Hamiltonian Dynamics

Lecture 1

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Content

Lecture 1

- Comparison of Newtonian, Lagrangian and Hamiltonian approaches.
- Hamilton's equations, canonical transformations, symplecticity, integrability.
- Poisson brackets and Lie transformations.

Lecture 2

- The "accelerator" Hamiltonian.
- Dynamic maps, symplectic integrators.
- Integrable Hamiltonian.

Newtonian Mechanics

In Newtonian mechanics the key function is the force $\mathbf{F}$ (a vector quantity). In general the force is a function of position $\mathbf{r}$, velocity $\dot{\mathbf{r}}$ and time t. The equation of motion of a particle of mass m subject to a force $\mathbf{F}$ is (for a non-inertial frame of reference)

$$\frac{d}{dt}(m\dot{\mathbf{r}}) = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \tag{1}$$

The dynamics are determined by solving N second order differential equations as a function of time. In a non-inertial frame we may need to consider *fictitious* forces.

Lagrangian Mechanics

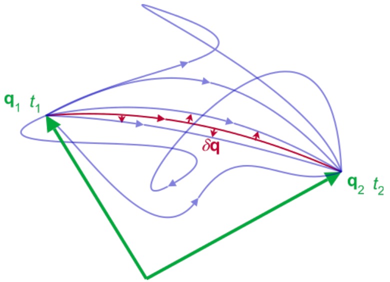
In Lagrangian mechanics the key function is the Lagrangian (a scalar quantity)

$$L = L(q, \dot{q}, t) \tag{2}$$

The solution to a given mechanical problem is obtained by solving a set of N second-order differential equations known as the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{3}$$

Principle of least action



The action S is the integral of L along the trajectory

$$S = \int_{t1}^{t2} L(q, \dot{q}, t)t \tag{4}$$

The *principle of least action* or *Hamilton's principle* holds that the system evolves such that the action S is stationary. It can be shown that the Euler-Lagrange equation defines a path for which.

$$\delta S = \delta \left[\int_{t1}^{t2} L(q, \dot{q}, t)t \right] = 0 \tag{5}$$

Conservative force

In the case of a convervative force field the Lagrangian is the difference of the kinetic and potential energies

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q) \tag{6}$$

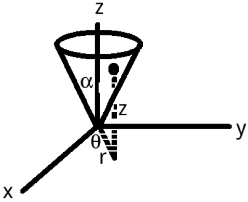
where

$$F = \frac{\partial V(q)}{\partial q} \tag{7}$$

Advantages of Lagrangian approach

- The Euler-Lagrange is true regardless of the choice of coordinate system (including non-inertial coordinate systems). We can transform to convenient variables that best describe the symmetry of the system.
- It is easy to incorporate constraints. We formulate the Lagrangian in a configuration space where ignorable coordinates are removed (e.g. a mass constrained to a surface), thereby incorporating the constraint from the outset.

Particle on a cone



Consider a particle rolling due to gravity in a frictionless cone. The cone's opening angle α places a constraint on the coordinates $\tan\alpha = r/z$. We may write the Lagrangian in cylindrical coordinates

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \tag{8}$$

Reduce the number of coordinates by eliminating z

$$z = \frac{r}{\tan\alpha}, \; \dot{z} = \frac{\dot{r}}{\tan\alpha} \tag{9}$$

Then the Lagrangian $L = T - V$ is given by

$$L = \frac{m}{2}\left[(1+\cot^2\alpha)\dot{r}^2 + r^2\dot{\theta}^2\right] - mgr\cot\alpha \tag{10}$$

Write down the Euler-Lagrange equation for each coordinate (r, θ). For r we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{11}$$

We obtain the first equation of motion

$$(1 + \cot^2\alpha)\ddot{r} - r\dot{\theta}^2 + g\cot\alpha = 0$$

Likewise for θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

leading to

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

giving the second equation of motion

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

General electromagnetic fields

The Lagrangian for a particule in an EM field $U(x, \dot{x}, t) = e(\phi - \boldsymbol{v} \cdot \boldsymbol{A})$

$$L(x, \dot{x}, t) = -mc^2\sqrt{1-\beta^2} - e\phi + e\boldsymbol{v} \cdot \boldsymbol{A}. \tag{12}$$

The conjugate (or canonical) momentum is

$$P_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{m\dot{x}_i}{\sqrt{1-\beta^2}} + eA_i \tag{13}$$

i.e. the field contributes to the conjugate momentum.

