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Accelerator Physics

Lecture 11: Momentum Effects

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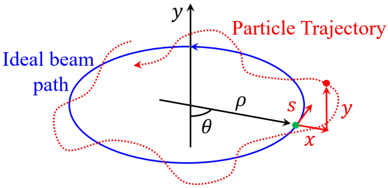
Contents

Curvilinear Co-ordinates



- (x, y, s), often called the standard co-ordinate system in accelerator physics
- The origin is defined by the vector $\vec{S}(s)$ following the ideal reference path
- $x = r - \rho \qquad s = \rho\theta$
- $X = r\sin\theta = (\rho+x)\sin\theta,\ Y = y,\ Z = r\cos\theta = (\rho+x)\cos\theta$

Transverse Equation of Motion - 1

- Start with the basics

$$\begin{aligned} F_x &= m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{mv^2}{r} \\ &= m\frac{\mathrm{d}^2(x+\rho)}{\mathrm{d}t^2} - \frac{mv^2}{x+\rho} = -eB_y v \end{aligned} \tag{1}$$

- Factorise the equation

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 + \frac{x}{\rho}\right)^{-1} = -eB_y v \tag{2}$$

- Utilise the binomial approximation

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -eB_y v \tag{3}$$

Transverse Equation of Motion - 2

- Replace t with s and rearrange

$$mv^2 \frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -eB_y v \tag{4}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{eB_y}{mv} \tag{5}$$

- Consider small displacements in x

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{e}{\color{red}{mv}}\left(B_0 + x\frac{\partial B_y}{\partial x}\right) \tag{6}$$

Transverse Equation of Motion - 3

- Set field gradient, $g = \frac{\partial B_y}{\partial x}$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{eB_0}{mv} - \frac{exg}{mv} \tag{7}$$

  This is a modified Hill's equation

- Consider small momentum offsets $\Delta p = p - p_0 \ll p_0$

$$\begin{aligned} \frac{1}{p_0 + \Delta p} &= \frac{1}{p_0}\left(1 + \frac{\Delta p}{p_0}\right)^{-1} \\ &\approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2} \end{aligned} \tag{8}$$

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Transverse Equation of Motion - 4

- Insert equation 8 into modified Hill's equation 7

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{eB_0}{p} - \frac{exg}{p}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = -\frac{eB_0}{p_0} + \frac{eB_0 \Delta p}{p_0^2} - \frac{exg}{p_0} + \frac{exg\Delta p}{p_0^2} \tag{9}$$

- Remember magnetic rigidity??, $B\rho = p/e$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + \frac{x}{\rho^2} = \frac{1}{\rho}\frac{\Delta p}{p_0} + kx \tag{10}$$

where $k = eg/p_0$ and the last term is the product of two small terms (≈ 0)

Transverse Equation of Motion - 5

- Finally a new **modified Hill's equation**

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + \left(\frac{1}{\rho^2} - k\right) x = \frac{1}{\rho} \frac{\Delta p}{p_0} \tag{11}$$

- Compare to the original Hill's equation from transverse lectures

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + \left(\frac{1}{\rho^2} - k\right) x = 0 \tag{12}$$

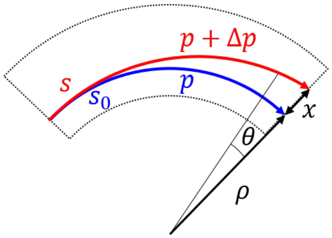
- **Particles with different momenta/energy have different orbits**

Dispersion

- General solution will be of the form $x(s) = x_h(s) + x_i(s)$

- From previous lecture, **dispersion** is defined as

$$D(s) = \frac{x_i(s)}{\Delta p/p_0} \qquad (13)$$



- It is just another orbit and is subject to the focusing properties of the lattice
- The orbit of any particle is the sum of the well-known x_h and dispersion

Matrix formalism

- Recall transfer matricies from transverse lectures and add dispersion

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p_0} \begin{pmatrix} D \\ D' \end{pmatrix} \tag{14}$$

where $C = \cos\sqrt{|k|}s$, $S = \frac{1}{\sqrt{k}}\sin\sqrt{|k|}s$, $C' = \frac{\mathrm{d}C}{\mathrm{d}s}$, $S' = \frac{\mathrm{d}S}{\mathrm{d}s}$ and $D'(s) = \frac{x_i'(s)}{\Delta p/p_0}$

- One can show that

$$D(s) = S(s)\int_{s_0}^{s_1} \frac{1}{\rho}C(s)\mathrm{d}s - C(s)\int_{s_0}^{s_1} \frac{1}{\rho}S(s)\mathrm{d}s \tag{15}$$

