transfer matrix for a FODO cell (see figure). We start from the centre of the focusing quadrupole where the betatron function is maximum. This exercise considers a general case where $f_F$ is not necessarily equal to $f_D$. Using the thin lens approximation for the FODO cell with drifts of length $L$ we get the following matrix:

$$\begin{aligned} M_{cell} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) & 2L + \frac{L^2}{f_D} \\ \frac{1}{f_D} - \frac{1}{f_F}(1 - \frac{L}{2f_F} + \frac{L}{f_D} - \frac{L^2}{4f_F f_D}) & 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) \end{pmatrix} \end{aligned} \tag{1}$$

Remember that, in terms of betatron functions and phase advance, the matrix of a FODO cell is given by:

$$M_{cell} = \begin{pmatrix} cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} \tag{2}$$

Since $\beta$ has a maximum at the centre of the focusing quadrupole, then $\alpha = -\beta'/2 = 0$, and we can also write:

$$M_{cell} = \begin{pmatrix} cos\mu & \beta \sin\mu \\ -\frac{\sin\mu}{\beta} & \cos\mu \end{pmatrix}$$

Equating Eq. (1) to Eq. (3) we obtain:

$$\cos\mu = \frac{1}{2}\mathrm{tr}(M_{cell}) = 1 + \frac{L}{f_D} - \frac{L}{f_F} - \frac{L^2}{2f_D f_F} = 1 - 2\sin^2\frac{\mu}{2}$$

or

$$2\sin^2\frac{\mu}{2} = \frac{L}{f_F} - \frac{L}{f_D} + \frac{L^2}{2f_D f_F} \tag{3}$$

Where we have applied the following trigonometric identity: $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$.

The maximum for the betatron function $\beta_{max}$ occurs at the focusing quadrupole. Since Eq. (1) is for a periodic cell starting at the centre of the focusing quadrupole, the $m_{12}$ component of the matrix gives us

$$\beta_{max}\sin\mu = 2L + \frac{L^2}{f_D}$$

Rearranging:

$$\beta_{max} = \frac{2L + \frac{L^2}{f_D}}{\sin\mu} \tag{4}$$

On the other hand, the minimum for the betatron function occurs at the defocusing quadrupole position. Therefore, interchanging $f_F$ with $-f_D$ for a FODO cell gives:

$$\beta_{min} = \frac{2L - \frac{L^2}{f_F}}{\sin\mu} \tag{5}$$

2. Calculate the natural chromaticities for this ring.

   **Answer.** Let us remember the definition of natural chromaticity. The so-called "natural" chromaticity is the chromaticity that derives from the energy dependence of the quadrupole focusing, i.e. the chromaticity arising only from quadrupoles. The chromaticity is defined in the following way:

$$\xi = \frac{\Delta Q}{\Delta P/P_0} \tag{6}$$

   where $\Delta Q$ is the tune shift due to the chromaticity effects and $\Delta P/P_0$ is the momentum offset of the beam or the particle with respect to the nominal momentum $p_0$.

   The natural chromaticity is defined as (remember from Lecture 4):

$$\xi_N = -\frac{1}{4\pi}\oint \beta(s)k(s)ds \tag{7}$$

   Sometimes, especially for small accelerators, the chromaticity is normalised to the machine tune Q and defined also as:

$$\xi' = \frac{\Delta Q/Q}{\Delta P/P_0} \tag{8}$$

$$\xi'_N = -\frac{1}{4\pi Q}\oint \beta(s)k(s)ds \tag{9}$$

   For this exercise, either you decide to use Eq. (7) or Eq. (9) it is fine! From now on let us use Eq. (7):

$$\begin{aligned}\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\ &= -\frac{1}{4\pi} \times N_{cell}\int_{cell} \beta(s)k(s)ds \\ &= -\frac{N_{cell}}{4\pi}\sum_{i\in\{quads\}} \beta_i(kl_q)_i\end{aligned}$$

   Here we have used the following approximation valid for thin lens:

$$\int_{cell} \beta(s)k(s)ds \simeq \sum_{i\in\{quads\}} \beta_i(kl_q)_i$$

   where we sum over each quadrupole $i$ in the cell. In the case of the FODO cell we have two half focusing quadrupoles and one defocusing quadrupole. Taking into account that $(kl_q)_i = 1/f_i$, we have:

$$
\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{4\pi} \sum_{i\in\{quads\}} \beta_i (kl_q)_i \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{2f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right) + \beta_{max}\left(\frac{1}{2f_F}\right)\right] \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right)\right] \\
&= -\frac{N_{cell}}{4\pi\sin\mu}\left[\left(2L + \frac{L^2}{f_D}\right)\frac{1}{f_F} - \left(2L - \frac{L^2}{f_F}\right)\frac{1}{f_D}\right] \\
&= -\frac{N_{cell}L}{2\pi\sin\mu}\left[\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{f_F f_D}\right]
\end{aligned}
$$

Here we have used the expressions (4) and (5) for $\beta_{max}$ and $\beta_{min}$.