Legendre transformation

The Legendre transform takes us from a *convex*[1] function $F(u_i)$ to another function $G(v_i)$ as follows. Start with a function

$$F = F(u_1, u_2, \ldots, u_n). \tag{14}$$

Introduce a new set of *conjugate* variables through the following transformation

$$v_i = \frac{\partial F}{\partial u_i}. \tag{15}$$

We now define a new function G as follows

$$G = \sum_{i=1}^{n} u_i v_i - F \tag{16}$$

[1] F is convex in u if $\frac{\partial^2 F}{\partial u^2} > 0$

Apply Legendre's transformation to the Lagrangian

Start with the Lagrangian

$$L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t), \tag{17}$$

and introduce some new variables we are going to call the p_is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{18}$$

We can then introduce a new function H defined as

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \tag{19}$$

We now have a function which is dependent on q, p and time.

$$H = H(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \tag{20}$$

L and H have a dual nature:

$$H = \sum p_i \dot{q}_i - L, \qquad L = \sum p_i \dot{q}_i - H,$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}.$$

Hamilton's canonical equations

Starting from Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q}$$

and combining with

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

leads to

$$\dot{p}_i = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i} \tag{21}$$

So we have

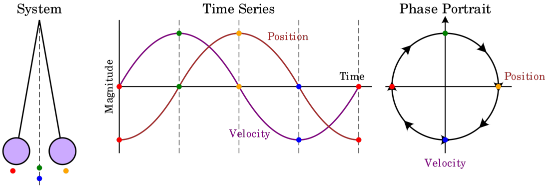
$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad (22) \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (23)$$

which are called Hamilton's canonical equations. They are the equations of motion of the system expressed as $2n$ first order differential equations.

In a conservative system the Hamiltonian represents the total energy

$$H = T + V$$

Phase space



In Hamiltonian mechanics, the canonical momenta $p_i = \delta L$ are promoted to coordinates on equal footing with the generalized coordinates q_i. The coordinates (q, p) are canonical variables, and the space of canonical variables is known as phase space.

Symmetry and Conservation Laws

A *cyclic* coordinate in the Langrangian is also cyclic in the Hamiltonian. Since $H(q,p,t) = \dot{q}_i p_i - L(q,\dot{q},t)$, a coordinate q_j absent in L is also absent in H.
A symmetry in the system implies a cyclic coordinate which in turn leads to a conservation law (*Noether's theorem*).

$$\frac{\partial L}{\partial q_j} = 0 \implies \frac{\partial H}{\partial q_j} = 0 \tag{24}$$

Hence

$$\dot{p}_j = 0 \tag{25}$$

so the momentum p_j is conserved.
Often we wish to simplify our problem by applying a transformation that exploits any symmetry in the system.

Canonical transformations

Transform from one set of canonical coordinates (p_i, q_i) to another (P_i, Q_i). The transformation should preserve the form of Hamilton's equations.

Old coordinates
Hamiltonian: $H(q, p, t)$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad (26)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (27)$$

New coordinates
Kamiltonian: $K(Q, P, t)$

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad (28)$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} \quad (29)$$

Preservation of Hamiltonian form

For the old Hamiltonian H it was true that

$$\delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H(q_i, p_i, t) \right) \mathrm{d}t = 0 \tag{30}$$

Likewise, for the new Hamiltonian K

$$\delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - K(Q_i, P_i, t) \right) \mathrm{d}t = 0 \tag{31}$$

This is true if

$$\lambda(p\dot{q} - H) = P\dot{Q} - K + \frac{\mathrm{d}F}{\mathrm{d}t} \tag{32}$$

where F is a generating function and we normally set $\lambda = 1$

The function F is called the generating function of the canonical transformation and it depends on old and new phase space coordinates. It can take 4 forms corresponding to combinations of (q_i, p_i) and (Q_i, P_i):

$$F = F_1(q_i, Q_i, t) \tag{33}$$

$$F = F_2(q_i, P_i, t) \tag{34}$$

$$F = F_3(p_i, Q_i, t) \tag{35}$$

$$F = F_4(p_i, P_i, t) \tag{36}$$

Generating function $F_1(q, Q, t)$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{\mathrm{d}F_1}{\mathrm{d}t} \tag{37}$$

$$= P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t} \tag{38}$$

$$\left(p_i - \frac{\partial F_1}{\partial q_i}\right)\dot{q}_i - \left(P_i + \frac{\partial F_1}{\partial q_i}\right)\dot{Q}_i + K - \left(H + \frac{\partial F_1}{\partial t}\right) = 0 \tag{39}$$

The old and new coordinates are separately independent so the coefficients of $\dot{q}_i$ and $\dot{Q}_i$ must each vanish leading to

$$p_i = \frac{\partial F_1}{\partial q_i} \tag{40}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} \tag{41}$$

$$K = H + \frac{\partial F_1}{\partial t} \tag{42}$$

F_1 example

$$F_1(q, Q, t) = qQ \tag{43}$$

This does not depend on time, so by equation 42 the new and original Hamiltonians are equal.

$$p = \frac{\partial F_1}{\partial q} = Q \tag{44}$$

$$P = -\frac{\partial F_1}{\partial Q} = -q \tag{45}$$

This generating function essentially swaps the coordinates and momenta.