Examples of Dispersion - 1

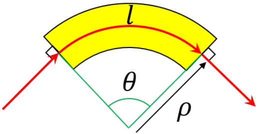
- Start with something simple, a **drift**!

$$M_{\text{drift}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \qquad C(s) = 1,\ S(s) = l \tag{16}$$

- Importantly $\rho = \infty$ so immediately $D_{\text{drift}} = 0$
- OK, how about a **pure sector dipole**?

$$M_{\text{dipole}} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

$$C(s) = \cos\frac{l}{\rho},\ S(s) = \rho\sin\frac{l}{\rho} \tag{17}$$



Examples of Dispersion - 2

- Putting this in the equation for dispersion

$$
\begin{aligned}
D_{\mathrm{dipole}}(s) &= \sin\frac{l}{\rho}\int_0^l \cos\frac{s}{\rho}\mathrm{d}s - \cos\frac{l}{\rho}\int_0^l \sin\frac{s}{\rho}\mathrm{d}s \\
&= \sin\frac{l}{\rho}\left[\rho\sin\frac{s}{\rho}\right]_0^l - \cos\frac{l}{\rho}\left[-\rho\cos\frac{s}{\rho}\right]_0^l \\
&= \rho\sin^2\frac{l}{\rho} + \rho\cos\frac{l}{\rho}\left(\cos\frac{l}{\rho} - 1\right) \\
&= \rho\left(1 - \cos\frac{l}{\rho}\right) \qquad (18)
\end{aligned}
$$

- And $D'_{\mathrm{dipole}}(s) = \sin\frac{l}{\rho}$

Examples of Dispersion - 3

- Assuming θ is small we can expand this

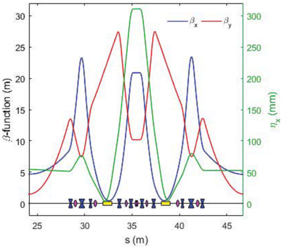
$$\begin{aligned} D(s)_{\text{dipole}} &= \rho\left(1 - \cos\frac{l}{\rho}\right) \\ &\approx \rho\left(1 - \left[1 - \frac{1}{2}\left(\frac{l}{\rho}\right)^2\right]\right) \\ &\approx \frac{\rho}{2}\left(\frac{l}{\rho}\right)^2 = \frac{\rho\theta^2}{2} \end{aligned} \tag{19}$$

Matrix formalism continued

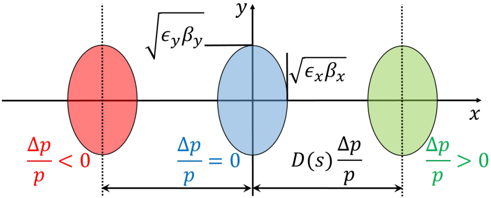
- Can now expand the transfer matrix to include dispersion

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_1 = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0 \tag{20}$$

- Dispersion can be calculated by an optics code for a real machine
- $D(s)$ is created by the **dipoles**…
- … and focused by the **quadrupoles**
- *Diamond DBA example* ⇒



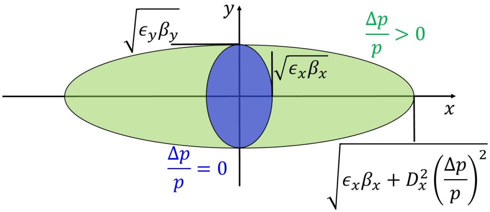
Dispersed Beam Orbits



- These are 2D ellipses defining the beam
- The central and extreme momenta are shown (there is a distribution in between)
- The vacuum chamber must accommodate the full spread
- With dispersion the **half height** and **half width** are (assuming $D_y = 0$)

$$a_y = \sqrt{\eta_y \beta_y}, \qquad a_x = \sqrt{\eta_x \beta_x} + D(s)\frac{\Delta p}{p} \tag{21}$$

Dispersed Beam Size



- Dispersion also contributes to the beam size
- Therefore we can **measure the dispersion** by measuring beam sizes at different locations with different amounts of dispersion and different βs

