CY and $SU(3)$ Structure Metrics and Stable Bundles

Lara B. Anderson

Virginia Tech

Work done in collaboration with:

J. Gray (VT), M.Gerdis, S. Krippendorf (LMU),

N. Raghuram (VT), and F. Ruehle (CERN) (arXiv:2012.04656)

D. Bloodworth, J. Gray and N. Tompkins (VT) (In progress)

String Data 2021

December 16th, 2021

Determining the 4D theory in string compactification

- Physical quantities in low energy string theory depend on the metric and gauge connections in the extra dimensions.
- For example:
  - Yukawa couplings in Heterotic string theory descend from a term in the 10-dimensional action of the form $\sim \int d^{10}\sqrt{-g}\bar{\psi}A\psi$. Normalization of fields and coefficients of the superpotential depend on g.
  - I.e. Defining the perturbative theory around a background:

$$\mathcal{A}_b = A_b^0 + \phi^I v_{Ib}^x T_x + \ldots$$

where v_I is the bundle valued harmonic 1-form on compact 6D space X which counts the multiplicity of the 4D fields ϕ^I.

  - E.g. Superpotential tri-linear coupling $\lambda_{IJK}\phi^I\phi^J\phi^K$ is given by the integral

$$\lambda_{IJK} \sim \int_X v_I \wedge v_J \wedge v_K \wedge \Omega$$

Particle masses/couplings

- Interested in
  - Particle masses/couplings.
  - Textures/hierarchies: Standard model (Heavy top quark), SUSY "mu problem" (want to forbid a mass term allowed by gauge symmetry, i.e. $\mu H_d H_u$), Forbid rapid proton decay operators, etc...
- Major obstacle: Matter field Kahler potential unknown except for special cases.

$$G_{IJ} = \frac{1}{2(Vol(X))} \int_X v_I \wedge \bar{\star}_V v_J$$

- Easiest class of solutions for X a complex, Kahler manifold: Ricci flat metric $\Rightarrow$ i.e. a Calabi-Yau manifold
- **How to determine the metric and the connection?**
- Only current viable approach via $\Rightarrow$ numeric approximation.

The Donaldson Algorithm

- Idea: Use projective embeddings to generate simple metrics that can be parametrically tuned to the Ricci-flat solution.
- Kodaira embedding: Given an ample line bundle $\mathcal{L}$ on X then an embedding

$$i_k : X \to \mathbb{P}^{n_k - 1}, \quad (x_0, \ldots, x_2) \mapsto [s_0(x) : \ldots : s_{n_k - 1}(x)]$$

exists for all $\mathcal{L}^k = \mathcal{L}^{\otimes k}$ with $k \geq k_0$ for some k_0, where $s_\alpha \in H^0(X, \mathcal{L}^k)$.

- What do we know about metrics on $\mathbb{P}^n$? Fubini-Study:

$$(g_{FS})_{i\bar{j}} = \frac{i}{2}\partial_i \bar{\partial}_{\bar{j}} K_{FS} \quad \text{where} \quad K_{FS} = \frac{1}{\pi} \ln \sum_{i\bar{j}} h^{i\bar{j}} z_i \bar{z}_{\bar{j}}$$

and $h^{i\bar{j}}$ is a non-singular, hermitian matrix.

- FS metric restricted to X is not Ricci-flat. But...

- Generalize: $K_{h,k} = \frac{1}{k\pi} \ln \sum_{\alpha,\bar{\beta}=0}^{n_k-1} h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}} = \ln ||s||^2_{h,k}$
- $h^{\alpha\bar{\beta}}$ is a hermitian fiber metric on $\mathcal{L}^{\otimes k}$.
- Such Kähler potentials are dense in the moduli space (Tian)
- Fixed point of Donaldson's "T-operator" $\leftrightarrow$ "balanced metric".