Generating function $F_2(q, P, t)$

Look for a function of the form

$$F = F_2(q, P, t) - Q_i P_i \tag{46}$$

can show

$$p_i = \frac{\partial F_2}{\partial q_i} \tag{47}$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \tag{48}$$

$$K = H + \frac{\partial F_2}{\partial t}. \tag{49}$$

F_2 example

$$F_2(q, P, t) = \sum_i q_i P_i \tag{50}$$

This example generating function also does not depend on time so the new and original Hamiltonians are again equal. So

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i \tag{51}$$

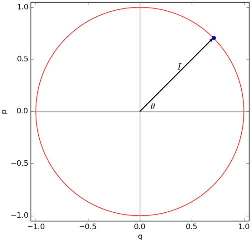
$$Q_i = \frac{\partial F_2}{\partial P_i} = q_i \tag{52}$$

This generating function is just the identity transformation, the coordinates and Hamiltonian are swapped into themselves.

Summary of generating functions

Generating function	Transformation equations	
$F = F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$	$P_i = -\frac{\partial F_1}{\partial Q_i}$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i}$	$Q_i = \frac{\partial F_2}{\partial P_i}$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i}$	$P_i = -\frac{\partial F_3}{\partial Q_i}$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i}$	$Q_i = \frac{\partial F_4}{\partial P_i}$

Action-angle variables



A Hamiltonian system can be written in action-angle form if there is a set of canonical variables (θ, I) such that H only depends on the action

$$H = H(I) \tag{53}$$

Then

$$\dot{\theta} = \nabla H(I) = \omega(I), \quad \dot{I} = 0 \tag{54}$$

Example: Harmonic oscillator

The Hamiltonian for a harmonic oscillator is given

$$H = \frac{\omega}{2}\left(q^2 + p^2\right). \tag{55}$$

This Hamiltonian is the sum of two squares, which suggest that one of the new coordinates is cyclic. Try a transformation to action-angle variables

$$q = \sqrt{\frac{2}{\omega}} f(P) \sin Q \tag{56}$$

$$p = \sqrt{\frac{2}{\omega}} f(P) \cos Q. \tag{57}$$

Then the new Hamiltonian

$$K = H = f^2(P)(\sin^2 Q + \cos^2 Q) = f^2(P). \tag{58}$$

Take the ratio of the transformation equations

$$p = q \cot Q. \tag{59}$$

This is independent of f(P), and has the form of the $F_1(q, Q, t)$ type of generating function

$$p = \frac{\partial F_1}{\partial q}. \tag{60}$$

The simplest form for F_1 agreeing with the above is

$$F_1(q, Q) = \frac{1}{2} q^2 \cot Q. \tag{61}$$

We can then find P using the other transformation equation for F_1

$$P = -\frac{\partial F_1}{\partial Q} = \frac{1}{2} q^2 \csc^2 Q = \frac{1}{2} \frac{q^2}{\sin^2 Q}. \tag{62}$$

Rearrange for q

$$q = \sqrt{2P\sin^2 Q} = \sqrt{2P}\sin Q. \tag{63}$$

Comparing this with equation 56 gives the function $f(P)$

$$f(P) = \sqrt{\omega P}. \tag{64}$$

The new Hamiltonian is therefore

$$K = \omega P. \tag{65}$$

This is cyclic in Q, so P is constant. The energy is constant and equal to K so

$$P = \frac{E}{\omega}. \tag{66}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \tag{67}$$

Symplecticity

A symplectic transformation M satisfies

$$M^T \Omega M = \Omega \quad (68)$$

where

$$\Omega = \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix} \quad (69)$$

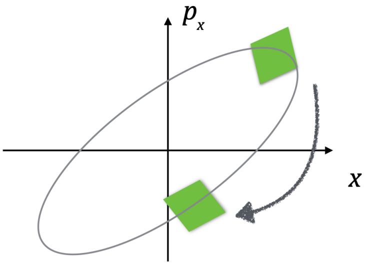
Hamilton's equations in matrix form are

$$\begin{pmatrix} \dot{q}_i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} \quad (70)$$

or in vector form

$$\dot{\zeta} = \Omega \nabla H(\zeta) \quad (71)$$

where ζ is the vector of phase space coordinates.



It can be shown that the corresponding map M given by

$$\zeta(t) = M\zeta(t_0) \tag{72}$$

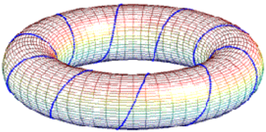
has the symplectic property

$$M^T(t)\Omega M(t) = \Omega \tag{73}$$

In Hamiltonian systems the equations of motion generate symplectic maps of coordinates and momenta and as a consequence preserve volume in phase space. This is equivalent to *Liouville theorem* which asserts that the phase space distribution function is constant along the trajectories of the system.

