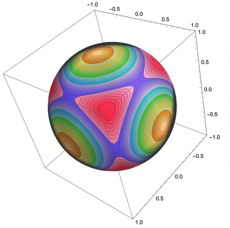
Metrics and Machine Learning



Challenger Mishra

Academic Fellow
The Computer Laboratory
University of Cambridge

String Data, Johannesburg
December 16, 2021

ACCELERATE PROGRAMME
FOR SCIENTIFIC DISCOVERY

UNIVERSITY OF CAMBRIDGE
Computer Laboratory

SCHMIDT FUTURES

Follow our activities:
https://acceleratescience.github.io/about.html

Machine Learning & String Theory: Datasets

Datasets; sizes; inception

- Complete Intersection Calabi-Yau threefolds; 10k; 1988[1].
- Kreuzer Skarke dataset of CY threefolds; 500m; 2002.
- Supersymmetric bundles over CY manifolds; string derived particle physics models; o(10k); 2007 and ongoing[†].
- (Free) Discrete Symmetries of Complete Intersection Calabi-Yau threefolds 10k; 2010.
- Complete Intersection Calabi-Yau fourfolds; 1m; 2013.
- Discrete Symmetries of Complete Intersection Calabi-Yau quotient threefolds; 10k; 2017[†].
- Transgressable bundles over Calabi-Yau threefolds; first examples, unknown size!; 2018 and ongoing[†].

New datasets are being (painstakingly) created. *Machine Learning String theorists* have been working on these datasets and others, with the first such papers appearing in 2017.

[1] possibly the first large dataset in algebraic-geometry

Research Directions

- Predicting topological properties of stringy geometries with high accuracy: $h^{1,1}, h^{2,1}, \chi$.
- Studying invariances of smooth Calabi-Yau threefolds (*Fantastically symmetric Calabi-Yaus and how to find them*); [2]
- Understanding String derived Standard Models.
- Study of moduli spaces of (stable) bundles over stringy geometries: $\mathcal{M}^{\mathrm{rk}}{}_{\mathrm{CY}}(\mathcal{V})$; start with the study of toy (yet powerful) moduli spaces of line bundle sums.
- Towards discovering the Calabi-Yau metric (g_{MN}) in three dimensions; start with toy geometries like the torus, or K3 surfaces.
- Towards discovering the correct string vacuum; start with ML algorithms to generate ad-hoc realistic string vacua.
- Understanding string landscape: the swampland conjectures; Reid's fantasy.
- Developing stochastic/ML algorithms to solve systems of nonlinear diophantine equations (see work due to Fabian Ruehle).

[2] possible hints from recent works on NN invariances ala Taco Cohen, Risi Kondor, and detecting symmetries by Sven Krippendorf and so on.

Machine Learning for Calabi–Yau Metrics

Neural Network Approximations for Calabi–Yau Metrics

Vishnu Jejjala[a] , **Damián Kaloni Mayorga Peña**[a,b] , **Challenger Mishra**[c]

[a] *Mandelstam Institute for Theoretical Physics, School of Physics, NITheP, and CoE-MaSS, University of the Witwatersrand, Johannesburg, WITS 2050, South Africa*

[b] *Data Laboratory, Universidad de Guanajuato, Loma del Bosque No. 103 Col. Lomas del Campestre C.P. 37150 Leon, Guanajuato, Mexico*

[c] *Department of Computer Science & Technology, University of Cambridge, 15 J.J. Thomson Ave., Cambridge CB3 0FD, United Kingdom*

E-mail: vishnu@neo.phys.wits.ac.za, damian.mayorgapena@wits.ac.za, cm2099@cam.ac.uk

Abstract: Ricci flat metrics for Calabi–Yau threefolds are not known analytically. In this work, we employ techniques from machine learning to deduce numerical flat metrics for the Fermat quintic, for the Dwork quintic, and for the Tian–Yau manifold. This investigation employs a single neural network architecture that is capable of approximating Ricci flat Kähler metrics for several Calabi–Yau manifolds of dimensions two and three. We show that measures that assess the Ricci flatness of the geometry decrease after training by three orders of magnitude. This is corroborated on the validation set, where the improvement is more modest.

