Symmetries of spin amplitudes:

applications for factorization and Monte Carlo solutions

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- **(1)** For phenomenology predictions one need to keep in mind what need to be taken into account for required precision and then, how distinct things need to be combined.
- **(2)** Detector acceptance, higher order matrix elements, structure functions, evolution kernels, Monte Carlo and or semi-analytic integration, algebraic manipulation programs. Lots of elements in the game.
- **(3)** How to prepare and combine all these activities?
- **(*) How in projects I participated, symmetry was used. It never was central of acivity, but may be it worth to be looked at?**

Examples

- **(1)** QED bremsstrahlung amplitudes for s-channel processes (useful for exponentiation).
- **(2)** Extension with t-channel contribution, contact interaction approx.
- **(3)** Spin amplitudes for bremsstrahlung in τ decays, QED versus scalar QED.
- **(4)** Spin amplitudes for double bremsstrahlung in QCD
- **(5)** Approximate methods
- **In practice I used independence from $\not\epsilon \to \not\epsilon + \lambda \not k$, amassed terms to the most singular ones to make gauge independent set. Then I looked at next to most singular term, etc.**
Difficulties and benefits of automated calculations.
- Some details, definitions/conventions can be found at
http://wasm.web.cern.ch/wasm/public/zakopane-2004b.pdf
My talk is sketchy. I was not able to do better ...

I will not talk about complete factorizations like:

Formalism for $\tau^+\tau^-$: phase space $\times$ M.E. squared

- Because narrow τ width (τ propagator works as Dirac δ), cross-section for $f\bar{f} \to \tau^+\tau^- Y; \tau^+ \to X^+\bar{\nu}; \tau^- \to \nu\nu$ reads (norm. const. dropped):

$$d\sigma = \sum_{spin} |\mathcal{M}|^2 d\Omega = \sum_{spin} |\mathcal{M}|^2 d\Omega_{prod}\; d\Omega_{\tau^+}\; d\Omega_{\tau^-}$$

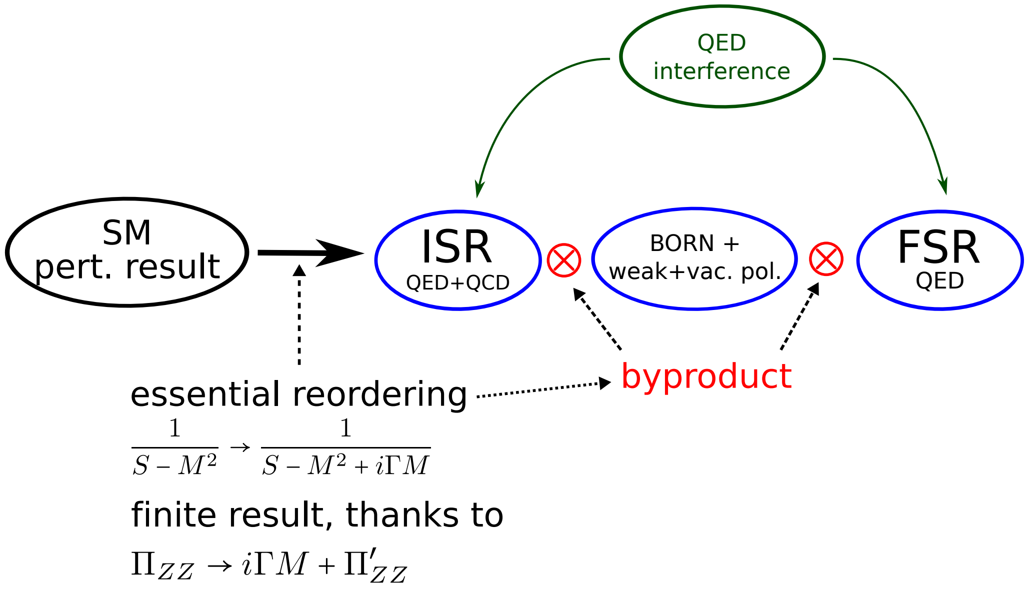
$$\mathcal{M} = \sum_{\lambda_1\lambda_2=1}^{2} \mathcal{M}^{prod}_{\lambda_1\lambda_2}\; \mathcal{M}^{\tau^+}_{\lambda_1} \mathcal{M}^{\tau^-}_{\lambda_2}$$

- Pauli matrices orthogonality $\delta^{\lambda'}_{\lambda}\delta^{\bar{\lambda}'}_{\bar{\lambda}} = \sum_\mu \sigma^\mu_{\lambda\bar{\lambda}}\sigma^{\lambda'\bar{\lambda}'}_\mu$ completes condition for production/decay separation with τ spin states.

