# Probing Dark Matter with Gravitational Waves

(Gravity Group)

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01 Introduction

02 Orbits Calculations

03 Gravitational Waves Calculations

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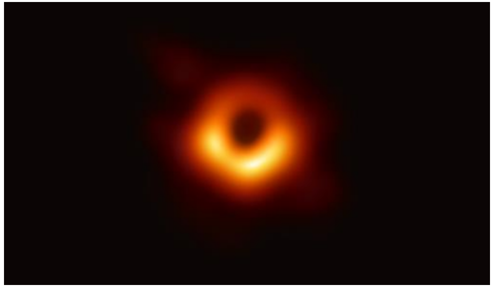
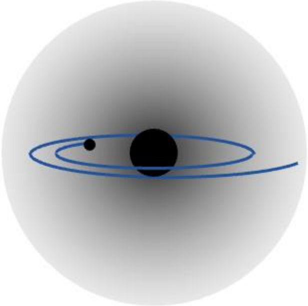
# Introduction

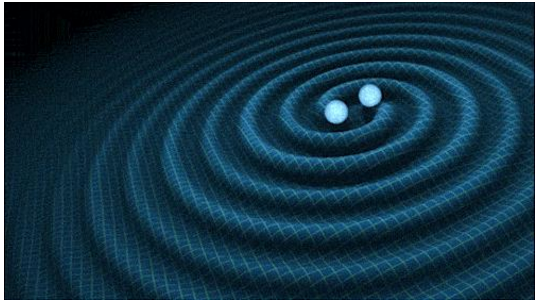
# Probing Dark Matter using Gravitational Waves



