



Dark Stars

EXPLORE - 2021

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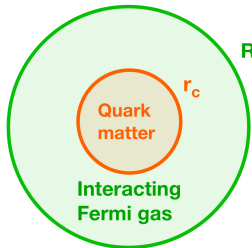
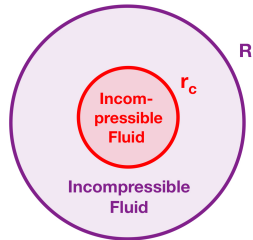
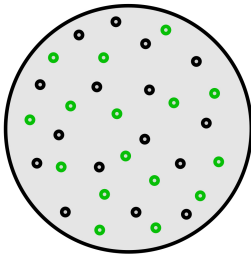
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EXPLORE Workshop, 2021

- 1 Introduction
- 2 Hydrostatic Equilibrium
- 3 Constant Density Solutions
- 4 Rescaling TOV for General Case
- 5 Non-constant Density : Fermi Gas
- 6 Conclusions
- 7 Extra slides

- Existence of unexplored matter: dark matter
- Possibility for stars consisting of dark matter
- Explore properties of combined compact stars, dark stars

Dark stars models



White Dwarf

White Dwarfs are compact stars, comprised of electrons and nucleons, supported by electron degeneracy pressure.

- Mass : 0.5-1 M_{\odot}
- Typical Radius : 10000 km
- Dominant Particles: Electrons, Nucleons

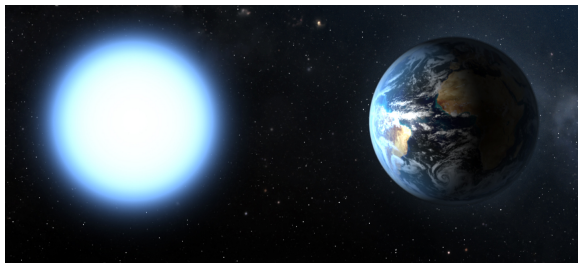


Figure: Comparison of the size of Sirius B and the Earth (Credit: ESA)

Neutron Star

Neutron Stars are compact stars comprised dominantly of neutrons, which are supported by the interaction pressure of the nucleons.

- Mass : $1-2 M_{\odot}$
- Typical Radius : 10 km
- Dominant Particles: Neutrons

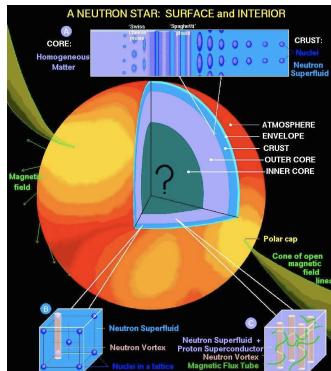


Figure: Neutron Star Structure (Credit: Dany Page)

Quark Star

Quark stars are a type of theoretical compact objects, more compact than a neutron star, made up of quark matter.

- Mass : $1-2 M_{\odot}$
- Typical Radius : 10 km
- Dominant Particles: Up quark, Down quark, Strange quark

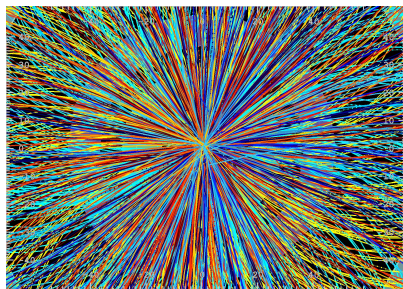
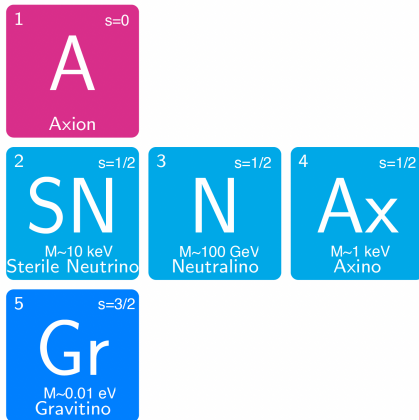


Figure: collision at the LHC producing quark matter (Alice collaboration)

Fermion stars are a class of compact stars that are made of mainly fermions.