3. Show that for short quadrupoles, if $f_F \simeq f_D$,

$$
\xi_N \simeq -\frac{N_{cell}}{\pi}\tan\frac{\mu}{2}.
$$

**Answer.** If $f_F \simeq f_D$, we have

$$
\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{2\pi\sin\mu}\frac{L^2}{f_F f_D} \\
&= -\frac{N_{cell}}{4\pi\sin\frac{\mu}{2}\cos\frac{\mu}{2}} 4\sin^2\frac{\mu}{2}
\end{aligned}
$$

where we have used the trigonometric identity: $\sin\mu = 2\sin\frac{\mu}{2}\cos\frac{\mu}{2}$
Considering Eq. (3), we have

$$
4\sin^2\frac{\mu}{2} = \frac{L^2}{f_F f_D}
$$

which finally gives:

$$
\xi_N \simeq -\frac{N_{cell}}{\pi}\tan\frac{\mu}{2}
$$

Q.E.D.!

4. Design the FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are $\beta_{max}$, $\beta_{min}$, $D_{max}$, $D_{min}$?

**Answer.** Lattice parameters: $L = 10$ m, $\theta = 5$ degrees$= 0.087266$ rad, $f = \frac{1}{\sqrt{2}}\frac{L}{2} = 3.535$ m

Maximum and minimum betatron functions:

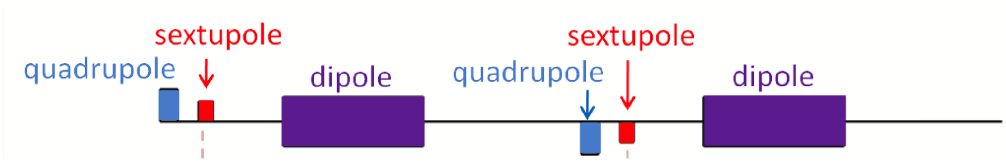
$$
\beta_{max} = \frac{L + \frac{L^2}{4f}}{\sin\mu} = L + \frac{L^2}{4f} = 17.07 \text{ m}, \quad \beta_{min} = \frac{L - \frac{L^2}{4f}}{\sin\mu} = L - \frac{L^2}{4f} = 2.93 \text{ m}
$$

Maximum and minimum dispersion:

$$
D_{max} = \frac{L\theta\left(1 + \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f + \frac{L}{2}\right)\theta = 0.59060 \text{ m}, \quad D_{min} = \frac{L\theta\left(1 - \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f - \frac{L}{2}\right)\theta = 0.28207 \text{ m}
$$

5. Add two sextupoles at appropriate locations to correct horizontal and vertical chromaticities. (hints: use 1 sextupole for the horizontal plane and 1 for the vertical plane; do not consider geometric aberrations).

**Answer.** By locating sextupoles with strength $K_s > 0$ where $\beta_x$ is large and $\beta_y$ is small, we can correct the horizontal chromaticity with relatively little impact on the vertical chromaticity. Similarly, by locating sextupoles with $K_s < 0$ where $\beta_y$ is large and $\beta_x$ is small, we can correct the vertical chromaticity with relatively little impact on the horizontal chromaticity. See figure below.




Let us assume the case of a FODO lattice where $f_F = f_D = f$. Then the natural chromaticity of this FODO cell is given by the expression (exercise 1.3):

$$\xi_N \simeq -\frac{1}{\pi}\tan\frac{\mu}{2}$$

For $\mu = 90$ it is $\xi_N \simeq -1/\pi$ in both horizontal and vertical plane. Therefore, we need to adjust the strength of the sextupoles to cancel this chromaticity:

$$-\frac{1}{4\pi}\left[K_{2F}D_{max}\beta_{max} + K_{2D}D_{min}\beta_{min}\right] \simeq -\frac{1}{\pi}$$

where $K_{2F} = k_{2F}l_s$ is the normalised integrated strength of the sextupole located near the focusing quadrupole, and $K_{2D} = k_{2D}l_s$ the normalised integrated strength of the sextupole near the defocusing quadrupole (with $l_s$ the effective length of the sextupole). For an effective cancellation of the chromaticity in both planes, usually $K_{2F} > 0$ and $K_{2D} < 0$. Substituting the values for the maximum and minimum dispersion and betatron function in terms of the total length of the lattice $L$ and the focal length of the quadrupoles $f$, one obtains the following expression:

$$-\frac{1}{4\pi}\frac{f}{L}\theta\left[K_{2F}\left(4f+\frac{L}{2}\right)\left(L+\frac{L^2}{4f}\right) + K_{2D}\left(4f-\frac{L}{2}\right)\left(L-\frac{L^2}{4f}\right)\right] \simeq -\frac{1}{\pi}$$

Considering the same absolute value for the strength of the sextupoles, $K_{2F} = -K_{2D} = K_s$, we can write then:

$$\frac{3}{4\pi}K_sLf\theta = \frac{1}{\pi}$$

The strength of the sextupole is given then by:

$$K_s = \frac{4}{3Lf\theta}$$

Then, substituting all the numerical values for the lattice parameters:

$K_{2F} = 0.865\ \mathrm{m}^{-2}$

$K_{2D} = -0.865\ \mathrm{m}^{-2}$

6. If the gradient of all focusing quadrupoles in the ring is wrong by $+10\%$, how much is the tune-shift with and without sextupoles?