Black holes



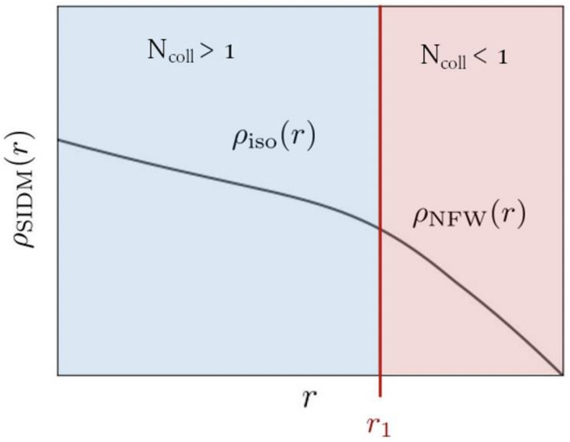
Dark matter halo



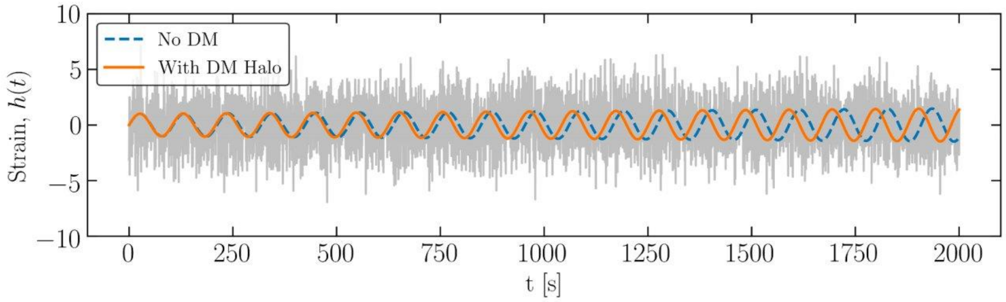
Gravitational waves

# Dark Matter Halo

## Dark matter density spike

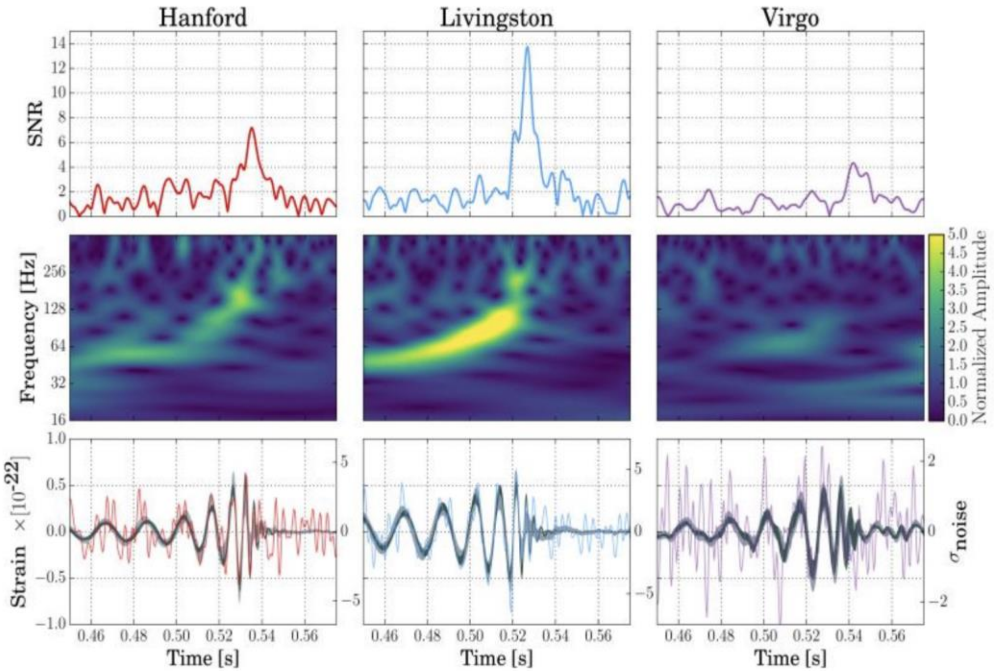



## Dephasing of gravitational waves

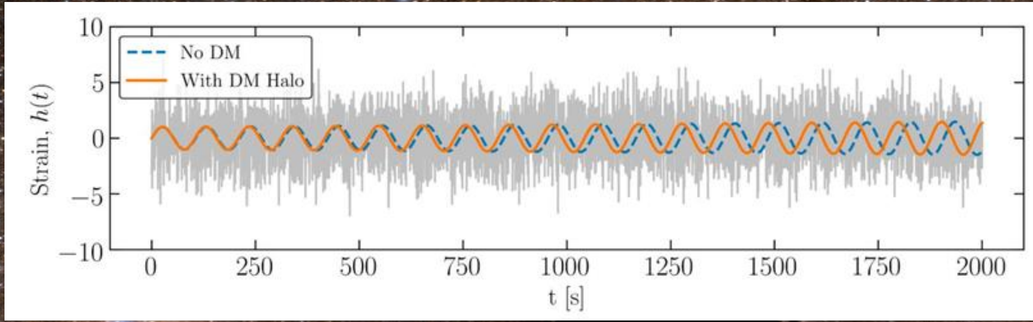



[Kazunari Eda, Yousuke Itoh, Sachiko Kuroyanagi, and Joseph Silk, 2014]

# Gravitational waves




[LIGO Scientific Collaboration and Virgo Collaboration, 2017]



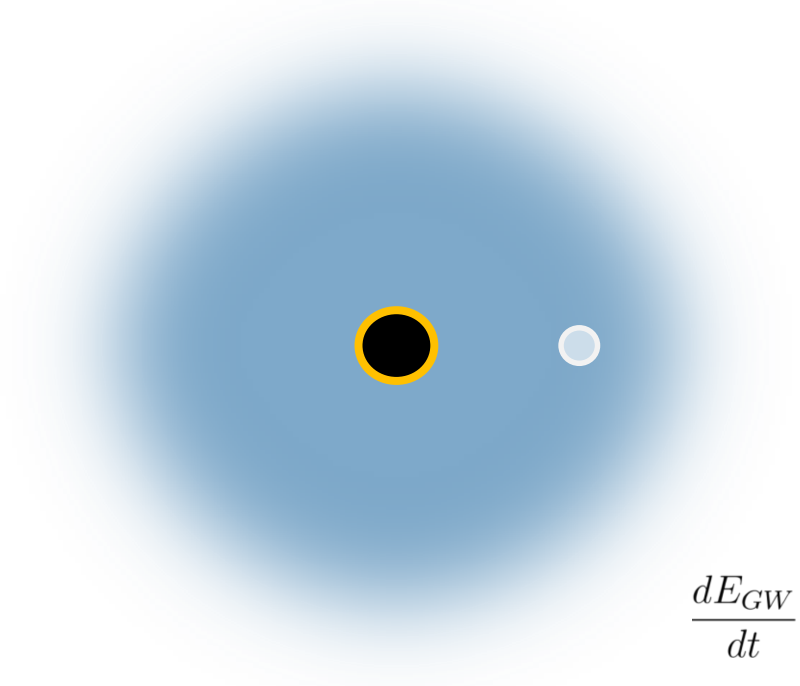

# Question for you:

**In the plot shown, what happens to the gravitational wave signal when dark matter is introduced?**

# Orbits Calculations

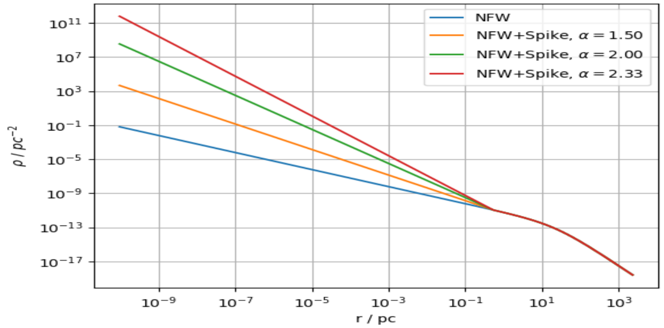
# Idea behind orbit calculations

3. Trajectory calculation



## Recipe for the Gravitational Wave:

1. Dark Matter density profile




2. Interaction between the DM halo and the orbiting object

$$-\frac{dE_{\text{orbit}}}{dt} = \frac{dE_{GW}}{dt} + \frac{dE_{DF}}{dt}$$

$$\frac{dE_{GW}}{dt} = v f_{GW} = \frac{32}{5}\frac{G\mu^2}{c^5} r^4 \omega_s^6 \qquad \frac{dE_{DF}}{dt} = v f_{DF} = 4\pi G^2 \frac{\mu^2 \rho_{DM}(r)}{v} \ln \Lambda$$