$T(h)_{\alpha\bar{\beta}} = \frac{n_k}{Vol_{CY}(X)} \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{\sum_{\gamma\bar{\delta}} h^{\gamma\bar{\delta}} s_\gamma \bar{s}_{\bar{\delta}}} \mathrm{dVol}_{CY}$

Theorem (Donaldson)

For each $k \geq 1$, the balanced metric, h, on $\mathcal{L}^{\otimes k}$ exists and is unique. As $k \to \infty$, the sequence of metrics

$$g_{i\bar{j}}^{(k)} = \frac{1}{k\pi} \partial_i \bar{\partial}_{\bar{j}} \ln \sum_{\alpha,\bar{\beta}=0}^{n_k-1} h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}}$$

on X converges to the unique Ricci-flat metric for the given Kähler class and complex structure.

A new approach

- Existing numeric implementations of Donaldson's algorithm (Douglas et al, Ovrut et al). Computationally intensive. (Accurate enough? Don't know...)
- Moduli dependence difficult to obtain.
- New Approach ⇒ Machine Learning. What we did:

1. Supervised learning of moduli dependence of Calabi-Yau metrics using the Donaldson algorithm to generate training data.
2. Direct learning of moduli dependent Calabi-Yau metrics both using the metric ansatz and without it.
3. Direct learning of metrics associated to SU(3) structures with torsion.

- I'll give a brief flavor of these results... (see also, (Douglas et al), (Jejjala, et al))(See more recent developments in talks of Fabian/Robin...).

Preliminaries

- One definition of a Calabi-Yau three-fold: A complex 3-fold admitting a nowhere vanishing real two-form, J, and a complex three-form, Ω, such that:

$$J \wedge \Omega = 0 \qquad J \wedge J \wedge J = \frac{3i}{4}\Omega \wedge \overline{\Omega}$$

$$dJ = 0 \qquad d\Omega = 0$$

- Metric is related to the two form as $ig_{a\bar{b}} = J_{a\bar{b}}$
- Example CY manifold: "Quintic" hypersurface: $X = \mathbb{P}^4[5]$
- e.g. $p(\vec{z}) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \psi z_0 z_1 z_2 z_3 z_4 = 0$
- The holomorphic $(3,0)$ form can explicitly constructed for such manifolds (Candelas, et al).

$$\Omega = \frac{1}{\partial p_\psi(\vec{z})/\partial z_b} \bigwedge_{\substack{c=0,\ldots 3,\\ c\neq a,b}} dz_c$$

Direct Learning of the Kähler Potential

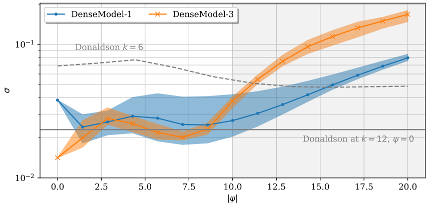
- The balanced metric output by Donaldson's algorithm at given finite k is not necessarily the most accurate approximation to the Ricci-flat metric - maybe we can do better?
- Can generate networks to find the parameters that are trained directly using a loss such as:

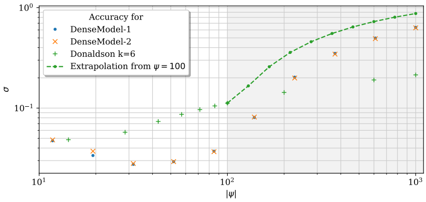
$$\mathcal{L}_{MA} = \left| 1 + \frac{4i}{3} \frac{J^3}{\Omega \wedge \overline{\Omega}} \right|$$

- C.f.: Headrick and Nassar (although note that we are obtaining moduli dependent results and using ML). Network Architecture:

Layer	Number of Nodes	Activation	Number of Parameters
input	17	–	–
hidden 1	100	leaky ReLU	1800
hidden 2	100	leaky ReLU	10 100
hidden 3	100	leaky ReLU	10 100
output	d^2	identity	101 d^2

- These networks were optimized for $0 \leq |\psi| \leq 10$





(shaded region denotes extrapolation of the networks).

- Note Donaldson algorithm with $k = 12$ takes order days to run even for the single case of $\psi = 0$. This network at $k = 6$ takes only minutes and gives comparable accuracy for a whole range of ψ.
- We do better than Donaldson Alg. at $k = 6$ and that this improvement extends up to $|\psi| \simeq 175$, nearly a factor of 2 beyond the regime used during training.