- Includes the aforementioned White Dwarfs, Neutron Stars, and Quark Stars
- This can be generalized to model an arbitrary fermion by modelling the equation of state as a free or interacting Fermi gas
- Dominant Particles: Fermions

Dark Matter Candidates



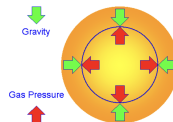
Hydrostatic Equilibrium

By definition hydrostatic equilibrium is obtained when the pressures of fluid and external forces such as gravity balance each other

$$dF_P + dF_g = 0$$

Two gravity regime cases where we can discuss hydrostatic equilibrium:

- Non-relativistic Case (Newtonian)
- Relativistic Case



Non-relativistic

The gravitational force on a mass element dm is

$$dF = -\Phi dm,$$

where $\Phi = -G\frac{M}{r}$ is the gravitational potential, and G is the gravitational constant.

Assuming a spherically symmetric object we arrive at an equation for hydrostatic equilibrium with $\rho(r)$ mass-density as function of r :

$$\boxed{\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r).} \quad (1)$$

Similarly from the spherical geometry assumption we get an equation for mass:

$$\boxed{\frac{dM}{dr} = 4\pi r^2 \rho(r).} \quad (2)$$

For the relativistic case we use General Relativity:

- Solve Einstein's field equations
- Schwarzschild metric for stars
- Energy momentum tensor of ideal fluid :

$$T_{\mu\nu} = \rho g_{\mu\nu} + (P + \rho)U_{\mu}U_{\nu},$$

with pressure P , restmass density ρ , velocity of fluid U_{μ}

Setting the condition of equilibrium, we get the structure of a compact star:

$$\frac{dP}{dr} = -\frac{G M(r)\rho(r)}{r^2} \left(1 + \frac{P}{\rho(r)c^2}\right) \frac{\left(1 + \frac{4\pi r^3 P}{M(r)c^2}\right)}{\left(1 - \frac{2GM(r)}{rc^2}\right)}, \quad (3)$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r), \quad (4)$$

which are the Tolman-Oppenheimer-Volkoff (TOV) equations.

- Throughout the presentation we will be using natural units by setting $\hbar = c = 1$
- G can be expressed in terms of Planck Mass M_p , $G = M_p^{-2}$.

Boundary Conditions

Initial and boundary conditions for solving differential equation for relativistic and non-relativistic case:

$$P(r = 0) = P_o \qquad P(r = R) = 0$$

$$M(r = 0) = 0 \qquad M(r = R) = M$$

- The central pressure P_0 is calculated from the Equation of State (EOS) (if known)
Central energy density $\rho(r = 0) = \rho_0$ is given as the initial condition.

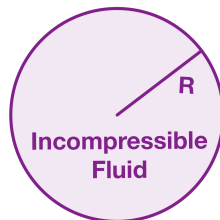
For example: EOS for ideal gas is $PV = nRT$

Model I: Constant Density Solution

Consider star with density $\rho(r) = \rho^*$ for all $r \leq R$ then there is an analytical solution for both the non-relativistic and relativistic cases.

The density profile reads as follows:

$$\rho(r) = \begin{cases} \rho^* & 0 \leq r \leq R, \\ 0 & r > R. \end{cases}$$



Model Ia: Non-relativistic Case with Constant Density

Solving equation (1) and (2) with constant density

$\rho(r) = \rho^*$ for all $r \leq R$

$$\left. \begin{aligned} \frac{dP}{dr} &= -\frac{GM(r)}{r^2} \rho^* \\ \frac{dM}{dr} &= 4\pi r^2 \rho^* \rightarrow M(r) = \frac{4}{3}\pi r^3 \rho^* \end{aligned} \right\} \quad \text{Non-relativistic}$$

We obtain the analytical solution:

$$\boxed{P(r) = P(0) \left(1 - \frac{r^2}{R^2} \right)} \quad (5)$$

where,

$$P(0) = \frac{2\pi}{3} GR^2 \rho^{*2}$$

Model Ib: Relativistic Case with Constant Density

In the relativistic case we have the TOV-equation for constant density $\rho(r) = \rho^*$ for all $r \leq R$

$$\left. \begin{aligned} \frac{dP}{dr} &= -G \frac{M(r)}{r^2} \rho^* \left(1 + \frac{P}{\rho^*}\right) \frac{\left(1 + \frac{4\pi r^3 P}{M(r)}\right)}{\left(1 - \frac{2GM(r)}{r}\right)} \\ \frac{dM}{dr} &= 4\pi r^2 \rho^* \rightarrow M(r) = \frac{4}{3}\pi r^3 \rho^* \end{aligned} \right\} \quad \text{Relativistic}$$