# Geodesics (First Attempts)

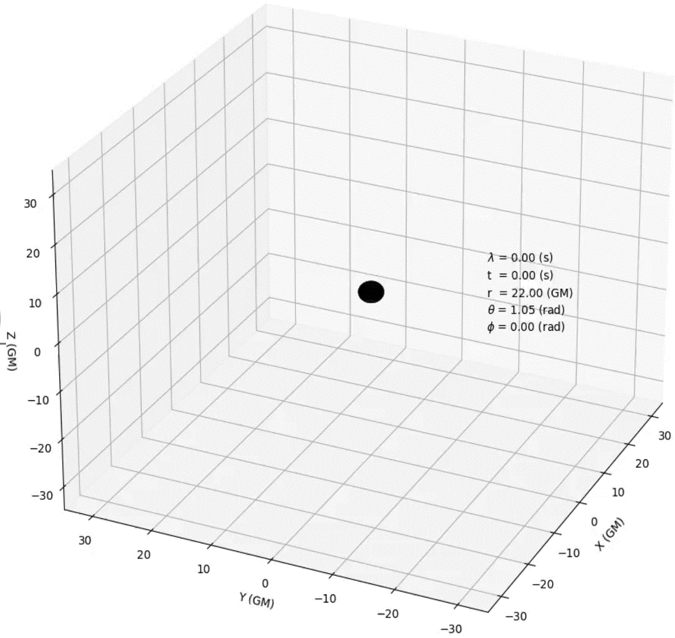
$$\frac{d^2x^\mu}{d\lambda^2} - \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

$$\frac{d^2}{d\lambda^2}t(\lambda) = \frac{2GM\frac{d}{d\lambda}r(\lambda)\frac{d}{d\lambda}t(\lambda)}{r(2GM-r)}$$

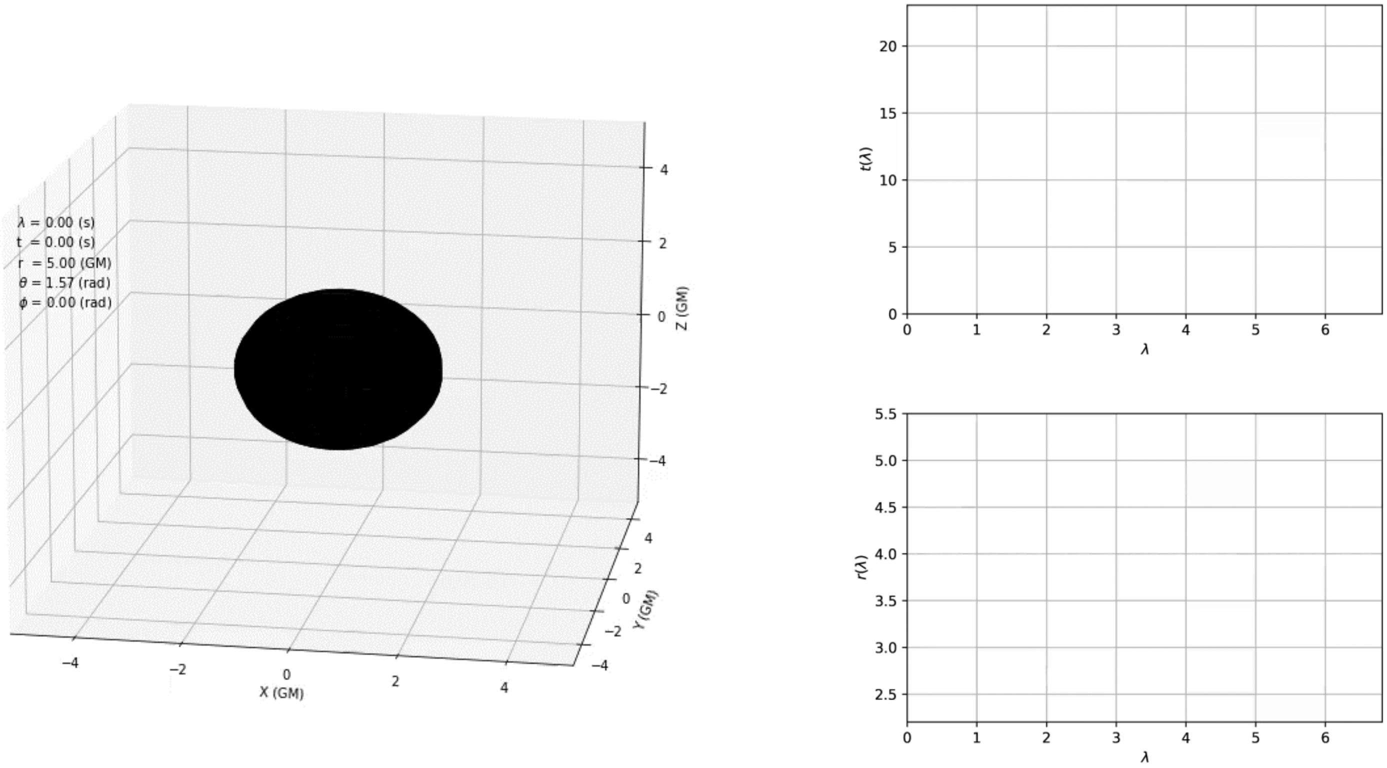
$$\frac{d^2}{d\lambda^2}r(\lambda) = \frac{-GMr^2\left(\frac{d}{d\lambda}r(\lambda)\right)^2 + GM(2GM-r)^2\left(\frac{d}{d\lambda}t(\lambda)\right)^2 - r^3(2GM-r)^2\left(\sin^2(\theta)\left(\frac{d}{d\lambda}\phi(\lambda)\right)^2 + \left(\frac{d}{d\lambda}\theta(\lambda)\right)^2\right)}{r^3(2GM-r)}$$

$$\frac{d^2}{d\lambda^2}\theta(\lambda) = \frac{\sin(2\theta)\left(\frac{d}{d\lambda}\phi(\lambda)\right)^2}{2} - \frac{2\frac{d}{d\lambda}\theta(\lambda)\frac{d}{d\lambda}r(\lambda)}{r}$$

$$\frac{d^2}{d\lambda^2}\phi(\lambda) = -\frac{2\left(\frac{r\frac{d}{d\theta}\theta(\lambda)}{\tan(\theta)} + \frac{d}{d\lambda}r(\lambda)\right)\frac{d}{d\lambda}\phi(\lambda)}{r}$$



# Geodesics (First Attempts)



# Question for you:

Do we expect the relativistic corrections to the orbits to be larger close to large black holes or small black holes?