Related Work

Numerical Calabi–Yau metrics

- M. Headrick and T. Wiseman, Numerical Ricci-flat metrics on K3, Class. Quant. Grav. 22 (2005) 4931–4960, [hep-th/0506129].
- S. Donaldson, Scalar curvature and projective embeddings, i, J. Differential Geom. 59 (11, 2001) 479–522.
- S. K. Donaldson, Some numerical results in complex differential geometry, arXiv preprint math/0512625 (2005).

Analytic K3 metrics

- S. Kachru, A. Tripathy and M. Zimet, K3 metrics from little string theory, 1810.10540.
- S. Kachru, A. Tripathy and M. Zimet, K3 metrics, 2006.02435.

A machine learning case for Calabi–Yau metrics

There is now an increasing effort in employing Machine Learning to construct approximations to Ricci flat Calabi–Yau metrics, an essential ingredient in connecting String theory to four dimensional particle physics.

- Jejjala, Mayorga Peña, CM: Neural Network Approximations for Calabi–Yau Metrics, [2012.15821].
- Larfors, Lukas, Ruehle, Schneider: Learning Size and Shape of Calabi-Yau Spaces, [2111.01436].
- Anderson, Gerdes, Gray, Krippendorf, Raghuram, and Ruehle: Moduli-dependent Calabi-Yau and SU(3)-structure metrics from Machine Learning, [2012.04656].
- Douglas, Lakshminarasimhan, and Qi: Numerical Calabi-Yau metrics from holomorphic networks, [2012.04797].
- Ashmore, He and Ovrut, Machine Learning Calabi–Yau Metrics, Fortsch. Phys. 68 (2020) 2000068, [1910.08605]

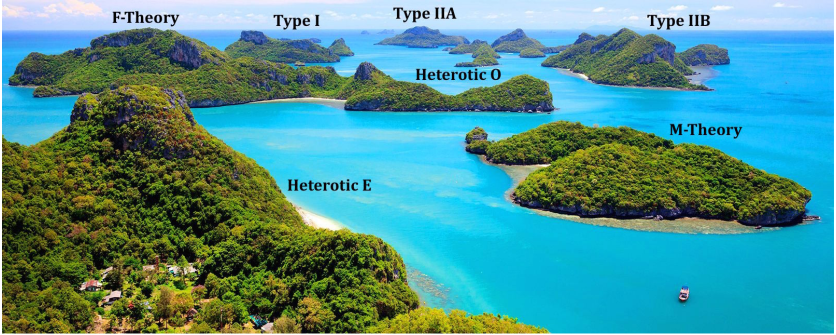
Unification

String theory is the only known consistent theory of quantum gravity.

- Postulates (6) extra-dimensions of space.
- Relies on a fundamental symmetry between matter particles and force carriers, called supersymmetry (SUSY).

String theory is also an organising principle for mathematics.

String Phenomenology

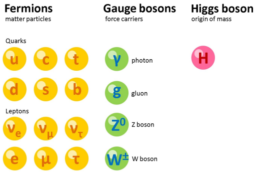


Superstring theory supplies an architectural framework for obtaining the real world from a consistent theory of quantum gravity.[3]

[3] Illustration is inspired from a talk by Liam McAllister.

String Phenomenology

The Holy grail: Embed the Standard Model (SM) of particle physics within the framework of string theory.



1. Reproduce the particle content, coupling constants, masses of particles of the Standard Model.
2. Explain the origin of discrete symmetries of SM that help explain unobserved couplings, the long lifetime of the proton, etc.
3. Other challenges: Explain fine tuning, supersymmetry breaking.
4. No such model till date, but there has been considerable progress building semi-realistic models over last decade or so.

Unification: Compactification

String theory unifies gravity and QM and reduces to the SM in the low energy limit, via an intermediate GUT.

$$\text{String Theory} \quad \longrightarrow \quad \text{GUT} \quad \longrightarrow \quad \text{SM}$$

This is called string ‘compactification’ where the low energy theory, SM, is recovered by hiding away or compactifying over the extra-dimensions of space.

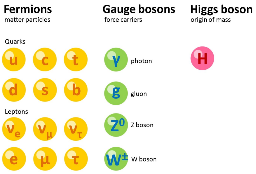
This places severe geometrical constraints on the extra-dimensions of string theory, called ‘Calabi-Yau’ manifolds.