We obtain analytical solution:

$$P(r) = \rho^* \left(\frac{R\sqrt{R-2GM} - \sqrt{R^3-2GMr^2}}{\sqrt{R^3-2GMr^2} - 3R\sqrt{R-2GM}} \right). \quad (6)$$

Model I: Constant Density Plot

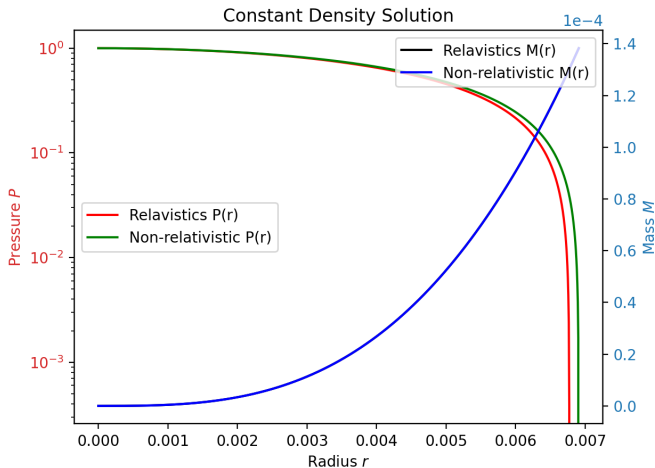
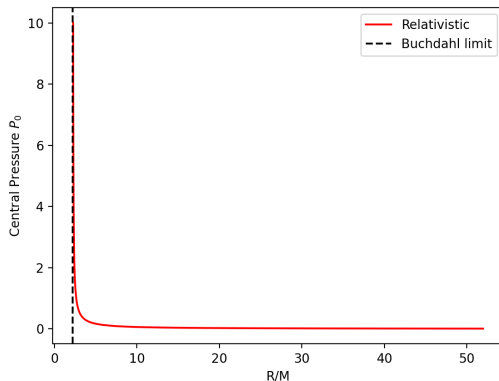


Figure: Pressure vs Radius & Mass vs Radius diagrams

Buchdahl's Limit

Notice in the solution of TOV (6) as $R \rightarrow \frac{9}{4} GM$ solution diverges we call this compactness limit or Buchdahl's limit.

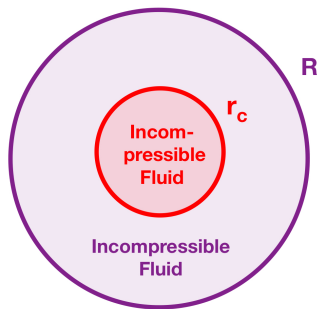
$$\frac{R}{M} > \frac{9G}{4} \quad \text{OR} \quad C = \frac{M}{R} < \frac{4}{9G}.$$



Model II: Two-fluid constant density solution

We can consider a fluid density profile for a star containing two incompressible fluids. The density profile reads as follows:

$$\rho(r) = \begin{cases} \rho_0 & 0 \leq r \leq r_c, \\ \rho_1 & r_c < r \leq R, \\ 0 & r > R. \end{cases}$$



Model II: Two-fluid constant density solution

Pressure and Mass vs Radius for constant densities and varying r_c :

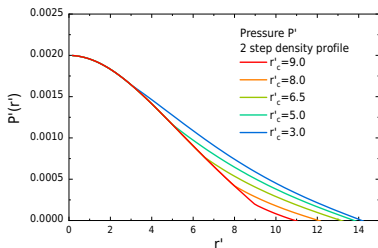


Figure: Pressure vs Radius

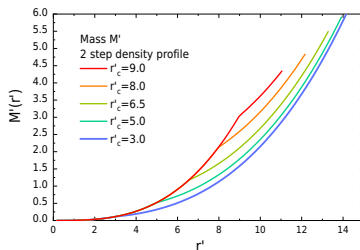
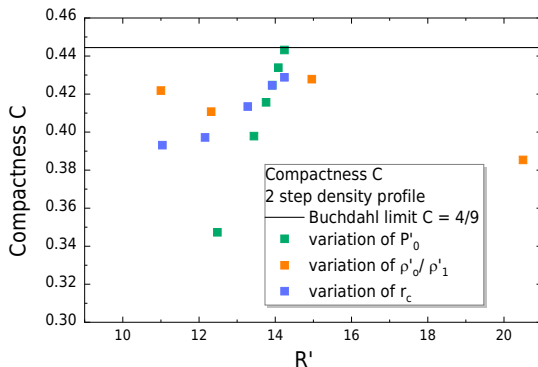


