Calabi-Yau Metrics, CFTs and Random Matrices

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arXiv: 2107.11061

Overview

Calabi–Yau metrics are important for both string phenomenology and CFT

The Laplacian encodes both geometry and the spectrum of operators in certain 2d CFTs

Numerical methods give us access to this data

The spectrum of these CFTs, averaged over moduli, is chaotic

2d conformal field theories

Most interacting CFTs understood near special points in moduli space, e.g. K3 as $T^4/\mathbb{Z}_2$

Most information is about quantities protected by supersymmetry, e.g. counts of BPS objects [Witten ‘82; ...,Keller, Ooguri ‘12; ...]

CYs appear as target spaces for CFTs:

- In large-volume limit, low-lying modes c.f. quantum mechanics with $H = \Delta$ [Witten ‘82]
- Spectrum of operators encoded in geometry

Statistics of 2d CFTs

Spectrum of a 2d CFT defined by

$$H|\mathcal{O}_i\rangle = D_i|\mathcal{O}_i\rangle, \qquad D_i \geq 0$$

Question

Given an *ensemble* of CFTs, what are the statistics of the scaling dimensions $\{D_i\}$?

Need spectrum of *generic interacting CFTs* (not solvable/rational/etc.) that come in families

- Not possible until now! (see [Afkhami-Jeddi et al. '06; Maloney, Witten '20; Benjamin et al. '21] for free theories)

Numerical CY metrics and spectra

Calabi–Yau basics

Calabi–Yau manifolds are Kähler and admit Ricci-flat metrics

- Existence but no explicit constructions
- Kähler + $c_1(X) = 0 \Rightarrow$ there exists a Ricci-flat metric [Yau '77]

Kähler $\Rightarrow$ Kähler potential K gives (real) closed two-form $J = \partial\bar{\partial}K$

$c_1(X) = 0 \Rightarrow$ (complex) nowhere-vanishing (3,0)-form Ω

Example: Fermat quintic

Quintic hypersurface Q in $\mathbb{P}^4$

$$Q(z) \equiv z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$$

(3,0)-form Ω determined by Q, e.g. in $z_0 = 1$ patch

$$\Omega = \frac{\mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4}{\partial Q / \partial z_1}$$

Metric g and Kähler form J determined by Kähler potential

$$g_{i\bar{j}}(z, \bar{z}) = \partial_i \bar{\partial}_{\bar{j}} K(z, \bar{z})$$

How to fix K?

Finding an approximate Ricci-flat metric amounts to finding K so that "distance" from Ricci-flat is minimised

Many approaches:

- Position space methods [Headrick, Wiseman '05]; iterative procedure [Donaldson '05; Douglas '06; Braun '07]; direct minimisation [Headrick, Nassar '09];
- K (or $g_{i\bar{j}}$) encoded by neural network [Douglas et al. 20; Anderson et al. '20; Jejjala '20; Larfors et al. '21]

In all cases, numerical integrals carried out by Monte Carlo

The Laplacian [Braun et al. '08, AA '20]

Eigenmodes are (p, q)-eigenforms of the Laplacian

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}, \qquad \Delta|\phi_n\rangle = \lambda_n|\phi_n\rangle$$

where λ_n are real and non-negative and can appear with multiplicity (c.f. continuous or finite symmetries)

Want to compute the spectrum and the eigenmodes

The Laplacian [Braun et al. '08, AA '20]

Given a (non-orthonormal) basis of functions $\{\alpha_A\}$, we can expand the eigenmodes as

$$|\phi\rangle = \sum_A \langle \alpha_A|\phi\rangle\,|\alpha_A\rangle = \sum_A \phi_A\,|\alpha_A\rangle, \qquad A = 1, \ldots, \dim\{\alpha_A\}$$

so that $\Delta|\phi\rangle = \lambda|\phi\rangle$ becomes eigenvalue problem for λ and ϕ_A

$$\begin{aligned} \langle \alpha_A|\Delta|\alpha_B\rangle\langle \alpha_B|\phi\rangle &= \lambda\langle \alpha_A|\alpha_B\rangle\langle \alpha_B|\phi\rangle \\ \Rightarrow \qquad \Delta_{AB}\phi_B &= \lambda O_{AB}\phi_B \end{aligned}$$

where

$$O_{AB} \equiv \langle \alpha_A|\alpha_B\rangle = \int \alpha_A \wedge \star\bar{\alpha}_B, \quad \text{etc.}$$

The Laplacian [Braun et al. '08, AA '20]