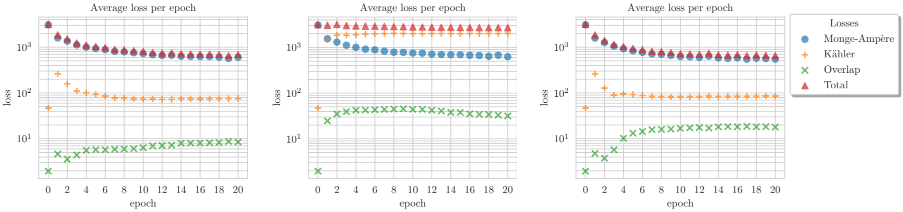
Direct learning of the metric

- Instead of learning parameters in an ansatz for the Kahler potential we can try to learn the CY metric directly.
- Why try?
    - Perhaps we can improve performance by not being tied to an ansatz at fixed k.
    - We will be able to generalize this approach to more complicated geometries.
- One disadvantage:
    - We now need loss functions to check that the metric is globally well defined and Kahler! We use $\mathcal{L} = \lambda_1 \mathcal{L}_{\mathrm{MA}} + \lambda_2 \mathcal{L}_{\mathrm{dJ}} + \lambda_3 \mathcal{L}_{\mathrm{overlap}}$
    - Here $\mathcal{L}_{\mathrm{MA}}$ is the loss described before and we add to this

$$\mathcal{L}_{dJ} = \frac{1}{2} ||dJ||_1$$

$$\mathcal{L}_{\mathrm{overlap}} = \frac{1}{d} \sum_{k,j} ||g^{(k)}_{\mathrm{NN}}(\vec{z}) - T_{jk}(\vec{z}) \cdot g^{(j)}_{\mathrm{NN}}(\vec{z}) \cdot T^{\dagger}_{jk}(\vec{\bar{z}})||_n$$

- Input: $Re(z_i)$, $Im(z_i)$ (homogeneous coords describing pt in CY) $Re(\psi)$, $Im(\psi)$. Output: d^2 real and imaginary parts of a metric at point.
- To give a concrete example: optimized at $\psi = 10$ on a data set of $10,000$ points. We split the points according to train:test=$90:10$ and we train for 20 epochs.
- Accuracy reaches same level as Donaldson Alg. at $k = 5$ (we expect more points and better architecture will easily improve this).



Left: Optimizing the NN with all three losses. *Middle:* Optimizing the NN without Kähler loss (i.e. $\lambda_2 = 0$). *Right:* Optimizing the NN without overlap loss (i.e. $\lambda_3 = 0$).

Learning $SU(3)$ structures from an anzatz

- Important reason to directly learn the metric: Can be generalized to non-Kähler geometry!
- One important class of geometries for $\mathcal{N} = 1$ compactifications: $SU(3)$ structure manifolds
- These are six-manifolds with a nowhere vanishing two form J and three form Ω obeying the same algebraic properties as the Calabi-Yau threefold case:

$$J \wedge \Omega = 0 \qquad\qquad J \wedge J \wedge J = \frac{3i}{4}\Omega \wedge \overline{\Omega}$$

But with different differential properties...

- An $SU(3)$ structure can be classified by its torsion classes:

$$\begin{aligned} dJ &= -\frac{3}{2}\mathrm{Im}(W_1\overline{\Omega}) + W_4 \wedge J + W_3 \\ d\Omega &= W_1 J \wedge J + W_2 \wedge J + W_5 \wedge \Omega \,, \end{aligned}$$

- Where torsion classes are given the defining forms:

$$W_1 = -\frac{1}{6} i\Omega \lrcorner dJ = \frac{1}{12} J^2 \lrcorner d\Omega \ , \quad W_4 = \frac{1}{2} J \lrcorner dJ \ , \quad W_5 = -\frac{1}{2} \Omega_+ \lrcorner d\Omega_+$$

- Given string theories place different constraints on the torsion classes for there to be an associated solution to the theory of the type we want.
- E.g. heterotic string theory: $W_1 = W_2 = 0$, $W_4 = \frac{1}{2} W_5 = d\phi$, W_3 free.
- Note that a CY structure is a special case: $W_i = 0 \ \forall i = 1, \ldots 5$.
- Need to start with some well-controlled/simple example.
- Observation: Some CY manifolds admit not only Ricci-flat metrics, but other $SU(3)$ structures as well.
- E.g. (generalization of work by Larfors, Lukas and Ruehle)

$$J = \sum_{i=1}^{h^{1,1}(x)} a_i J_i \qquad\qquad \Omega = A_1 \Omega_0 + A_2 \bar{\Omega}_0$$