- **core formula of spin algorithms,** wt **is product of density matrices of production and decays**, $0 < wt < 4$, $< wt >= 1$ useful properties.

$$d\sigma = \Big(\sum_{spin} |\mathcal{M}^{prod}|^2\Big)\Big(\sum_{spin} |\mathcal{M}^{\tau^+}|^2\Big)\Big(\sum_{spin} |\mathcal{M}^{\tau^-}|^2\Big) wt\; d\Omega_{prod}\; d\Omega_{\tau^+}\; d\Omega_{\tau^-}$$

Purpose of my talk:



I will talk neither about such big pictures, nor about PDFs.

- **Detail: think of** spin amplitudes as vectors (sums of vectors) from reducible representations of gauge × Lorentz symmetry group, with its subgroups and layers.
- $\mathcal{M} = \mathcal{M}_A \times \mathcal{M}_B + \mathcal{M}_C$, useful when $\mathcal{M}_A \times \mathcal{M}_B$ big and $\mathcal{M}_C$ small.
- Non trivial: what *big, small* mean and when. This is application dependent.
- I am not the first. Similar approaches are/were used already. Somebody pointed me to the Leningrad winter schools of theoretical physics, but without quotable reference.
- Let's look into examples. I stay in language of γ matrices because I played at that level, close to input Kleiss-Stirling spinor techniques.

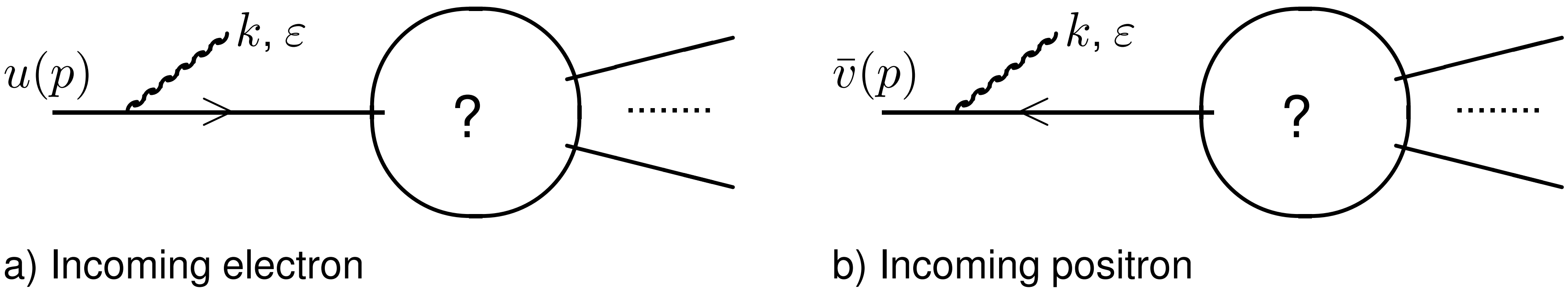


Figure 1: *Feynman diagrams for photon emission in initial state respectively from electron and positron. Dots represent all other fields entering amplitude (initial or final). Note that in case of positron arrow points in opposite direction, even though it is also initial state particle.*

$$\mathcal{M} = \; ... \frac{\not{p} - \not{k} + m}{-2pk} e \not{\epsilon} u(p,s). \tag{1}$$

The part $\sim \not{k}$ is gauge invariant individually for contribution from left and right plot. It leads e.g. to real photon emission part of Yennie-Frautchi-Suura exponentiation β_1.

$$
\begin{aligned}
\mathcal{M} &= \frac{-e}{2pk} \; ... \; (\not{p} + m) \; \not{\varepsilon} u(p,s) \\
&= \frac{-e}{2pk} ... \Big(2p\varepsilon + \not{\varepsilon}(- \not{p} + m) \Big) \\
& \quad u(p,s) \\
&= \frac{-e}{2pk} \; ... \; (2p\varepsilon) u(p,s) \\
&= e \frac{-\varepsilon p}{pk} \; ... \; u(p,s) \\
&= -e \frac{\varepsilon p}{pk} \mathcal{M}_B, \qquad (2)
\end{aligned}
$$

- If big blob is (essentially) point-like we get eikonal factor which is gauge invariant and process independent.
- Note factorization of Born-like (lower pert. level) amplitude. Care for energy-momentum conservation constraints important.
- Care for $\mathcal{M}_B$ def. extrapolation needed, especially if it depend strongly on kinematic.
- Then formula is valid all over phase-space.
- It iterate nicely in exponentiation and after partial integration contributes to double logarithms. It is the part singular in both collinear and infrared limits.