Basis $\{\alpha_A\}$ is infinite dimensional – truncate to a finite approximate basis at degree k in z_i

$$\{\alpha_A\} = \frac{(\text{degree } k \text{ in } z)(\text{degree } k \text{ in } \bar{z})}{\left(|z_0|^2 + \ldots |z_4|^2\right)^k}$$

(c.f. harmonic functions on $\mathbb{P}^4$)

Strategy

1. Specify the CY by $Q = 0$ and compute metric numerically
2. Compute matrices Δ_{AB} and O_{AB} numerically at degree k for fixed (p, q)
3. Find eigenvalues and eigenvectors

CY CFTs and RMT

σ-models and CFTs

Consider CFT defined by σ-model with Calabi–Yau target X (irrational, not solvable)

$$c = 3 \dim_{\mathbb{C}} X$$

Well-understood using mirror symmetry, supersymmetry, etc. – but now want non-BPS data!

These CFTs come in families labelled by

$$(\text{Kähler moduli}, \text{complex structure moduli})$$

Varying moduli gives an ensemble of CFTs

Large-volume limit

In large-volume limit, spectrum of operators

$$\mathcal{O} = \mathcal{O}_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} \lambda^{i_1} \ldots \lambda^{i_p} \bar{\psi}^{\bar{j}_1} \ldots \bar{\psi}^{\bar{j}_q}$$

corresponds to (p, q)-eigenforms of Δ for Calabi–Yau metric on X

Quantum numbers are

$$D = \lambda + \frac{p+q}{2}, \qquad J = \frac{p-q}{2}.$$

$\lambda \sim \mathrm{vol}^{-1/\dim_{\mathbb{C}} X}$ so at large volume, light operators come from scalar eigenmodes of Δ

Ensembles of CYs

Generic quintic threefold given by quintic equation in $\mathbb{P}^4$

$$Q \equiv \sum_{m,n,p,q,r} c_{mnpqr} z_m z_n z_p z_q z_r = 0$$

101 complex structure parameters

Choose the c_{mnpqr} randomly from unit disk in complex plane

$$c_{mnpqr} \in \mathbb{C}, \qquad |c_{mnpqr}| < 1$$

Plan

1. Numerically compute the CY metric for some choice of moduli
2. Numerically compute the spectrum of Δ (lowest ~ 100 eigenvalues)
3. Repeat for different choice of complex structure moduli $\rightarrow$ ensemble of CFT data
4. Compare statistics of ensemble to random matrices

Random matrix theory

Random matrix statistics in spectrum of physical system is a hallmark of quantum chaos [Bohigas, Giannoni, Schmit '84; ...]

Energy spectrum exhibits level repulsion and long-range rigidity

RMT has appeared in nuclear physics, billiards, SYK model and black hole physics / quantum gravity [Maldacena '01; Cotler et al. '16; Saad et al. '18; ...]

Holography suggests that generic CFTs might display chaos

Random matrix theory

Gaussian orthogonal ensemble (GOE) $=$ $N \times N$ real symmetric matrices

Density of eigenvalues (large N) given by Wigner's semicircle

$$\rho(\lambda) = \frac{1}{\pi}\sqrt{2N - \lambda^2}$$

This is not a universal feature of chaotic system – want to compare CFT statistics with universal features of RMT

- Statistics after normalising $\rho(\lambda) = 1$
- Unfolded spectrum focuses on fluctuations

Level repulsion and spectral rigidity

Nearest-neighbour level spacing – probability of distance s between consecutive eigenvalues

$$p_1(s) = \frac{\pi}{2} s\, e^{-\frac{\pi}{4}s^2}$$

Number variance – fluctuation of the number of eigenvalues in a typical interval L

$$\Sigma^2(L) \sim \log L$$

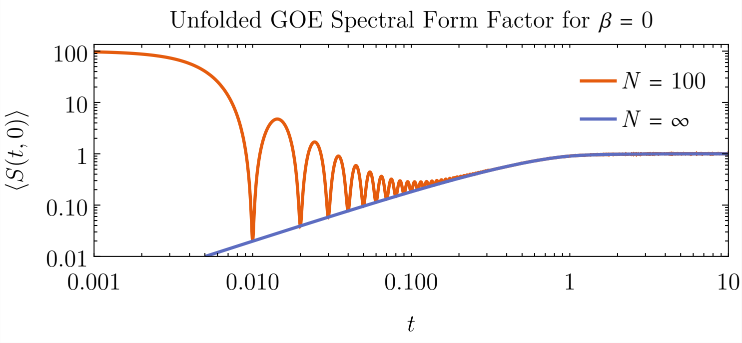
RMT displays both level repulsion and spectral rigidity