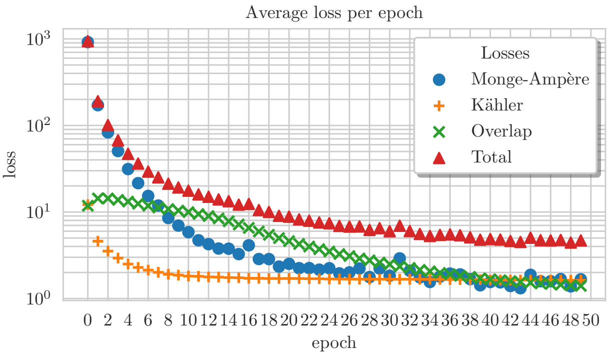
- The a_i are real functions and A_1 and A_2 are complex functions. Quintic e.g.

$$a_1 = \frac{1}{\pi^3} \frac{|\nabla p|^2}{\sigma^4} \ , \quad A_1 = a_1^2 \ , \quad A_2 = 0 \ , \text{(where} \quad \sigma = \sum_{a=0}^{4} |z_a|^2 \ )$$

- Can use the same losses as in the Calabi-Yau case, then, with the exception of replacing the Kahler loss by the following.

$$\mathcal{L}_{W_4} = \left\| dJ - d\ln(a_1 \wedge J) \right\|_n$$

- We ran this for the $\psi = 10$ quintic, using multiplicative boosting from g_{FS}.



On to connections...

- In the heterotic context, it is natural to ask how much of this approach can be readily applied to gauge fields $\Rightarrow$ i.e. Hermitian Yang-Mills connections?
- For stable bundles can encode connection via a Hermitian structure, i.e. fiber metric $G = G^{\dagger} \Rightarrow A = G^{-1}\partial G$
- A solution to HYM $\leftrightarrow$ a Hermitian bundle metric, G, such that

$$\mu(V) \cdot \mathbf{1} = g^{i\bar{j}} F_{i\bar{j}} = g^{i\bar{j}} \bar{\partial}_{\bar{j}} A_i = g^{\bar{j}i} \bar{\partial}_{\bar{j}} (G^{-1} \partial_i G)$$

  G is called Hermitian-Einstein.
- G induces an inner product on sections, $s_\alpha \in H^0(X, V)$ of the bundle:

$$< s_\alpha | s_\beta >= \int_X \left(s^b_\beta G_{b\bar{a}} \bar{s}^{\bar{a}}_{\bar{\alpha}} \right) dVol$$

Generalized Donaldson Algorithm: Computing Connections

- **Old idea: Generalize line bundle algorithm to rank n bundle**
- Metrics: Line bundle: $\mathcal{L}^k$ defines an embedding of X into $\mathbb{P}^{n_k-1}$ via global sections $s_\alpha \in H^0(X, \mathcal{L}^k)$:

$$x \to [s_0(x) : \cdots : s_{n_k-1}(x)] \Rightarrow i_k : X \to \mathbb{P}^{n_k-1} = G(1, n_k - 1)$$

  where $n_k = h^0(X, \mathcal{L}^k)$.
- Connections: Define an embedding via the global sections of a twist of some holomorphic vector bundle, $\mathcal{V}$, with non-Abelian structure group. A map:

$$x \to \left[\begin{pmatrix} S_1^1(x) \\ \vdots \\ S_1^n(x) \end{pmatrix} : \cdots : \begin{pmatrix} S_{N_k}^1(x) \\ \vdots \\ S_{N_k}^n(x) \end{pmatrix} \right] \Rightarrow i_k : X \to G(n, N_k - 1)$$

  S_α^a are global sections of $\mathcal{V} \otimes \mathcal{L}^k$, $\alpha = 0 \ldots N_k - 1 = h^0(X, \mathcal{V} \otimes \mathcal{L}^k)$.