**Answer.**

If the gradient of the focusing quadrupole has and error of 10%, then the corresponding quad. strength error is also 10%. We calculate the number of cells of a ring made of these FODO cells, $N_{cell} = 72$ cells, and then we calculate the total tune-shift in both planes:

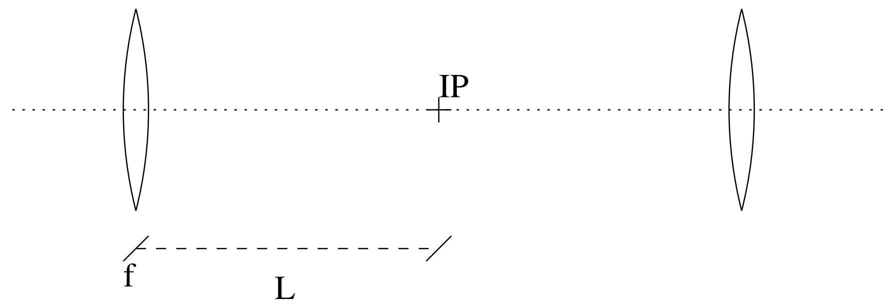
$\Delta Q_x = N_{cell} \frac{\Delta K_F \beta_{max}}{4\pi} = 9.78$

$\Delta Q_y = N_{cell} \frac{\Delta K_F \beta_{min}}{4\pi} = 1.68$

When the sextupoles correct for the chromaticity, the particles have, in principle, no tune-shift with energy. In real machines, one wants to have a non-zero residual chromaticity to stabilise the beam against resonant imperfections.

# 8 Exercise: Low-Beta Insertion

Consider the following low-beta insertion around an interaction point (IP). The quadrupoles are placed with mirror-symmetry with respect to the IP:




The beam enters the quadrupole with Twiss parameters $\beta_0 = 20$ m and $\alpha_0 = 0$. The drift space has length $L = 10$ m.

(i) Determine the focal length of the quadrupole in order to locate the waist at the IP.

(ii) What is the value of $\beta^\star$?

(iii) What is the phase advance between the quadrupole and the IP?

**Solution.**

$$M = \begin{pmatrix} 1 - \frac{L}{f} & L \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{\mathrm{IP}} = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

$$\begin{pmatrix} \beta_{\mathrm{IP}} & 0 \\ 0 & 1/\beta_{\mathrm{IP}} \end{pmatrix} = M \cdot \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \cdot M^T$$

We get a system of equations:

$$\begin{cases} \beta_{\mathrm{IP}} = & \beta_0 \left(1 - \frac{L}{f}\right)^2 + \frac{L^2}{\beta_0} \\ \frac{1}{\beta_{\mathrm{IP}}} = & \frac{\beta_0}{f^2} + \frac{1}{\beta_0} \end{cases}$$

multiplying them:

$$1 = \left(\beta_0 \left(1 - \frac{L}{f}\right)^2 + \frac{L^2}{\beta_0}\right)\left(\frac{\beta_0}{f^2} + \frac{1}{\beta_0}\right)$$

and solving for $f$:

$$f = \frac{\beta_0 \sqrt{(\beta_0^2 - 4L^2)} + \beta_0^2}{2L}$$

from which one finds:

$$f = 20 \text{ m}$$

and substituting back into one of the equations in the system:

$$\beta_{\mathrm{IP}} = 10 \text{ m.}$$

The phase advance can be computed remembering that

$$M_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\psi_s - \alpha_s\sin\psi_s\right) \end{pmatrix}$$

In this case, $\alpha_0 = \alpha_{\mathrm{IP}} = 0$,

$$\mathrm{Trace}\,(M) = \frac{3}{2} = \left(\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}\right)\cos\Delta\mu$$

$$\Delta\mu = \arccos\left(\frac{3}{2}\cdot\frac{1}{\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}}\right) = \arccos\left(\frac{3}{2}\cdot\frac{1}{2.1213}\right) = 45 \text{ degrees}$$

Alternatively, given that the system:

$$M = Q\cdot D\cdot D\cdot Q$$

is indeed periodic, one can say:

$$M = \begin{pmatrix} 1-\frac{2L}{f} & 2L \\ \frac{2L}{f^2}-\frac{2}{f} & 1-\frac{2L}{f} \end{pmatrix}$$

$$\cos\Delta\mu_{\mathrm{twice}} = \frac{1}{2}\mathrm{Trace}\,(M) = \frac{1}{2}\mathrm{Trace}\left(2-\frac{4L}{f}\right) = 0$$

$$\Delta\mu_{\mathrm{twice}} = 90 \text{ degrees} \qquad \Rightarrow \Delta\mu = 45 \text{ degrees}$$