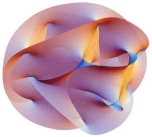
Calabi-Yau Manifolds: A centerpiece in String theory

- CY compactifications of the Heterotic String is one of the most promising avenues for string model building.
- The space-time for the effective field theory is the direct product: $\mathcal{M}_4 \times X_6$, where $\mathcal{M}_4$ is a maximally symmetric space.
- If X_6 is Riemannian, irreducible and we demand $N = 1$ supersymmetry in the 4-dimensional theory (SM), then $\mathrm{Hol}(X_6) = SU(3)$. Do such manifolds exist?
- Calabi conjecture (proved by Yau): An n-dimensional complex Kähler manifold with vanishing first Chern class admits a metric with $SU(n)$ holonomy. This leads us to the class of Calabi-Yau manifolds. Thus X_6 is a CY threefold.

Standard Model from topology of Calabi–Yau geometries

- In the simplest setting, $N_{\text{gen}} \sim$ Euler characteristic (a topological invariant) of the extra-dimensions.
- More generally, bundle valued cohomology groups compute particle spectrum of a string derived Standard Model of particle physics.
- Heterotic string models can be obtained by considering a compactification space that is a Calabi–Yau threefold (X) admitting a vector bundle $V \longrightarrow X$.





Non-standard embeddings

String Theory $\longrightarrow$ GUT $\longrightarrow$ SM

- Variations of the 10-dimensional heterotic effective action leads to stringent constraints on the bundle V. Additional constraints come from demanding an 'anomaly free' theory.
- These constraints are topological in nature. Satisfying such conditions lead to an *anomaly free* theory and involves *stable* bundles.
- In order to obtain a GUT group that is closer to the SM gauge group, one must choose the structure group G of the vector bundle V that is larger than SU(3), the structure group of the tangent bundle TX of the Calabi–Yau X.
- Given a vector bundle V with structure group $G \subset E_8$, the GUT group this leads to is the commutant of G in E_8.
- The GUT group should contain the standard model gauge group $SU(3) \times SU(2) \times U(1)$, and admit complex representations.

The breaking of the E_8 happens as follows:

$$\begin{array}{rcccc} & E_8 & \longrightarrow & \mathrm{GUT} & \times & G \\ rk(V) = 4: & E_8 & \longrightarrow & SO(10) & \times & S(U(1)^{\times 4}) \\ rk(V) = 5: & E_8 & \longrightarrow & SU(5) & \times & S(U(1)^{\times 5}) \end{array}$$

Line Bundle Models

- Prior to c.2008, only a handful of string derived standard models existed.
- Since then tens of thousands of semi-realistic string standard models were constructed by choosing the vector bundle $V \longrightarrow X$ to be a sum of line bundles (Anderson, Gray, Lukas, He, Constantin, etc.).

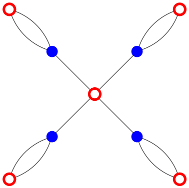
$$V = \bigoplus_{a=1}^{\mathrm{n}} L_a, \quad \text{where}$$

  n is the rank of the bundle.
- V is then completely specified by $h^{1,1}(X) \times rk(V)$ integers

A vanilla string model

Example: A CICY $\subset (\mathbb{P}^1)^{\times 4} \times \mathbb{P}^3$ with $h^{1,1} = 5$ *Constantin, Lukas and CM*

$$X = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{array} \right]^{5,37}_{-64}$$



.

$$\Gamma = \mathbb{Z}_4 \rtimes \mathbb{Z}_4 \quad , \quad V = \begin{bmatrix} -36 & -1 & 9 & 28 \\ 1 & -5 & 4 & 0 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

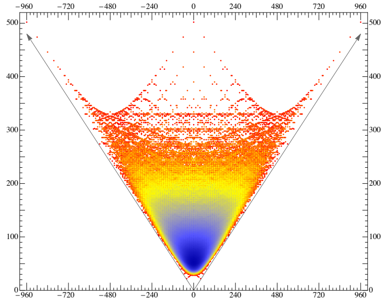
This is one of 46,534 string models that lead to a GUT with gauge group $SO(10)$ and reproduces many features of the SM. Note that none of these models give a complete picture, but simply lead to semi-realistic string models that agree with the Standard Model in broad terms.

Why study Calabi–Yau metrics?