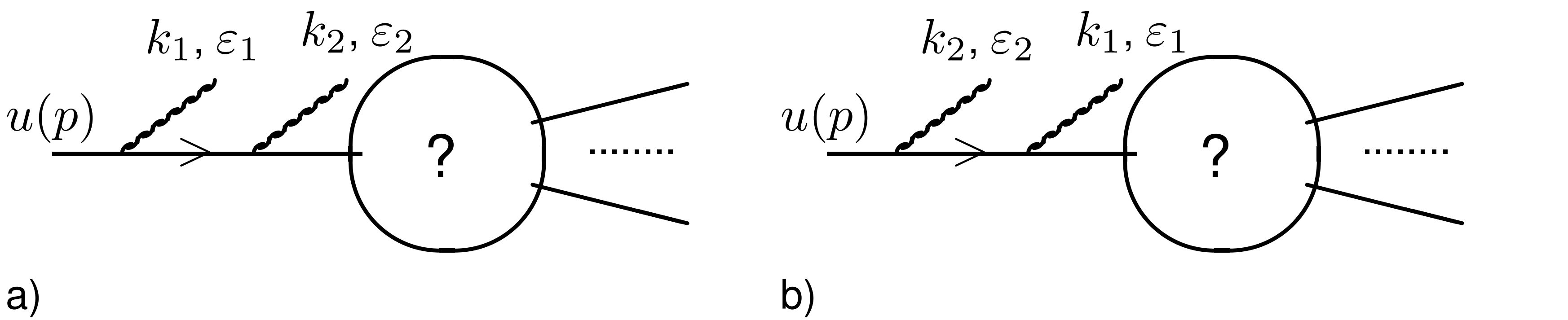


Figure 2: *Feynman diagrams for double photon emission in the initial state from electron.*

$$\begin{aligned}\mathcal{M}_a &= e^2 \frac{-1}{2k_1p} \frac{-1}{2k_1p + 2k_2p - 2k_1k_2} \ldots (\not{p} - \not{k}_1 - \not{k}_2 + m) \not{\varepsilon}_2 (\not{p} - \not{k}_1 + m) \not{\varepsilon}_1 u(p,s) \\ &\to \{\not{k}_1, \not{k}_2 \; out\} \to e^2 \frac{-1}{2k_1p} \frac{-1}{2k_1p + 2k_2p - 2k_1k_2} \ldots (\not{p} + m) \not{\varepsilon}_2 (\not{p} + m) \not{\varepsilon}_1 u(p,s)\end{aligned} \quad (3)$$

$$\mathcal{M}_a + \mathcal{M}_b \to \{\not{k}_1, \not{k}_2 \; out\} \to \left(e \frac{-\varepsilon_1 p}{k_1 p}\right)\left(e \frac{-\varepsilon_2 p}{k_2 p}\right) \mathcal{M}_B. \quad (4)$$