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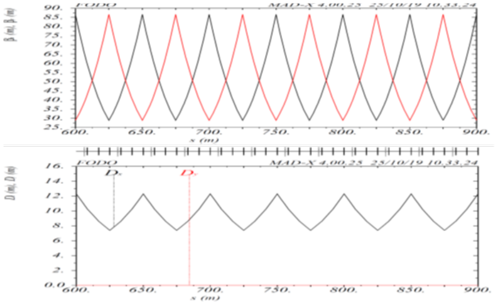
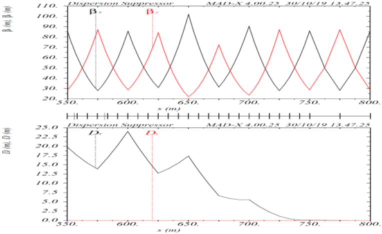
Dispersion Suppression



- Given a periodic lattice what can we do about dispersion?
- We can't get rid of it completely as it's produced by the dipoles
- Answer . . . **suppress the dispersion elsewhere**

Dispersion Suppression: Easy option

- Use **extra quadrupoles** to match $D(s)$ and $D'(s)$
- Given an optical solution in the arc, suppressing dispersion can be achieved with **2 additional quadrupoles**
- **But that's not enough!** Need to match the Twiss, optical parameters too
- An **extra 4 quadrupoles** are needed to match α and β



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Dispersion Suppression: Easy option

Advantages:

- Straight forward
- Works for any phase advance per cell
- Ring geometry is unchanged
- Flexible! Can match between different lattice structures

Disadvantages:

- Additional quadrupole magnets and power supplies required
- The extra quadrupoles are, in general stronger
- The β function increases so the aperture increases

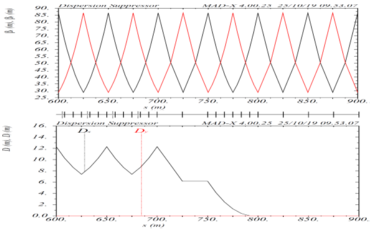
Dispersion Suppression: Missing Bend

- Start with $D = D' = 0$ and create dispersion such that the conditions are matched in the first regular quadrupoles
- Utilise n cells **without dipole magnets** at the end of an arc, followed by m arc cells
- … hence **“missing bend” dispersion suppression**

- Condition:

$$\frac{2m+n}{2}\Phi_C = (2k+1)\frac{\pi}{2} \qquad (22)$$

where Φ_C = cell phase advance,
$\sin\frac{m\Phi_C}{2} = \frac{1}{2},\ k = 0, 2, \ldots$ or
$\sin\frac{m\Phi_C}{2} = -\frac{1}{2},\ k = 1, 3, \ldots$



Dispersion Suppression: Missing Bend

Advantages:

- No additional quadrupoles or new power supplies
- Aperture requirements are the same as those in the arc as β is unchanged

Disadvantages:

- Only works for certain phase advances restricting optics options in the arc
- The geometry of the ring is changed

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Dispersion Suppression: Half Bend

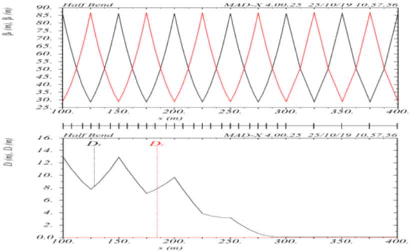
- How about inserting different strength dipoles? Does it help?
- Assume you have a FODO arc cell, a lattice insertion and then a dispersion free section without dipoles
- Condition for vanishing disperion can be calculated for n cells with dipole strength δ_{sup}

$$2\delta_{\text{sup}}\sin^2\left(\frac{n\Phi_C}{2}\right) = \delta_{\text{arc}} \tag{23}$$

- So if we require $\delta_{\text{sup}} = \frac{1}{2}\delta_{\text{arc}}$ we get

$$\sin^2\left(\frac{n\Phi_C}{2}\right) = 1 \quad \Rightarrow \quad \sin(n\Phi_C) = 0$$
$$\Rightarrow \quad n\Phi_C = k\pi, \quad k = 1, 3, \ldots \tag{24}$$

Dispersion Suppression: Half Bend



Advantages and **Disadvantages** are the same as for the missing bend only there is an extra **disadvantage**:

A special half strength dipole is required which may add extra cost to the design

N.B. This is not an exhaustive list of dispersion suppression techniques, just a taster!