- Luckily, topology on its own is a sufficiently powerful tool to enable string model building.
- To go beyond an analysis of the particle spectrum (and break supersymmetry in a controlled manner), we must obtain the Kähler potential, that leads to the metric on a Calabi–Yau.
- Knowledge of the metric will ultimately lead to computing the mass of fundamental particles in the context of String theory.
- Determining physical quantities from first principles therefore requires an understanding of the geometry as well as the topology of Calabi–Yau spaces.

Many possible Calabi-Yau geometries: The Hodge Plot

Different approaches to string theory involve different geometries: Manifolds with G_2 holonomy, CICY threefolds, hypersurfaces in reflexive polytopes, CY-fourfolds.



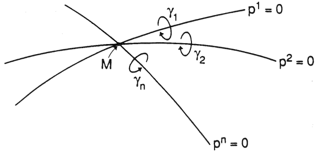
CY threefolds: x-axis: Euler Characteristic, y-axis: 'Height' $(h^{1,1} + h^{2,1})$

473,800,776 geometries

Complete Intersection Calabi-Yau Manifolds

The following is the form of a *configuration matrix* of a Complete Intersection Calabi–Yau manifold or a CICY:

$$\begin{array}{c} \mathbb{P}^{n_1} \\ \vdots \\ \mathbb{P}^{n_m} \end{array} \left[\begin{array}{ccc} q_1^1 & \cdots & q_n^1 \\ \vdots & \ddots & \vdots \\ q_1^m & \cdots & q_n^m \end{array} \right]^{h^{1,1}(X)}_{\chi(X)}$$



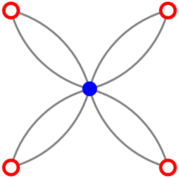
The CICY is given by the vanishing locus of a set of n homogeneous polynomials over a product of m complex projective spaces. Vanishing 1^{st} Chern class implies sum of each row is equal to the number of co-ordinates in the projective space,

$$\sum_a q_a^r = n_r + 1, \quad \forall r \in \{1, \dots, m\}$$

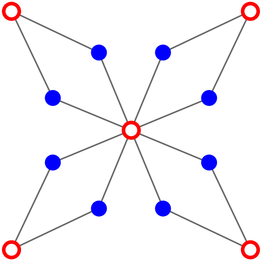
Example: $\quad X = \mathbb{P}^4[5] : \quad X = \{x \in \mathbb{P}^4 \mid \sum_{a=0}^{4} c_a x_a^5 = 0\}$

Calabi-Yau Manifolds: Examples

$$\begin{array}{c} \mathbb{CP}^1 \\ \mathbb{CP}^1 \\ \mathbb{CP}^1 \\ \mathbb{CP}^1 \end{array} \left[\begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \end{array} \right]^{4,68}_{-128}$$

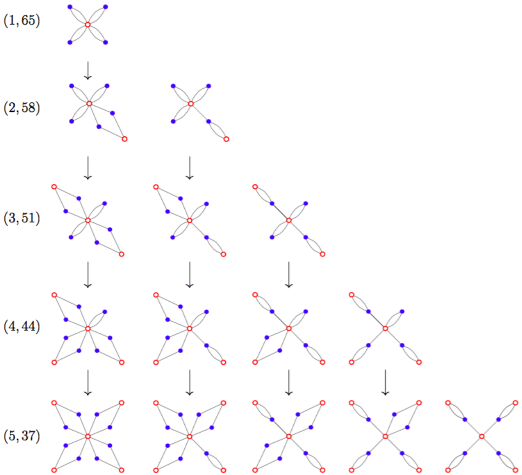


$$\begin{array}{c} \mathbb{CP}^1 \\ \mathbb{CP}^1 \\ \mathbb{CP}^1 \\ \mathbb{CP}^1 \\ \mathbb{CP}^7 \end{array} \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right]^{5,37}_{-64}$$



Vanishing 1^{st} Chern class implies sum of each row is equal to the number of co-ordinates in the projective space.

Conifold Transitions between CICYs



Calabi–Yau manifolds: General Remarks

As a complex Kähler manifold, the metric of a CY $\mathcal{M}$ is a Hermitian matrix that can be derived from a Kähler potential $K(z, \bar{z})$:

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K(z, \bar{z}) \ .$$

The metric can be used to construct the Kähler form as

$$J = \frac{\mathrm{i}}{2} g_{a\bar{b}} \, dz^a \wedge d\bar{z}^{\bar{b}} \ .$$

This is a closed $(1,1)$-form. The corresponding Ricci tensor is given by:

$$R_{a\bar{b}} = \partial_a \partial_{\bar{b}} \log \det g \ .$$

A Calabi–Yau admits a metric with vanishing Ricci curvature. This metric is unique in each Kähler class. Till date, no analytic expression has been found for a Ricci-flat metric of a Calabi–Yau threefold.