# Equations of Motion

In terms of generalized forces:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = Q_r^{\mathrm{GW}} + Q_r^{\mathrm{DF}}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = Q_\phi^{\mathrm{GW}} + Q_\phi^{\mathrm{DF}}$$

Where

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{G\left(m_1+m_2\right)\mu}{r}$$

$$Q_r^{\mathrm{GW}} = -\frac{32}{5}G\frac{\mu^2}{r^2}\sin^6(\phi)\dot{r}^2\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{\frac{5}{2}}$$

$$Q_\phi^{\mathrm{GW}} = -\frac{32}{5}G\mu^2\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{\frac{5}{2}}\sin^6(\phi)\dot{\phi}$$

… etc

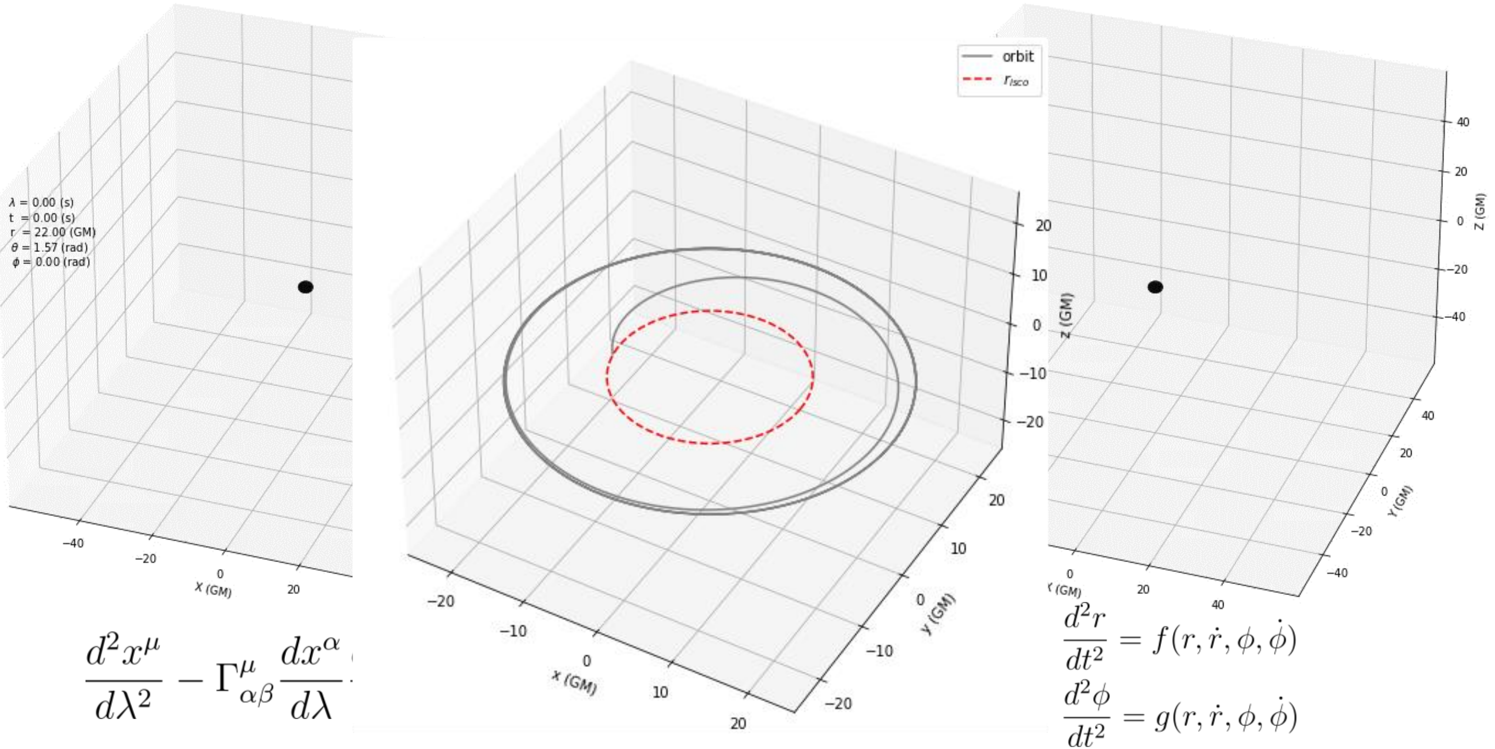
Equations of Motion:

$$\mu\ddot{r} - \mu r\dot{\phi}^2 + \frac{G\left(m_1+m_2\right)\mu}{r^2} = -\frac{64}{5}\frac{G\mu^2}{r^2}\sin^6(\phi)\dot{r}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{\frac{5}{2}} - 8\pi G^2\mu^2\rho_{\mathrm{DM}}(r)\ln(\Lambda)\frac{\dot{r}}{\dot{r}^2 + r^2\dot{\phi}^2}$$

$$\mu r^2\ddot{\phi} + 2\mu r\dot{r}\dot{\phi} = -\frac{64}{5}G\mu^2\sin^6(\phi)\dot{\phi}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{\frac{5}{2}} - 8\pi G^2\mu^2\rho_{\mathrm{DM}}(r)\ln(\Lambda)\frac{r^2\dot{\phi}}{\dot{r}^2 + r^2\dot{\phi}^2}$$