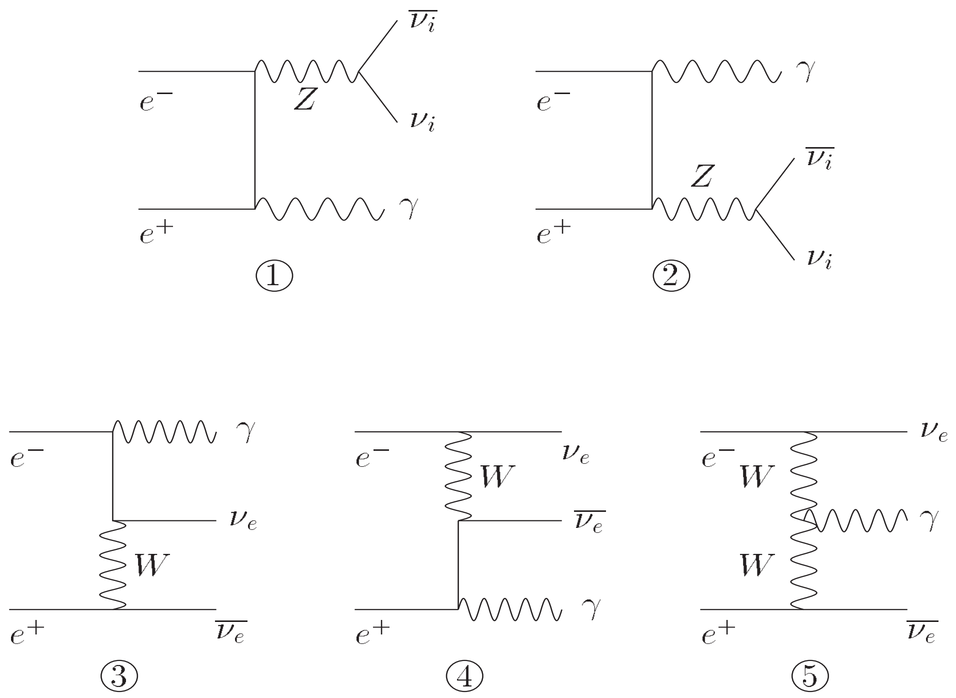
Again, each parts; $\sim \not{k}_1$, $\sim \not{k}_2$ and $\sim \not{k}_1 \ \not{k}_2$ form gauge invariant contributions once all diagrams contributions are summed. It leads to real photon emission part of Yennie-Frautchi-Suura exponentiation $\beta_1(k_1)$, $\beta_1(k_2)$ and $\beta_2(k_1,k_2)$ for example.

$$
\begin{aligned}
\mathcal{M} &= \left(e\Big(\frac{\varepsilon_1 p_2}{k_1 p_2} - \frac{\varepsilon_1 p_1}{k_1 p_1}\Big)\right)\left(e\Big(\frac{\varepsilon_2 p_2}{k_2 p_2} - \frac{\varepsilon_2 p_1}{k_2 p_1}\Big)\right)\mathcal{M}_B \\
&+ \beta_1(k_1)\left(e\Big(\frac{\varepsilon_2 p_2}{k_2 p_2} - \frac{\varepsilon_2 p_1}{k_2 p_1}\Big)\right) \\
&+ \beta_1(k_2)\left(e\Big(\frac{\varepsilon_1 p_2}{k_1 p_2} - \frac{\varepsilon_1 p_1}{k_1 p_1}\Big)\right) \\
&+ \beta_2(k_1,k_2)
\end{aligned}
$$

This was quite sketchy, lot of details were missing, but it **CAN** be done rigorously. In fact it was done already in 60's and is known under YFS.

QED bremsstrahlung amplitudes for s-channel processes

- Separation of eikonal parts of amplitudes first.
- Then powers powers of $1/kp$ and $\not{k}$ (resulting from structure of Lorentz group representations; boosts) helps organization into β_i functions of Yennie Frautchi Suura exponentiation.
- Separation of gauge invariant parts appeared naturally, all inheritable from YFS.
- Additional conformal symmetry, useful also for photons phase space.
- All this helped sizably improve perturbation convergence.
- Full phase space cover remained.
- That works nicely if either s-channel or t-channel exchange dominate.
- Interferences need to be taken into account too, but this is then no problem.
- S. Jadach, B.F.L. Ward Z.Was Phys.Rev.D 63 (2001) 113009 Comput.Phys.Commun. 130 (2000) 260



Issues:

- t-channel W exchange impose necessity to take into account emission from t-channel W
- W is massive, thus expansion with respect to contact interaction possible, corrections are not excessive.
- The structure of singularities remain as for s-channel only, emissions from W do not bring new singularities.

Double photon emissions:

Figure 3: Four boson coupling and coupling for nonphysical χ field.

- Enforce taking into account non QED-like diagrams !
- W is massive, thus expansion with respect to contact interaction possible, corrections are not excessive.
- Again the structure of singularities remain as for s-channel only.

Extension, t-channel contribution and contact interaction

- Things get somewhat complicated if t- contribution appear like in $ee \to \nu_e \nu_e$
- Because no new singularities with respect to $ee \to \nu_\mu \nu_\mu$ appear, one can use established solutions and for t-channel W exchange, contact interaction first. The t-dependence introduce perturbatively through β_i functions.
- The $W - W - \gamma$ vertex is required for gauge cancellations, also virtual χ propagator.
- It works nicely because corrections beyond contact interaction are small.
- But this is no limitation of principle. As previously full phase space remain.
- D. Bardin, S. Jadach, T. Riemann, Z. Was, Eur.Phys.J.C 24 (2002) 373
  Z. Was Eur.Phys.J.C 44 (2005) 489
- there is more of gauge invariant parts, than needed for exponentiation. Could be useful for other processes.