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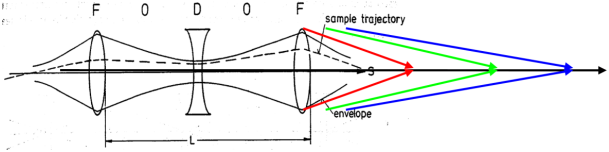
Chromaticity - 1

- What about off-momentum effects through quadrupoles?
- The focusing strength of a quadrupole depends on the momentum of the particle $1/f \propto 1/p$



- Particles with $\Delta p > 0$, $\Delta p < 0$, ideal momentum
- Off-momentum particles oscillate around a **chromatic closed orbit** NOT the design orbit

Chromaticity - 2

- Normalised quadrupole strength $k = \frac{g}{p/e}$
- In case of a momentum spread

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{eg}{p_0}\left(1 - \frac{\Delta p}{p_0}\right) = k_0 + \Delta k \qquad (25)$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0 \qquad (26)$$

- This acts like a quadrupole error in the machine and leads to a **tune spread**

$$\Delta Q = \frac{1}{4\pi}\int \Delta k(s)\beta(s)\,\mathrm{d}s = -\frac{1}{4\pi}\frac{\Delta p}{p_0}\int k_0(s)\beta(s)\,\mathrm{d}s \qquad (27)$$

Chromaticity - 3

- This spread in tune is expressed via **chromaticity, Q'** or the **normalised chromaticity, ξ**

$$Q' = \frac{\Delta Q}{\Delta p/p_0}, \qquad \xi = \frac{\Delta Q/Q}{\Delta p/p_0} \tag{28}$$

- Note that chromaticity is produced by the lattice itself
- It is determined by the focusing strength of all the quadrupoles
- The **"natural" chromaticity** is negative and can lead to a large tune spread and consequent instabilities
- For example, for a FODO lattice $\xi \approx -1$

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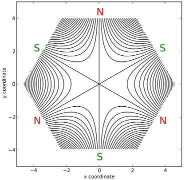
Correcting Chromaticity - 1

- Want to **"sort"** the particles by their momentum
- Utilise dispersive trajectory! Apply magnetic field that is zero at small amplitudes and rises quickly outward
- Use **sextupoles**!



$$B_x = \tilde{g}xz, \qquad B_y = \frac{1}{2}\tilde{g}(x^2 - y^2) \qquad (29)$$

- This results in a linear gradient in x, $\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x$
- And a normalised quadrupole strength $k_{\mathrm{sext}} = \frac{\tilde{g}x}{p/e} = m_{\mathrm{sext}}x = m_{\mathrm{sext}}D\Delta p/p$

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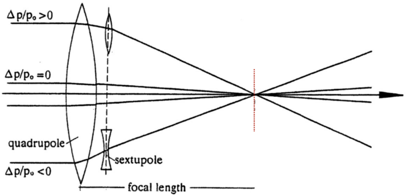
Correcting Chromaticity - 2



- This all results in a corrected chromaticity

$$Q' = -\frac{1}{4\pi} \oint \beta(s) \left[k(s) - mD(s) \right] \, \mathrm{d}s \qquad (30)$$

- **Chromatic sextupoles**: Sextupoles at nonzero dispersion can correct natural chromaticity
- Usually **2 families**, one horizontal and one vertical
- Place where $\beta_{x/y} D$ is large to minimise their strength

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Summary

- Reminder of co-ordinate system
- Transverse equation of motion: **modified Hill's equation** with momentum spread
- **Dispersion** revisited in matrix form
- Effect of dispersion on beam orbit and beam size
- **Dispersion suppression**
- **Chromaticity** and chromatic tune spread
- Chromatic sextupoles and **chromaticity correction**

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References

- Bernard Holzer, **Lattice Design and Transverse Dynamics Course**, CERN Accelerator School 2015, `https://cas.web.cern.ch/schools/warsaw-2015`
- S.Y. Lee, **Accelerator Physics**, 2$^{\text{nd}}$ edition, World Scientific, 2007
- H. Wiedemann, **Particle Accelerator Physics I**, 2$^{\text{nd}}$ edition, Springer, 2003
- Edward J. N. Wilson, **An Introduction to Particle Accelerators**, Oxford University Press, 2001
- A. Chao, K.H. Mess, M. Tigner, F. Zimmerman, **Handbook of Accelerator Physics and Engineering**, 2$^{\text{nd}}$ edition, World Scientific, 2013
- Mario Conte, William W MacKay, **An Introduction to the Physics of Particle Accelerators**, 2$^{\text{nd}}$ edition, World Scientific, 2008
- Klaus Wille, **The Physics of Particle Accelerators: An Introduction**, Oxford University Press, 2005
- D.A. Edwards, M.J. Syphers, **An Introduction to the Physics of High Energy Accelerators**, Wiley-VCH, 2004
- J.D. Lawson, **The Physics of Charged-Particle Beams**, Oxford University Press, 1978
- Klaus G. Steffen, **High Energy Beam Optics**, Interscience Publishers, 1965