Figure: Mass vs Radius

Model II: Two-fluid constant density solution

Compactness for 2 Step Profile when varying central pressure P'_0 , ratio of densities ρ'_0/ρ'_1 and core radius r_c



→ All cases stay under the Buchdahl limit

Scaling of the TOV equation for General Case I

Consider dimensionless quantities:

$$P = \epsilon_0 \cdot P' \quad \text{and} \quad \rho = \epsilon_0 \cdot \rho', \quad (7)$$

and rescale radius r and mass M such that

$$r = b \cdot r' \quad \text{and} \quad M = a \cdot M', \quad (8)$$

With conditions we have:

$$a = \frac{M_p^3}{\sqrt{\epsilon_0}} \quad \text{and} \quad b = \frac{M_p}{\sqrt{\epsilon_0}} \quad (9)$$

Dimensionless TOV

Then dimensionless TOV:

$$\frac{dP'}{dr'} = -\frac{M'(r')}{r'^2} \rho'(r') \left(1 + \frac{P'}{\rho(r')}\right) \frac{\left(1 + \frac{4\pi r'^3 P'}{M'(r')}\right)}{\left(1 - \frac{2M'(r')}{r'}\right)}, \quad (10)$$

$$\frac{dM'(r')}{dr'} = 4\pi r'^2 \rho'(r'). \quad (11)$$

Model III: Fermi gas Equation of state

An ideal Fermi gas is a gas ensemble of many fermions

- Fermions are particles such as electrons, protons, and neutrons
- Particles with half-integer spin

Why Fermionic matter ?

Candidates for dark matter particles such as:

- Super-symmetric particles
- Neutralino
- Gravitino
- Axino

EOS: Fermi gas with interactions

The equation of state for an interacting Fermi gas $P(\rho)$ can be calculated via explicit expressions for the energy density ρ and pressure P from thermodynamics:

$$\rho' \equiv \frac{\rho}{m_f^4} = \rho_{ff} + \rho_{int} \quad (12)$$

$$P' \equiv \frac{P}{m_f^4} = P_{ff} + P_{int} \quad (13)$$

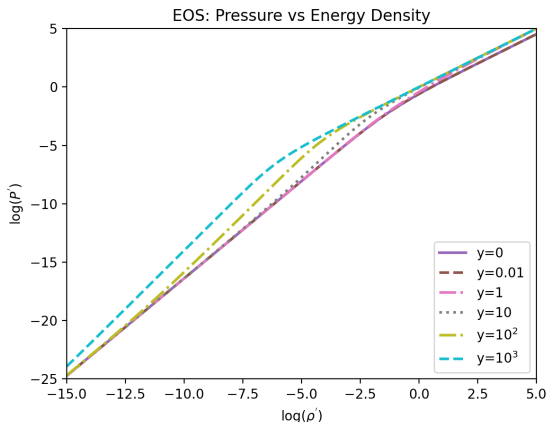
with P_{int} and $\rho_{int} \propto y$

where y is interaction strength and m_f is mass of fermion

- We make EOS dimensionless by dividing by m_f^4 so $\epsilon_o = m_f^4$
- Note: when $y = 0$ we get the EOS for Free Fermi gas (Fermi gas with no interactions)

Model III: Fermi gas Equation of state

Figure below depicts the resulting dimensionless pressure P versus the dimensionless energy density ρ in a logarithmic of P' and ρ' for different interaction strengths (y) ranging from 0 to 10^3



Model IIIa: Free Fermi gas

For interaction strength $y = 0$ we have a Free Fermi gas (non-interacting fermions):

Mass vs Radius plot for a Free Fermi Gas :