# Geodesics (First Attempts+Changes)

General relativistic orbits in contrast to Newtonian orbits:




$$\frac{d^2x^\mu}{d\lambda^2} - \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}$$

$$\frac{d^2r}{dt^2} = f(r, \dot{r}, \phi, \dot{\phi})$$

$$\frac{d^2\phi}{dt^2} = g(r, \dot{r}, \phi, \dot{\phi})$$

# Gravitational Waves Calculations

# Gravitational Waves Calculations

## 1. Finding the mass quadrupole moment

**Mass quadrupole moment**

$$
\begin{aligned}
Q^{ij} &\equiv M^{ij} - \frac{1}{3}\delta^{ij} M_{kk} \\
&= \int d^3x \rho(t, \mathbf{x}) \left( x^i x^j - \frac{1}{3} r^2 \delta^{ij} \right)
\end{aligned}
$$

**Reduced mass quadrupole moment**

$$
Q_{xx} = 2MR^2 \left( \cos^2(2\pi f t) - \frac{1}{3} \right)
$$

$$
Q_{yy} = 2MR^2 \left( \sin^2(2\pi f t) - \frac{1}{3} \right)
$$

$$
Q_{xy} = Q_{yx} = 2MR^2 (\sin(2\pi f t) \cos(2\pi f t))
$$

# Gravitational Waves Calculations

## 2. Taking second time derivative of mass quadrupole moment and plugging into strain equation

**Perturbations in the metric (strain)** ⇨

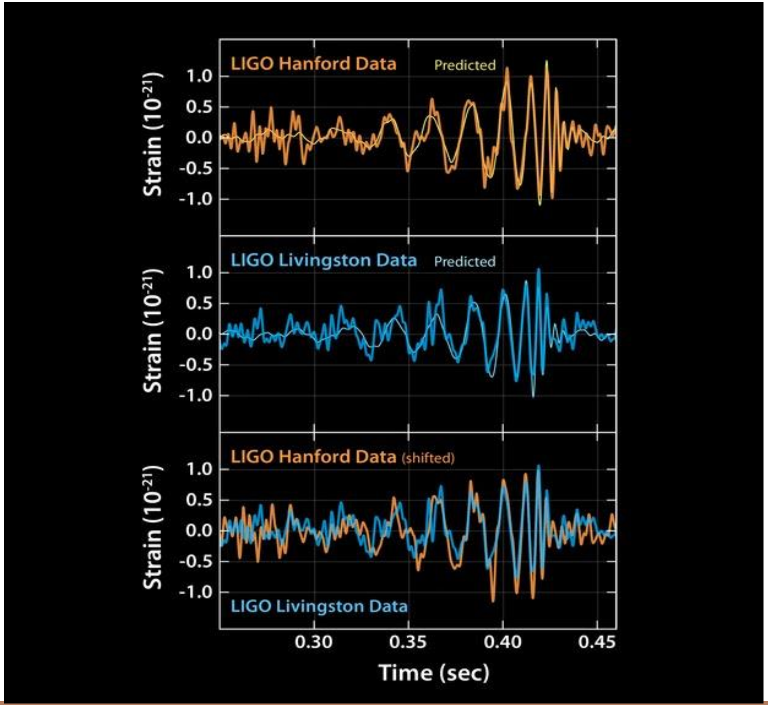
$$h_{ij} \sim \frac{G}{c^4} \frac{\ddot{Q}_{ij}}{\mathrm{R}}$$

**Final strain equation** ⇨

$$\boxed{h_{ij}(t, \mathbf{x}) = \frac{2G}{c^4 r} \frac{d^2}{dt^2} Q_{ij}(t - r/c)}$$

# Gravitational Waves Calculations

## 2. Taking second time derivative of mass quadrupole moment and plugging into strain equation




$$h_{ij} \sim \frac{G}{c^4}\frac{\ddot{Q}_{ij}}{\mathrm{R}}$$

$$\boxed{h_{ij}(t,\mathbf{x}) = \frac{2G}{c^4 r}\frac{d^2}{dt^2}Q_{ij}(t - r/c)}$$

[https://www.ligo.caltech.edu/image/ligo20160211a]

# Gravitational Waves Calculations

GW propagating in z direction:

$$h_+ = \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right)$$

$$h_\times = \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}$$

To obtain arbitrary direction:

$$n_i = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$$

# Gravitational Waves Calculations

$$h_+(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}[\ddot{M}_{11}\left(\cos^2\phi - \sin^2\phi\cos^2\theta\right)$$
$$+ \ddot{M}_{22}\left(\sin^2\phi - \cos^2\phi\cos^2\theta\right)$$
$$- \ddot{M}_{33}\sin^2\theta$$
$$- \ddot{M}_{12}\sin 2\phi \mid \left(1+\cos^2\theta\right)$$
$$+ \ddot{M}_{13}\sin\phi\sin 2\theta$$
$$+\ddot{M}_{23}\cos\phi\sin 2\theta]$$

$$h_\times(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}[\left(\ddot{M}_{11} - \ddot{M}_{22}\right)\sin 2\phi\cos\theta$$
$$+ 2\ddot{M}_{12}\cos 2\phi\cos\theta$$
$$- 2\ddot{M}_{13}\cos\phi\sin\theta$$
$$+2\ddot{M}_{23}\sin\phi \mid \sin\theta]$$

$$\xrightarrow{\varphi = 0,\, \theta = \pi}$$

$$h_+(t) = \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right)$$
$$h_\times(t) = \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}$$