Generalized Donaldson Algorithm

- Note: We need to twist $\mathcal{V}$ so that it has global sections! ($H^0(V) = 0$ for stable $SU(n)$ bundles).
- As before, choose a starting ansatz for fiber metric on $\mathcal{V} \otimes \mathcal{L}^k$: $(G^{-1})^{a\bar{b}} = \sum_{\alpha,\beta=0}^{N_k-1} H^{\alpha\bar{\beta}} S^a_\alpha (\bar{S})^{\bar{b}}_{\bar{\beta}}$
- Inner product on the space of sections, S^a_α, given by

$$< S_\beta | S_\alpha >= \frac{N_k}{\mathrm{Vol}_{CY}} \int_X S^a_\alpha (G^{a\bar{b}})^{-1} \bar{S}^{\bar{b}}_{\bar{\beta}} dVol_{CY} = \frac{N_k}{\mathrm{Vol}_{CY}} \int_X S^a_\alpha (S^a_\gamma H^{\gamma\bar{\delta}} \bar{S}^{\bar{b}}_{\bar{\delta}})^{-1} \bar{S}^{\bar{b}}_{\bar{\beta}} dVol_{CY} \ .$$

- Generalized T-operator

$$T(H)_{\alpha\bar{\beta}} = \frac{N_k}{\mathrm{Vol}_{CY}} \int_X S_\alpha \left(S^\dagger H S \right)^{-1} \bar{S}_{\bar{\beta}} dVol_{CY} \ ,$$

- Note when $\mathcal{V}$ is a line bundle this reduces to the metric T-operator!
- If $\mathcal{V}$ stable $\rightarrow$ generalized T-operator has a fixed point for each k.

Theorem (Wang)

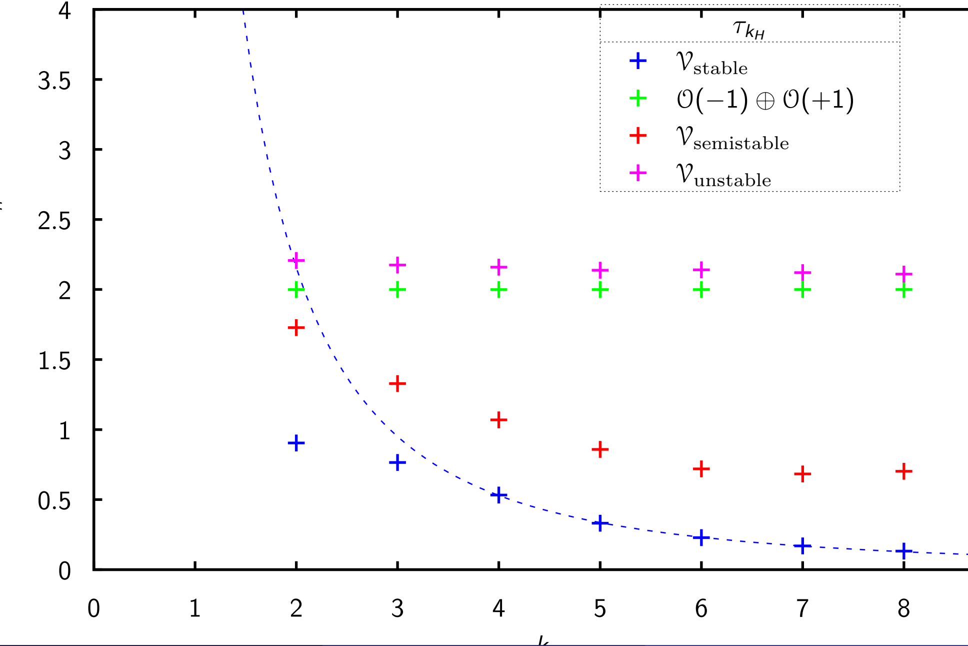
Let $\mathcal{V}$ be a rank n, Gieseker stable bundle. If $G_k = G^{(k)} \otimes G_{\mathcal{L}}^{-k} \to G_\infty$ as $k \to \infty$, then the metric G_∞ solves the Hermitian-Einstein equation

$$g^{i\bar{j}} F_{i\bar{j}} = (\mu)\, \mathbf{1}_{n\times n}$$

where $G^{(k)}$ is the fiber metric on $\mathcal{V} \otimes \mathcal{L}^k$ and $G_{\mathcal{L}}$ is the metric on $\mathcal{L}$.