- e.g. Poisson has $p_1(s) \sim e^{-s}$ and $\Sigma^2(L) \sim L$

Spectral form factor

Spectral form factor defined by thermal partition function

$$S(t,\beta) \sim \left| \sum_i e^{-(\beta+2\pi i t)\lambda_i} \right|^2$$



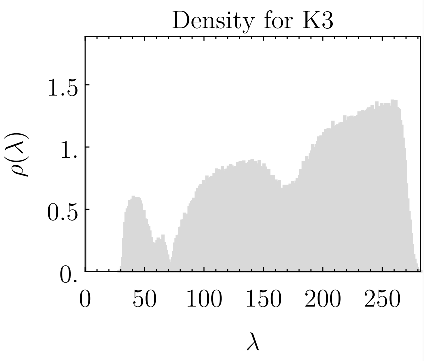
"Dip" → "ramp" → "plateau"

Results

Can then compare RMT statistics to spectra of Calabi–Yau CFTs

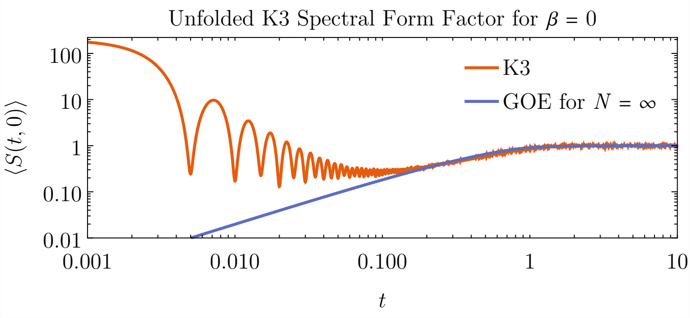
- 1000 K3's as quartic equations in $\mathbb{P}^3$
- 1000 Quintic threefolds as quintic equations in $\mathbb{P}^4$

e.g. eigenvalue density for K3

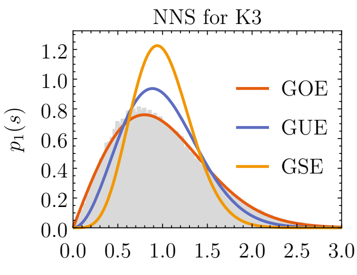


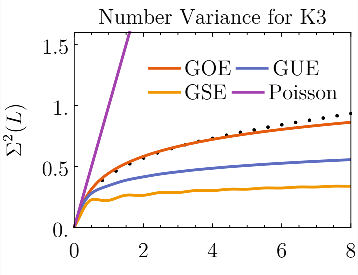
Not a semicircle! Fine, since that is not a *universal* feature

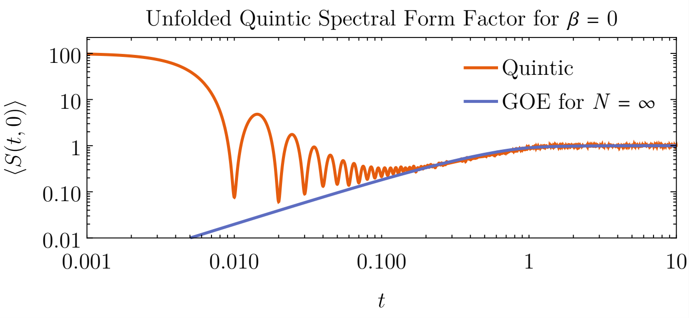
K3 statistics



SFF shows dip, ramp and plateau expected from GOE







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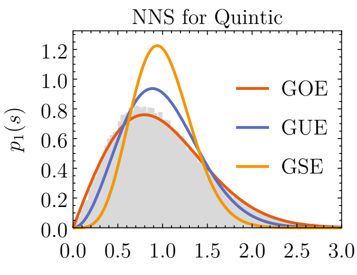

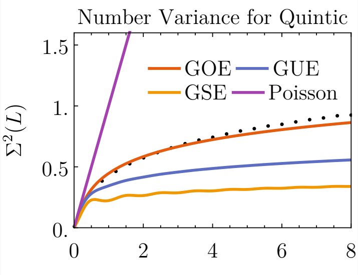

Summary and outlook

Calabi–Yau metrics are accessible with numerical methods

Source of new non-BPS "data" with uses in geometry and CFT

Spectrum of light operators in large-volume CFTs is chaotic and described by GOE statistics

- Other spectral statistics – spectral gap? eigenvalue density?
- Mirror symmetry in non-BPS spectrum? Modularity of 2d CFTs? Complements CFT and geometric bootstrap
- Distribution of Yukawa couplings? "Typical" compactifications? [Denef, Douglas '04; ...; Balasubramanian et al. '21]