# Gravitational Waves Calculations

$$h_+(t) = \frac{1}{D}\frac{4G\mu\omega_s^2 R^2}{c^4}\frac{1+\cos^2\iota}{2}\cos\left(\omega_{\mathrm{GW}} t\right)$$
$$h_\times(t) = \frac{1}{D}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\iota\sin\left(\omega_{\mathrm{GW}} t\right)$$

$$h_+(t) = \frac{1}{D}\frac{4G\mu\omega_s(t)^2 R(t)^2}{c^4}\frac{1+\cos^2\iota}{2}\cos[\Phi(t)]$$
$$h_\times(t) = \frac{1}{D}\frac{4G\mu\omega_s(t)^2 R(t)^2}{c^4}\cos\iota\sin[\Phi(t)]$$
$$\Phi(t) \equiv \int^t \omega_{\mathrm{GW}}\left(t'\right) dt'$$

Frequency Space:
Fourier transformation

$$\tilde{h}_{+,\times}(f) = \int_{-\infty}^{\infty} h_{+,\times}(t) e^{2\pi i f t} dt$$

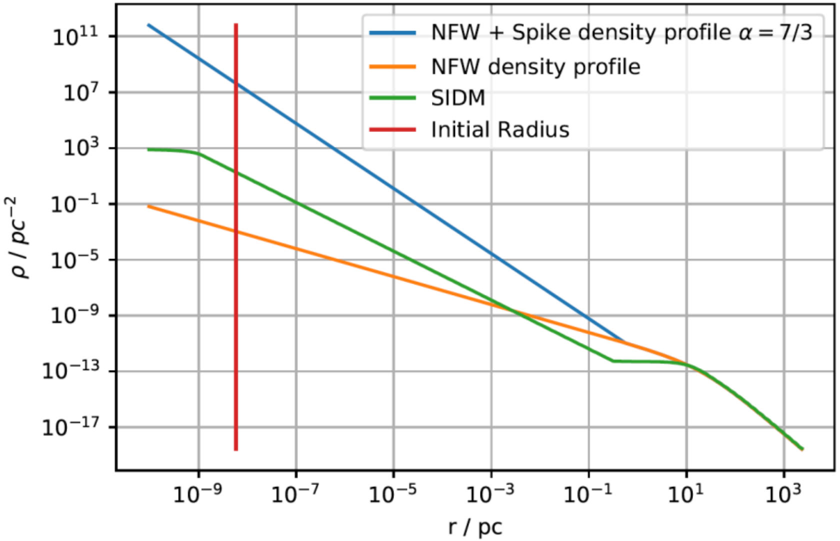
# Question for you:

Can a perfectly spherical object rotating around itself emit gravitational waves?

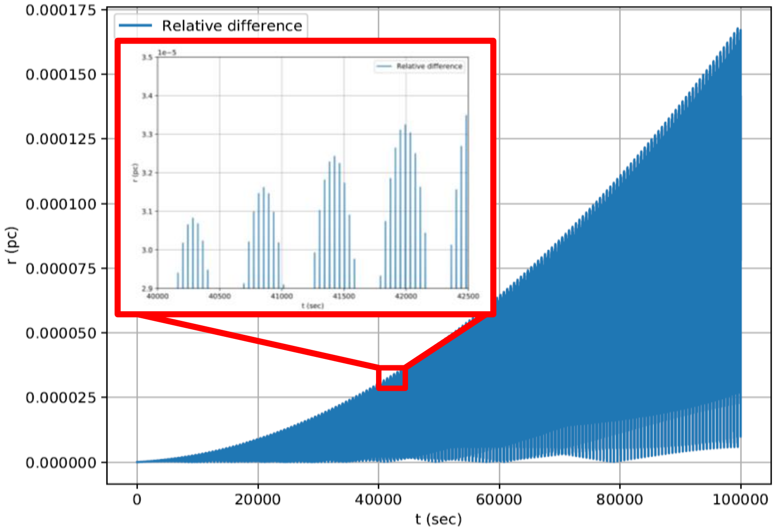
# Final Results

# Results

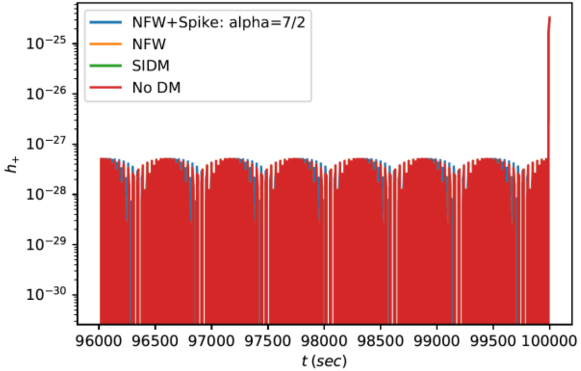
Types of densities used:

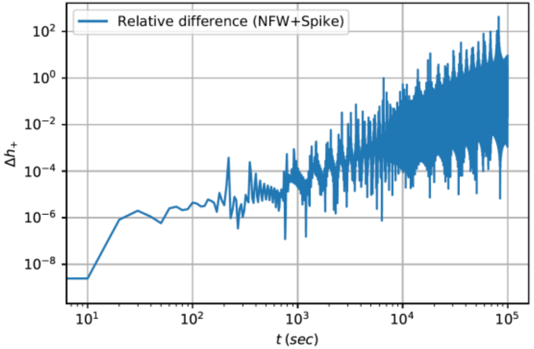


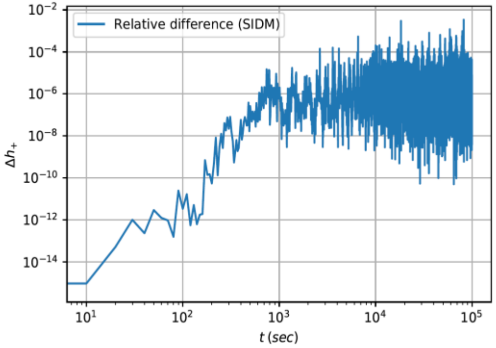

Deviations from no DM:

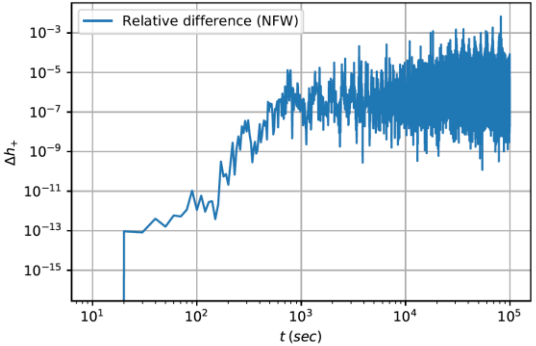


# Results

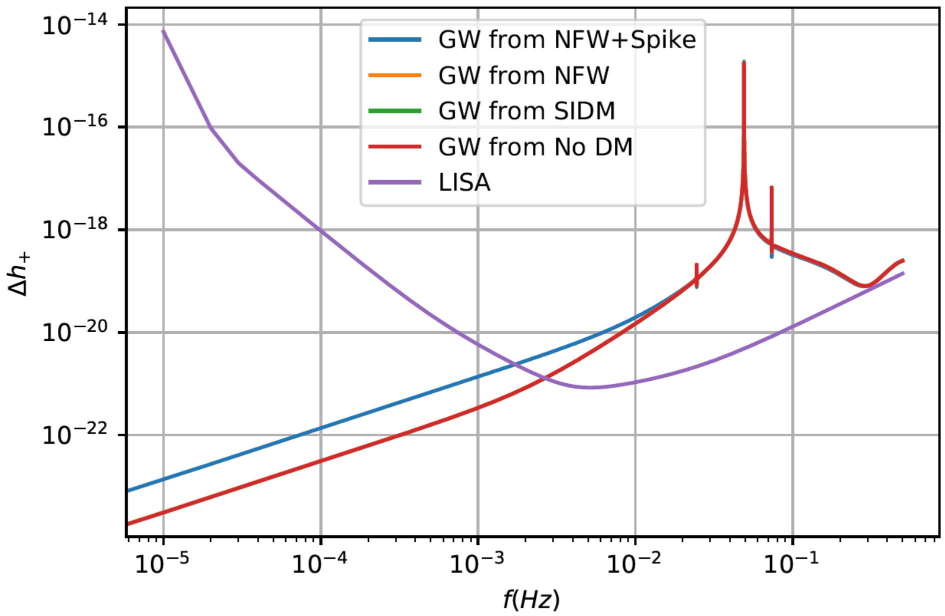

NFW+Spike: alpha=7/2
NFW
SIDM
No DM
$h_+$
$t\ (sec)$


Relative difference (NFW+Spike)
$\Delta h_+$
$t\ (sec)$

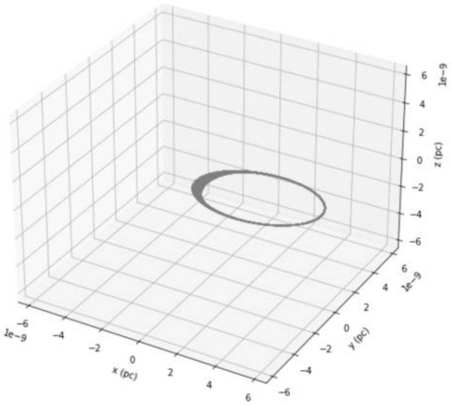

Relative difference (SIDM)
$\Delta h_+$
$t\ (sec)$


Relative difference (NFW)
$\Delta h_+$
$t\ (sec)$

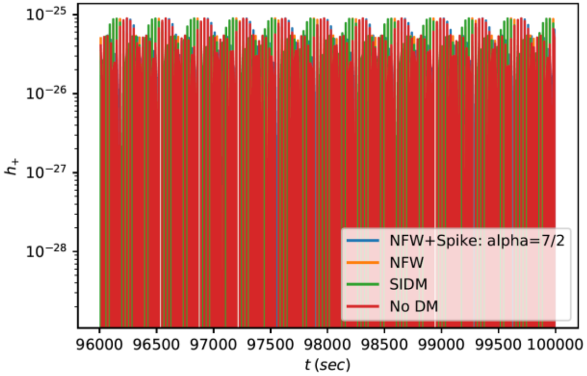
# Results

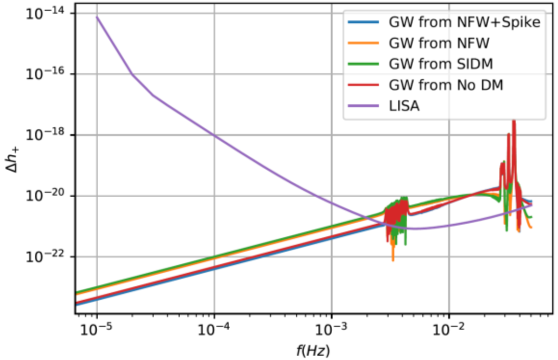



But what about highly elliptical orbits?