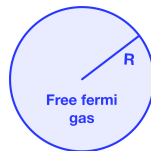
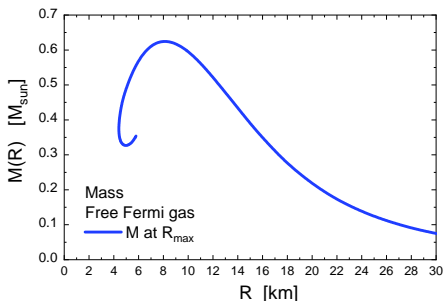
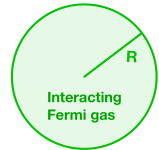
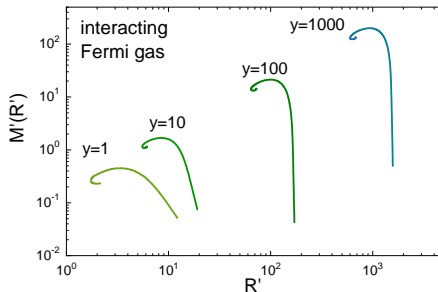


Figure: Mass vs Radius

Model IIIb: Fermi gas with Interactions

Mass vs Radius for an interacting Fermi gas
(varying interaction strength, y):

On double logarithmic scale ($y = 1$ to $y = 1000$) :



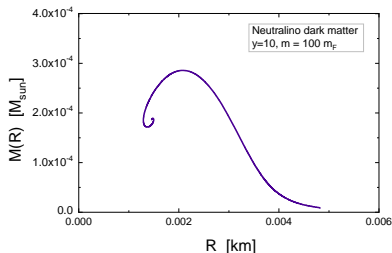
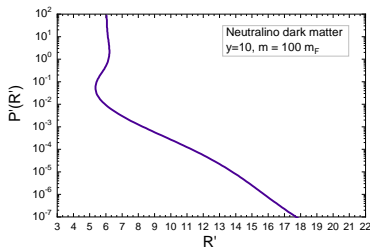
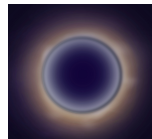
→ Maximal mass and
Minimum Radius increases with interaction strength

Model IV: Neutralino dark matter

Dark matter star : $y = 10$

Neutralino : $m = 100 \text{ GeV}$

(Narain et. al. 2006 [astro-ph/0605724])



→ For Neutralinos the radius of the star is under 1 km

Model IV: Neutralino dark matter

Compactness for the dark matter:

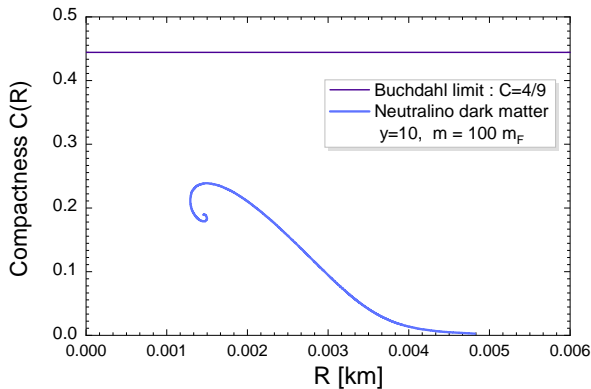


Figure: Compactness vs Radius

Model V: Axino dark matter and fermi solutions

Combined dark matter fermi star:

Core: Free fermi gas of (dark) neutrons,

Shell: Selfinteracting axino dark matter $y = 10$

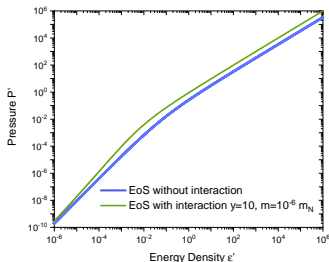
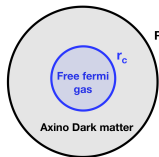


Figure: EoS

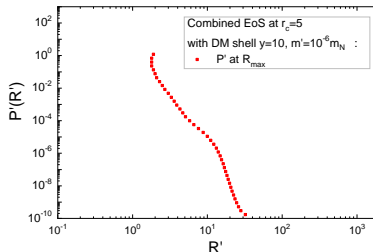
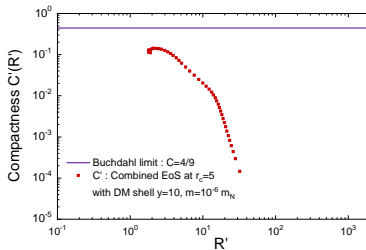
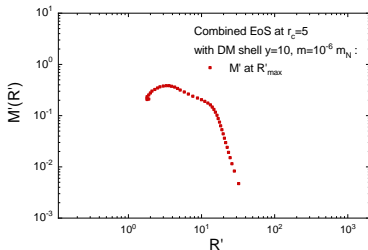