Calabi–Yau manifolds: Case studies

$$\begin{aligned} \text{K3}: &\quad z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0 \subset \mathbb{P}^3 , \\ \text{Fermat quintic}: &\quad z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 \subset \mathbb{P}^4 , \\ \text{Dwork family}: &\quad z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 - 5\psi z_1 z_2 z_3 z_4 z_5 = 0 \subset \mathbb{P}^4 , \quad \psi^5 \neq 1 . \end{aligned}$$

$$\text{Tian–Yau}: \quad \left[\begin{array}{c||ccc} \mathbb{P}^3 & 3 & 0 & 1 \\ \mathbb{P}^3 & 0 & 3 & 1 \end{array} \right]^{14,\ 23}_{\chi=-18} \qquad \Longleftrightarrow \qquad \left\{ \begin{array}{ccc} \alpha^{ijk} z_i z_j z_k & = & 0 , \\ \beta^{ijk} w_i w_j w_k & = & 0 , \\ \gamma^{ij} z_i w_j & = & 0 . \end{array} \right.$$

A freely acting $\mathbb{Z}_3$ quotient of the Tian–Yau yields a Calabi–Yau manifold with $\chi = -6$ [G. Tian and S. Yau; B. R. Greene, K. H. Kirklin, P. J. Miron and G. G. Ross, 1986-1987]

Calabi–Yau manifolds: Case studies

$$\text{Tian–Yau}: \quad \left[\begin{array}{c||ccc} \mathbb{P}^3 & 3 & 0 & 1 \\ \mathbb{P}^3 & 0 & 3 & 1 \end{array}\right]^{14,\ 23}_{\chi=-18} \qquad \Longleftrightarrow \qquad \left\{\begin{array}{ccl} \alpha^{ijk} z_i z_j z_k & = & 0\ , \\ \beta^{ijk} w_i w_j w_k & = & 0\ , \\ \gamma^{ij} z_i w_j & = & 0\ . \end{array}\right.$$

$$\text{Schoen}: \quad \left[\begin{array}{c||cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 3 & 0 \\ \mathbb{P}^2 & 0 & 3 \end{array}\right]^{19,19}_{\chi=0} \qquad \simeq \qquad \left[\begin{array}{c||cccccc} \mathbb{P}^1 & 0 & 0 & 0 & 0 & 1 & 1 \\ \mathbb{P}^2 & 1 & 1 & 0 & 0 & 1 & 0 \\ \mathbb{P}^2 & 1 & 1 & 0 & 0 & 1 & 0 \\ \mathbb{P}^2 & 0 & 0 & 1 & 1 & 0 & 1 \\ \mathbb{P}^2 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]^{19,19}_{\chi=0} .$$

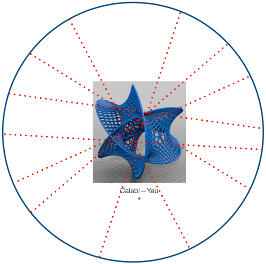
Methodologies for Calabi–Yau metrics

- Donaldson's algorithm
- Neural networks for PDEs
- Kähler–Ricci Flow
- Energy Functionals

I. Neural Networks for PDEs

- Generating points uniformly over a geometry.
- Benchmarks for a flat metric: flatness measures and loss function.
- Machine Learning architectures.
- Machine driven approximate Ricci-flat metrics of Calabi–Yau manifolds.