# Results



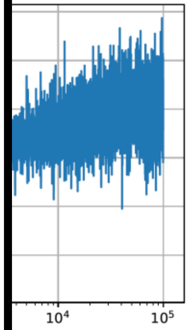



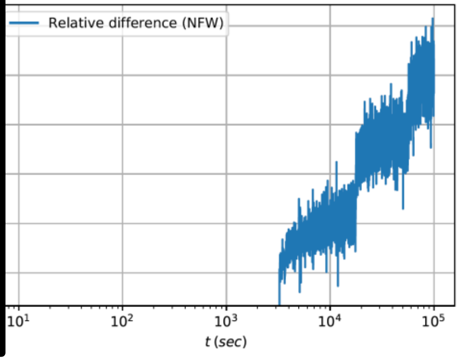

## Summary:

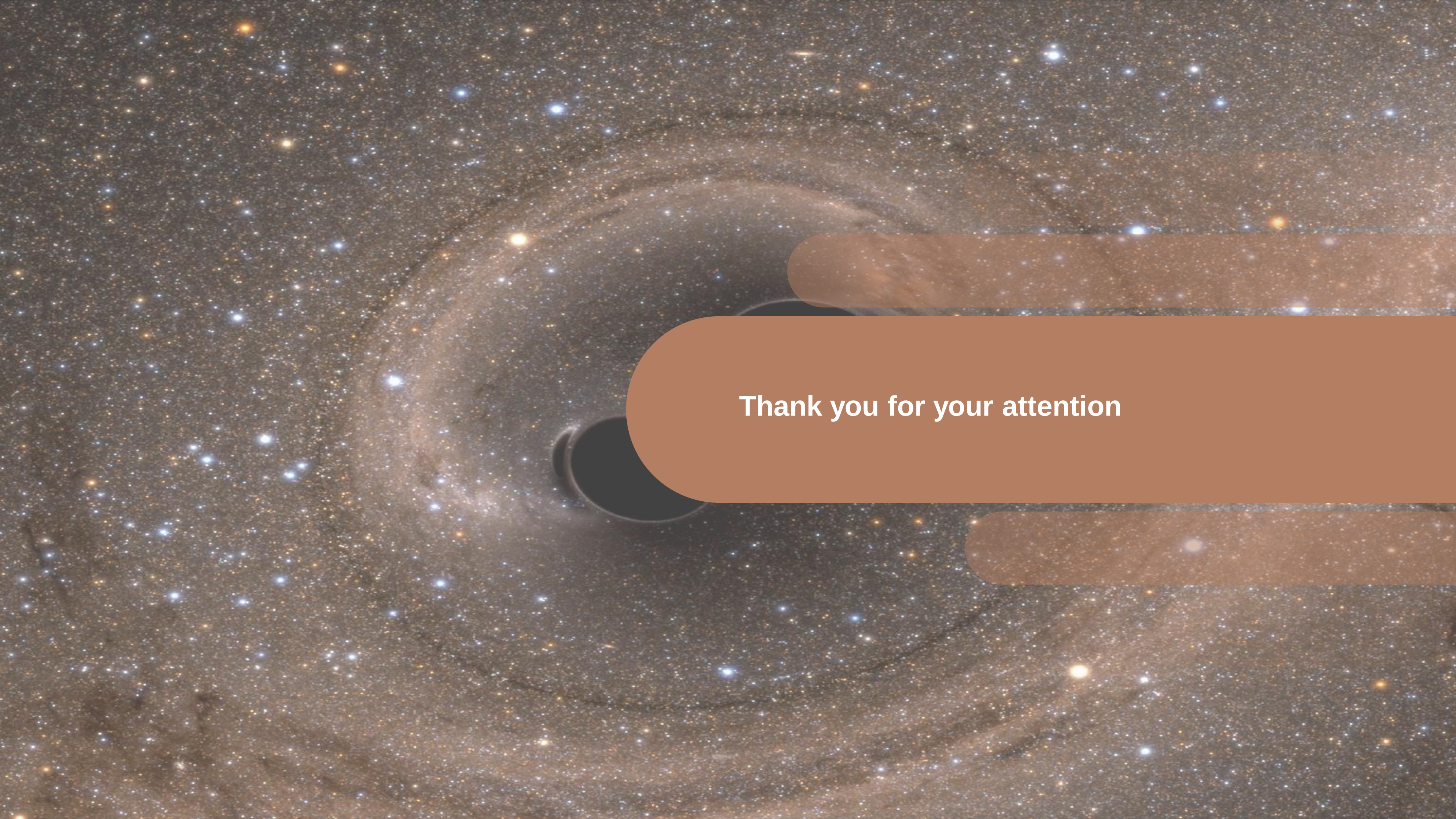
- Computed general classical orbits in the presence of dark matter friction and gravitational wave radiation.
- Computed the modifications to the gravitational wave due to different dark matter profiles.

## Future Work:

- Computing a wider variety of density profiles on bigger time scales and different orbit shapes.
- Finding an appropriate metric and applying full relativistic effects.





# Thank you for your attention