- Error measure: $g^{i\bar{j}} F_{i\bar{j}} \sim \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad (= 0 \quad ?)$
- Integrated error measure: $\tau(A_{\mathcal{V}}) = \frac{1}{2\pi} \frac{k_g}{Vol_{k_g}\, rank(\mathcal{V})} \int_X \left(\sum |\lambda_i| \right) \sqrt{g}\, d^{2d}x$

  Implementation of generalized algorithm: Anderson, Braun, Ovrut



τ_{k_H}
$\mathcal{V}_{\text{stable}}$
$\mathcal{O}(-1) \oplus \mathcal{O}(+1)$
$\mathcal{V}_{\text{semistable}}$
$\mathcal{V}_{\text{unstable}}$
0 0.5 1 1.5 2 2.5 3 3.5 4
0 1 2 3 4 5 6 7 8

A better way to apply ML techniques to a pair $(g_{\mu\nu}, A_\mu)$?

- Computing CY metric, then HYM connection is laborious. Question: Can we group the two computations into one?
- Several approaches to this... Basic idea: Composite metric on total space of the bundle
- E.g.

$$\bar{g}_{AB} \sim \begin{pmatrix} g_{\mu\nu}(x) - \tilde{g}_{mn}(y) B_\mu{}^m B_\nu{}^n & B_\mu{}^n \\ B_\nu{}^m & -\tilde{g}_{mn}(y) \end{pmatrix}$$

- With $B_\mu{}^n = \xi^n{}_a(y) A^a{}_\mu(x)$ and ξ a Killing vector of the compact fiber space (e.g. $SU(N) \to \mathbb{CP}^{N-1}$ fiber with infinitesimal isometries $y^n \to y^n + \xi^n_a(y)\epsilon^a(x) \Rightarrow$ gauge transformations)
- Idea: Kaluza-Klein type metric (but over compact directions only in heterotic).

- Several choices for how to build a total space and $\bar{g}_{AB}$
- E.g. principle bundle (e.g. for $SU(N)$, compact fiber is group manifold of dimension $N^2 - 1$)
- E.g. projectivized vector bundle $\mathcal{M} = \mathbb{P}(\pi : V \to X)$. Comes equipped with a canonical line bundle, $\mathcal{L}_{\mathcal{M}}$ such that $\pi_* \mathcal{L}_{\mathcal{M}} = V$.
- For an $SU(N)$ vector bundle, fibers are $\mathbb{CP}^{N-1}$ as above.
- Nice results known for properties of $\bar{g}_{AB}$ in this case.
- Thm (Hong): If X admits a constant scalar curvature (CHSC) metric and V is stable, then $\mathcal{M}$ also admits a CHSC metric in the class $[F_L + k\pi^* \omega_X]$.
- Uniqueness and iff? True in some circumstances...
- **Central idea:** Apply Donaldson anzatz to Hermitian fiber metric of (twists) of $\mathcal{L}_{\mathcal{M}} \Rightarrow$ similar ML implementation to CY case.

Actually Strominger system solutions?

- Interesting extension: Presence of generic HYM bundle (with $V \neq TX$) deforms metric on X away from Ricci-flat one $\Rightarrow$ Strominger system solution (i.e $SU(3)$ structure).
- Can apply ML techniques directly to imposing the vanishing of the β-functions in this case, rather than solve for X CY first, then HYM.
- Use similar approaches for non-Kähler metrics (i.e. Loss functions include transition functions, etc).
- Greater accuracy for the matter field Kähler potential?

Results and Future Work

- Control of the metric is necessary to specify the 4D theory in compactification $\Rightarrow$ ML techniques can provide a powerful new tool.
- In particular, we have provided the first numeric approx. to $SU(3)$ structure metrics.
- We are exploring novel ways to package – and then approximate – the metric and connection data in a heterotic compactification.
- Open questions: What next? $\Rightarrow$ Pushing the control of the heterotic effective theory further (i.e. matter field Kähler potentials, moduli stabilization scenarios, etc)