Figure: Pressure vs Radius

Model V: Axino dark matter and fermi solutions

Mass vs Radius and compactness plots for a combined dark matter fermi star:



→ With dark matter we obtain different shapes of M-R-curve

Model VI: Quark matter and fermi solutions

Combined quark fermi star:

Core: quark matter

MIT-BAG-model : $P = (\epsilon - \epsilon_{vac})/3$

Shell: interacting fermi gas $y = 10$

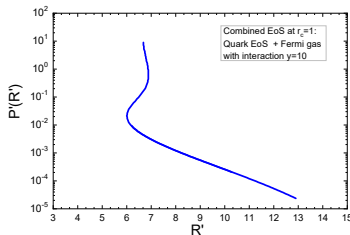
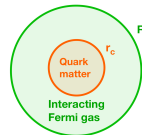


Figure: Pressure vs Radius

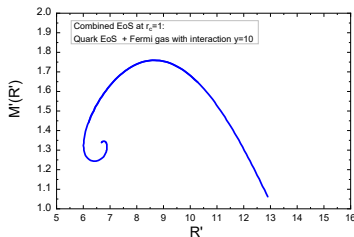


Figure: Mass vs Radius

Model VI: Quark matter and Fermi solutions

Compactness plot for a combined quark fermi star:

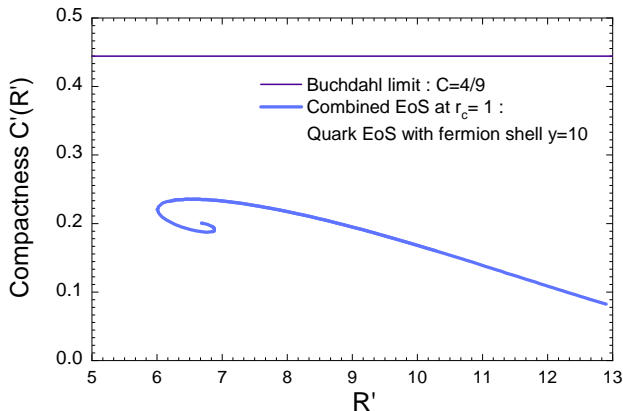


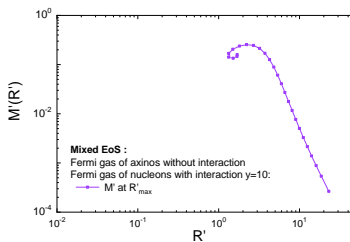
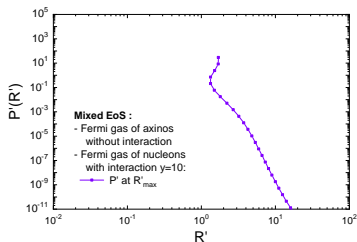
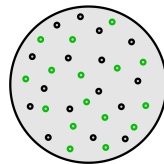
Figure: Compactness vs Radius

Model VII: Mixed dark matter and fermi solutions

Mixed dark matter fermi star:

→ Interacting fermi gas of neutrons $y = 10$

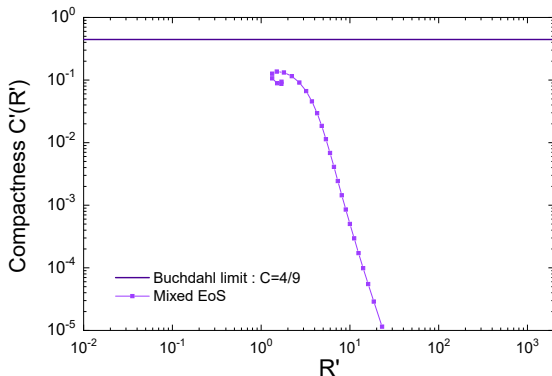
→ Axino dark matter



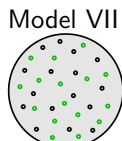
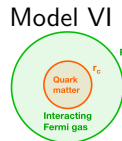
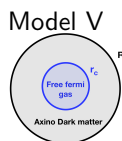
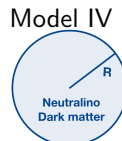
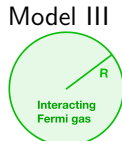
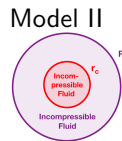
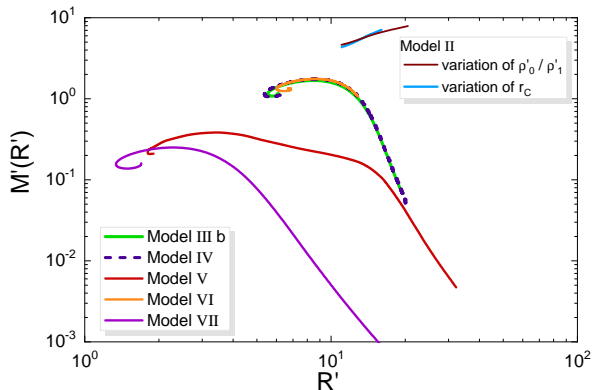
→ A new star configuration

Model VII: Mixed dark matter fermi solutions

Compactness for a mixed dark matter fermi star:



Results of combined stars



Summary

- find various solutions for combined star configurations
- solutions are stable, even Dark Matter solutions
- produce different mass-radius solutions, different shapes
- in all cases the Buchdahl limit is fulfilled

Outlook: exploration not yet finished

- Explore parameter range
- Classify dark stars
- Realization with microphysics



Artistic logo

Question to audience :

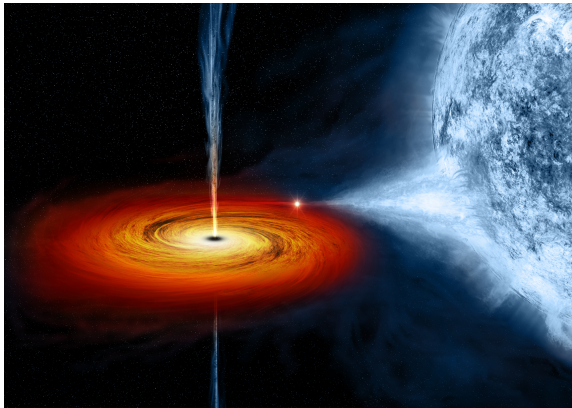
What could be beyond the Buchdahl limit ?

Question to audience :

What could be beyond the Buchdahl limit ?

- Answer :

Objects with higher compactness : Black holes !



NASA/CXC/M.Weiss



EXPLORE

Thank you for your attention!

Dark Stars

Appendix:

- Derivation of analytical expressions

The gravitational force on a mass element dm is

$$d\vec{F} = -\Phi dm, \quad (14)$$

where Φ is the gravitational potential given by:

$$\Phi = -G \frac{M}{r}, \quad (15)$$

where G is the gravitational constant.

Non-relativistic II

We can write an expression for the force density and since the pressure is the force over the area we get:

$$\frac{d\vec{F}}{dV} = -\rho\vec{\nabla}\Phi \implies \vec{\nabla}P = -\rho(r)\vec{\nabla}\Phi(r), \quad (16)$$

where ρ is the mass-density of the object.

For a spherically symmetric object we can rewrite eq. (11) as

$$\frac{dP}{dr} = -\rho(r)\frac{d\Phi}{dr}, \quad (17)$$

Thus from the derivative of the gravitational potential we obtain the equation for hydrostatic equilibrium:

$$\boxed{\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r).} \quad (18)$$

Mass differential Equation

Assuming spherical geometry, the equation of mass follows from eq. (??)

$$\nabla\Phi = 4\pi r^2\rho(r). \quad (19)$$

From above we get :

$$\boxed{\frac{dM(r)}{dr} = 4\pi r^2\rho(r).} \quad (20)$$

For the relativistic case we use General Relativity:

- Solve Einstein's field equations
- Schwarzschild metric for stars
- Energy momentum tensor of ideal fluid :

$$T_{\mu\nu} = \rho g_{\mu\nu} + (P + \rho)U_{\mu}U_{\nu}, \quad (21)$$

with pressure P , restmass density ρ , velocity of fluid U_{μ}

Setting the condition of equilibrium, $U_\mu = (1, 0)$ we get for a compact star:

$$\frac{dP}{dr} = -\frac{G M(r)\rho(r)}{r^2} \left(1 + \frac{P}{\rho(r)}\right) \frac{\left(1 + \frac{4\pi r^3 P}{M(r)}\right)}{\left(1 - \frac{2GM(r)}{r}\right)}, \quad (22)$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r), \quad (23)$$

which are the Tolman-Oppenheimer-Volkoff (TOV) equations.