Liouville Integrability

The Liouville-Arnold theorem states that existence of n invariants of motion is enough to fully characterize a for an n degree-of-freedom system. In that case a canonical transformation exists to action angle coordinates in which the Hamiltonian depends only on the action.



Liouville integrability means that there exists a regular foliation of the phase space by invariant manifolds such that the Hamiltonian vector fields associated to the invariants of the foliation span the tangent distribution.

Poisson brackets

Let p and q be canonical variables and let u and v be functions of p and q. The Poisson bracket of u and v is defined as

$$[u, v]_{p,q} = \frac{\partial u}{\partial q}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial q}. \tag{74}$$

From the definition of the Poisson bracket

$$[q_i, q_j] = [p_i, p_j] = 0 \tag{75}$$

$$[q_i, p_j] = -[p_i, q_j] = \delta_{i,j}. \tag{76}$$

A Poisson bracket is invariant under a change in canonical variables

$$[u, v]_{p,q} = [u, v]_{P,Q}. \tag{77}$$

In other words, Poisson brackets are canonical invariants, which gives us an easy way to determine whether a set of variables is canonical.

Equations of motion with brackets

Hamilton's equations may be written in terms of Poisson brackets
For a function $u = u(q_i, p_i, t)$ the total differential is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t}. \tag{78}$$

We can replace $\dot{q}_i$ and $\dot{p}_i$ with their Hamiltonian solutions to obtain

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t} \tag{79}$$

which is just

$$\frac{\mathrm{d}u}{\mathrm{d}t} = [u, H] + \frac{\partial u}{\partial t}. \tag{80}$$

If u is constant, then $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$ and $[u, H] = -\frac{\partial u}{\partial t}$. If u does not depend explicitly on t $[u, H] = 0$.
If $u = q$

$$\dot{q} = [q, H]. \tag{81}$$

If $u = p$

$$\dot{p} = [p, H]. \tag{82}$$

Which are just the equations of motion in terms of Poisson brackets.

Lie Transformations

Suppose we have some function of the phase space variables

$$f = f(x_i, p_i) \tag{83}$$

which has no explicit dependence on the independent variable, s. However if we evaluate f for a particle moving along a beamline, the value of f will evolve with s as the dynamical variables evolve.
The rate of change of f with s is

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \sum_{i=1}^{n} \frac{\mathrm{d}x_i}{\mathrm{d}s}\frac{\partial f}{\partial x_i} + \frac{\mathrm{d}p_i}{\mathrm{d}s}\frac{\partial f}{\partial p_i}. \tag{84}$$

Using Hamilton's equations

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial f}{\partial p_i}. \tag{85}$$

We now define the Lie operator $: g :$ for any function $g(x_i, p_i)$

$$: g := \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \frac{\partial}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial}{\partial x_i}. \tag{86}$$

Compare with the definition of a Poisson bracket

$$[u, v]_{p,q} = \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial q}. \tag{87}$$

If the Hamiltonian H has no explicit dependence on s we can write

$$\frac{\mathrm{d}f}{\mathrm{d}s} = - : H : f. \tag{88}$$

We can express f at $s = s_0 + \Delta s$ in terms of f at $s = s_0$ in terms of a Taylor series

$$f\Big|_{s=s_0+\Delta s} = f\Big|_{s=s_0} + \Delta s \frac{\mathrm{d}f}{\mathrm{d}s}\Big|_{s=s_0} + \frac{\Delta s^2}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}s^2}\Big|_{s=s_0} + \ldots \tag{89}$$

$$= \sum_{m=0}^{\infty} \frac{\Delta s^m}{m!}\frac{\mathrm{d}^m f}{\mathrm{d}s^m}\Big|_{s=s_0} \tag{90}$$

$$= e^{\Delta s \frac{\mathrm{d}}{\mathrm{d}s}} f\Big|_{s=s_0}. \tag{91}$$

This suggests the solution for equation 88 can be written as

$$f\Big|_{s=s_0+\Delta s} = e^{-\Delta s :H:} f\Big|_{s=s_0}. \tag{92}$$

The operator $e^{-\Delta s :g:}$ is known as a Lie transformation, with generator g. In the context of accelerator beam dynamics, applying a Lie transformation with the Hamiltonian as the generator to a function f produces a transfer map for f.

- f can be any function of the dynamical variables
- Any Lie transformation represents the evolution of a conservative dynamical system, with Hamiltonian corresponding to the generator of the Lie transformation
- The map represented by a Lie transformation must be symplectic