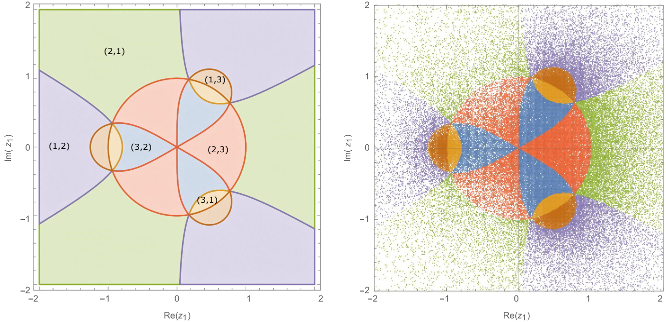
Generating points uniformly distributed on a Calabi–Yau



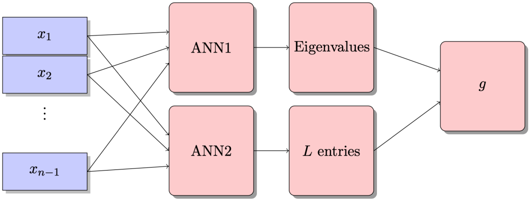
Case: CICY Hypersurfaces $\subset \mathbb{P}^n$, e.g., Fermat Quintic, Dwork family; also, quartic K3, complex T^2.

- Lines in $\mathbb{P}^n$ are uniformly distributed with respect to the $SU(n+1)$ symmetry of the Fubini–Study metric.
- Sampling the manifold with points at the intersection of each line with the hypersurface allows one to evaluate numerical integrations in a straightforward manner, taking the Fubini– Study metric as a measure of point distribution.

Generating points uniformly distributed on a Calabi–Yau



Neural Network Architecture



Flatness measures and Loss functions

$$\text{Loss} = \alpha_\sigma \sigma + \alpha_\kappa \kappa + \alpha_\mu \mu \,.$$

where,

$$\sigma = \frac{1}{\mathrm{Vol}_\Omega} \int_{\mathcal{M}} d\mathrm{Vol}_\Omega \left| 1 - \frac{\mathrm{Vol}_\Omega}{\mathrm{Vol}_J} \cdot \frac{J^n}{\Omega \wedge \bar{\Omega}} \right| \,.$$

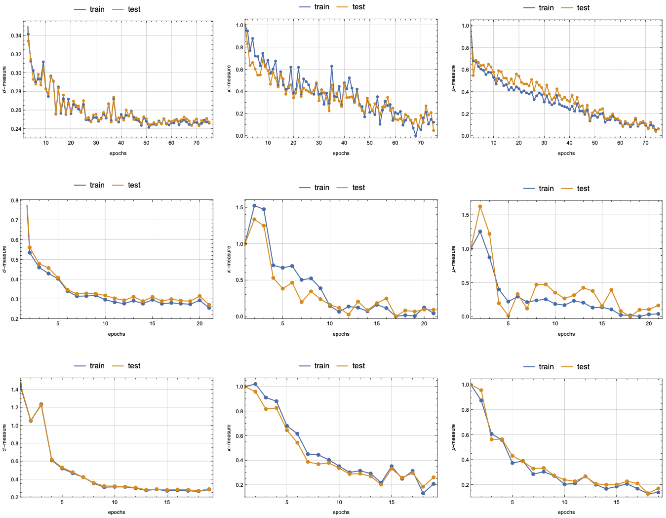
$$\kappa = \frac{\mathrm{Vol}_J^{1/n}}{\mathrm{Vol}_\Omega} \int_{\mathcal{M}} d\mathrm{Vol}_J \, |k|^2, \quad |k|^2 = \sum_{a,b,\bar{c}} |k_{ab\bar{c}}|^2, \qquad k_{ab\bar{c}} = \partial_a g_{b\bar{c}} - \partial_b g_{a\bar{c}} \,.$$

$$\mu = \frac{1}{N_p!} \sum_{m',l'} \sum_{m.l \neq m',l'} \frac{1}{\mathrm{Vol}_\Omega} \int_{\mathcal{M}} d\mathrm{Vol}_J \, |M(m',l';m,l)|^2,$$

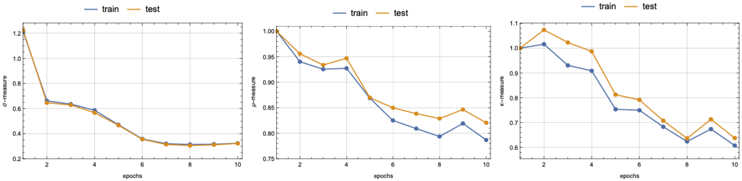
where M captures the disagreement of the metric over different patches.

These three measures capture Ricci flatness (σ), Kählericity (κ), and agreement over different patches (μ).