- Throughout the presentation we will be using natural units by setting $\hbar = c = 1$

Boundary Conditions

-> Initial and boundary conditions for solving TOV :

- The radius of the star, R , is found by using the condition that the pressure vanishes at the surface of the star. ($P(R) = 0$)
- The mass $M(0)$ must be zero at $r = 0$ and $M(R)$ gives the total mass of the star at $r = R$
- The central pressure P_0 is calculated from the equation of state (EOS) (if known) once the central energy density $\rho(0) = \rho_0$ is given as the initial condition.

Model Ia: Non-relativistic Case with Constant Density

Solving equation (1) and (2) with constant density

$$\rho(r) = \rho^* \text{ for } r \leq R$$

$$\left. \begin{aligned} \frac{dP}{dr} &= -\frac{GM(r)}{r^2} \rho^*, \\ \frac{dM(r)}{dr} &= 4\pi r^2 \rho^*. \end{aligned} \right\} \quad \text{Non-relativistic} \quad (24)$$

We obtain the analytical solution:

$$P(r) = P(0) \left(1 - \frac{r^2}{R^2} \right), \quad (25)$$

where,

$$P(0) = \frac{2\pi}{3} GR^2 \rho^{*2}.$$

Model Ib: Relativistic Case with Constant Density

In the relativistic case we have the TOV-equation for constant density $\rho(r) = \rho^*$ for $r \leq R$

$$\text{Relativistic: } \left\{ \begin{array}{l} \frac{dP}{dr} = -G \frac{M(r)}{r^2} \rho^* \left(1 + \frac{P}{\rho^*} \right) \frac{\left(1 + \frac{4\pi r^3 P}{M(r)} \right)}{\left(1 - \frac{2GM(r)}{r} \right)}, \\ \frac{dM(r)}{dr} = 4\pi r^2 \rho^*. \end{array} \right\} \quad (26)$$

We obtain analytical solution:

$$P(r) = \rho^* \left(\frac{R\sqrt{R-2GM} - \sqrt{R^3-2GMr^2}}{\sqrt{R^3-2GMr^2} - 3R\sqrt{R-2GM}} \right). \quad (27)$$

Scaling of the TOV equation for General Case I

Consider dimensionless quantities:

$$P = \epsilon_0 P' \text{ and } \rho = \epsilon_0 \rho', \quad (28)$$

and rescale radius r and mass M such that

$$r = br' \text{ and } M = aM', \quad (29)$$

- Note: G can be expressed in terms of Planck Mass, M_p :
 $G = M_p^{-2}$.

Scaling of the TOV equation for General Case

Rescaled TOV equation in terms of dimensionless quantities:

$$\frac{\epsilon_0 dP'}{a dr'} = -G \frac{b M' \epsilon_0 \rho'}{a^2 r'^2} \left(1 + \frac{\epsilon_0 P'}{\epsilon_0 \rho'} \right) \frac{\left(1 + \frac{4\pi a^3 r'^3 \epsilon_0 P'}{b M'} \right)}{\left(1 - \frac{2G b M'}{a r'} \right)}, \quad (30)$$

With conditions:

$$a = b^3 \epsilon_0 \quad \text{and} \quad b = \frac{a}{M_p^2}, \quad (31)$$

We have:

$$a = \frac{M_p^3}{\sqrt{\epsilon_0}} \quad \text{and} \quad b = \frac{M_p}{\sqrt{\epsilon_0}}. \quad (32)$$

Dimensionless TOV

Then dimensionless TOV:

$$\frac{dP'}{dr'} = -\frac{M'(r')}{r'^2} \rho'(r') \left(1 + \frac{P'}{\rho(r')} \right) \frac{\left(1 + \frac{4\pi r'^3 P'}{M'(r')} \right)}{\left(1 - \frac{2M'(r')}{r'} \right)}, \quad (33)$$

$$\frac{dM'(r')}{dr'} = 4\pi r'^2 \rho'(r'). \quad (34)$$

Why dimensionless?