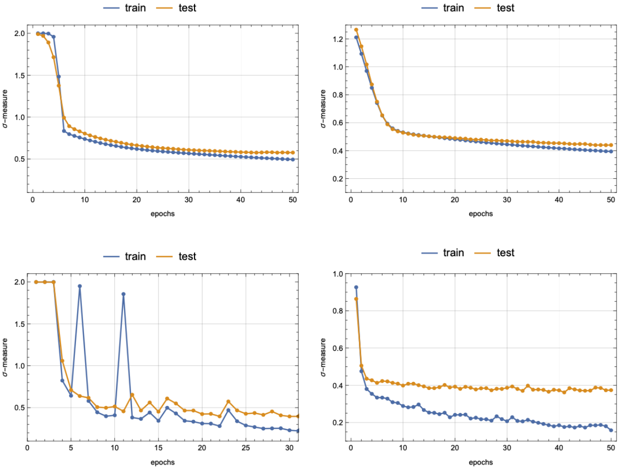
Results: Fermat Quintic



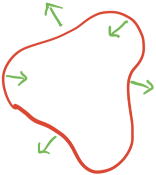
Results: Dwork Family of Quintics



Results: Tian–Yau



II. Ricci Flow: A geometric picture



II. Kähler–Ricci Flow

- Ricci flow gives a partial differential equation for a Riemannian metric g.
- It was introduced by Hamilton and famously employed by Perelman to prove the Poincaré conjecture[4] in three dimensions.
- The differential equation tells us that

$$\frac{\partial}{\partial\lambda} g_{a\bar{b}}(\lambda) = -\mathrm{Ric}_{a\bar{b}}(\lambda) = \frac{\partial^2}{\partial z^a \partial \bar{z}^{\bar{b}}} \log\det g$$

- Perelman functional:

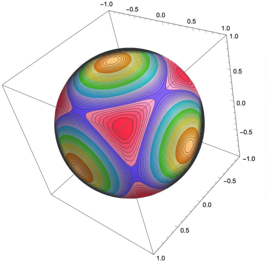
$$\mathcal{F}(g, f) \quad = \quad \int_{\mathcal{M}} d\mu \; e^{-f} (R + |\nabla f|^2)$$

where R is the scalar curvature.

[4] Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

The Fubini-Study metric:

The Fubini Study metric described below is a starting point for Kähler Ricci flow.

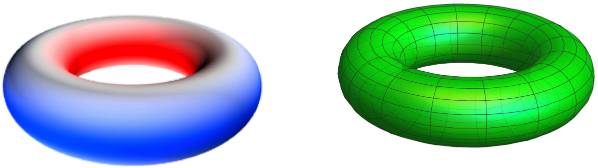


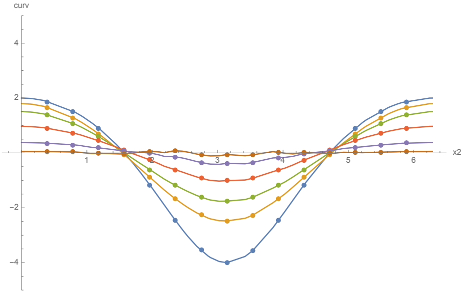
$$K(z,\bar{z}) = \frac{1}{\pi}\log(|z|^2) \; , \qquad |z|^2 = \sum_{a=1}^{n+1} z^a \bar{z}^{\bar{a}} \; .$$

This potential leads to the following metric:

$$g_{a\bar{b}}(z,\bar{z}) = \frac{|z|^2 \delta^{a\bar{b}} - \bar{z}^a \bar{z}^{\bar{b}}}{\pi |z|^4} \; .$$

III. Energy Functional Approach: Real Torus

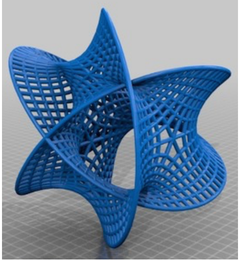




Future Directions

- Emulating Kähler Ricci Flow on Neural Networks
- Energy Functional Approaches
- How Calabi–Yau metrics transform under geometric transitions
- Discovering flat metrics on other CY constructions

Thank you!



Acknowledgments

Andrew Dancer, Oxford

Mario Garcia-Fernandez, ICMAT

Arghya Chattopadhyay, Wits

Justin Tan, Cambridge

Francisco Vargas, Cambridge

Carl Henrik Ek, Cambridge

Pietro Lio, Cambridge

Per Berglund, New Hampshire

Tristan Hübsch, Howard