- Cases of $B^\pm \to K^\pm\pi^0(\gamma)$ or $B^0 \to K^\pm\pi^\mp(\gamma)$ look simple.
- Simpler than previously discussed cases. Matrix element essentialy consist of eikonal part only.
- Scalar QED simpler than QED: Eur.Phys.J.C 51 (2007) 569, !?
- Also the indication of amplitudes similarity between those of spin 0 and 1/2 amplitude structures
- Unfortunately, for seemingly similar, scalar QED process, ...

For the process $e^+e^- \to \gamma^*(p) \to \pi^+(q_1)\pi^-(q_2)(\gamma(k,\epsilon))$, at Born level amplitude reads $M = V^\mu H_\mu$, $V_\mu = \bar{v}(p_1,\lambda_1)\gamma_\mu u(p_2,\lambda_2)$, and $H_0^\mu(p,q_1,q_2) = \frac{eF_{2\pi}(p^2)}{p^2}(q_1-q_2)^\mu$. If photon is present, H^μ reads:

$$H^\mu = \frac{e^2 F_{2\pi}(p^2)}{p^2}\left\{(q_1+k-q_2)^\mu \frac{q_1\cdot\epsilon^*}{q_1\cdot k} + (q_2+k-q_1)^\mu \frac{q_2\cdot\epsilon^*}{q_2\cdot k} - 2\epsilon^{*\mu}\right\}, \quad (5)$$

The amplitude can be decomposed into a sum of two gauge invariant parts:

$$H_I^\mu = \frac{e^2 F_{2\pi}(p^2)}{p^2}(q_1-q_2)^\mu \left(\frac{q_1\cdot\epsilon^*}{q_1\cdot k} - \frac{q_2\cdot\epsilon^*}{q_2\cdot k}\right), \quad (6)$$

$$H_{II}^\mu = \frac{e^2 F_{2\pi}(p^2)}{p^2}\left(k^\mu\left(\frac{q_1\cdot\epsilon^*}{q_1\cdot k} + \frac{q_2\cdot\epsilon^*}{q_2\cdot k}\right) - 2\epsilon^{*\mu}\right), \quad (7)$$

or alternatively

$$H_{I'}^\mu = \frac{e^2 F_{2\pi}(p^2)}{p^2}\left((q_1-q_2)^\mu + k^\mu \frac{q_2\cdot k - q_1\cdot k}{q_2\cdot k + q_1\cdot k}\right)\left(\frac{q_1\cdot\epsilon^*}{q_1\cdot k} - \frac{q_2\cdot\epsilon^*}{q_2\cdot k}\right), \quad (8)$$

$$H_{II'}^\mu = \frac{2e^2 F_{2\pi}(p^2)}{p^2}\left(\frac{k^\mu}{q_2\cdot k + q_1\cdot k}(q_1\cdot\epsilon^* + q_2\cdot\epsilon^*) - \epsilon^{*\mu}\right). \quad (9)$$

- In this case eikonal part appear, as in QED, but the remaining part H_{II} does not naturally split to sum of β_1-like terms for emissions from π^+ and π^-.

- Factorization terms can be restored at cross section level:

$$\sum_{\lambda,\epsilon} |M|^2 = 4\pi\alpha \left\{ \frac{-m_\pi^2}{(q_1\cdot k)^2} A + \frac{-m_\pi^2}{(q_2\cdot k)^2} B + \frac{S-2m_\pi^2}{2(q_1\cdot k)(q_2\cdot k)} (C+D) \right\} + E\,. \tag{10}$$

where $A=\sum_\lambda |M_{Born}|^2(S,T',U)$, $B=\sum_\lambda |M_{Born}|^2(S,T,U')$,
$C=\sum_\lambda |M_{Born}|^2(S,T,U)$, $D=\sum_\lambda |M_{Born}|^2(S,T',U')$,
$E=32(4\pi\alpha)^3 m_\pi^2 \frac{F_{2\pi}^2(S)}{S^2}$.

- Spin amplitudes parts differ from QED, spin-less particles do not bring simplification. Full phase space coverage remain.

- G. Nanava, Z. Was Eur.Phys.J.C 51 (2007) 569 G. Nanava, Qingjun Xu, Z. Was Eur.Phys.J.C 70 (2010) 673

- One may expect approach to break for QCD, let us nonetheless investigate two gluon emission amplitudes.

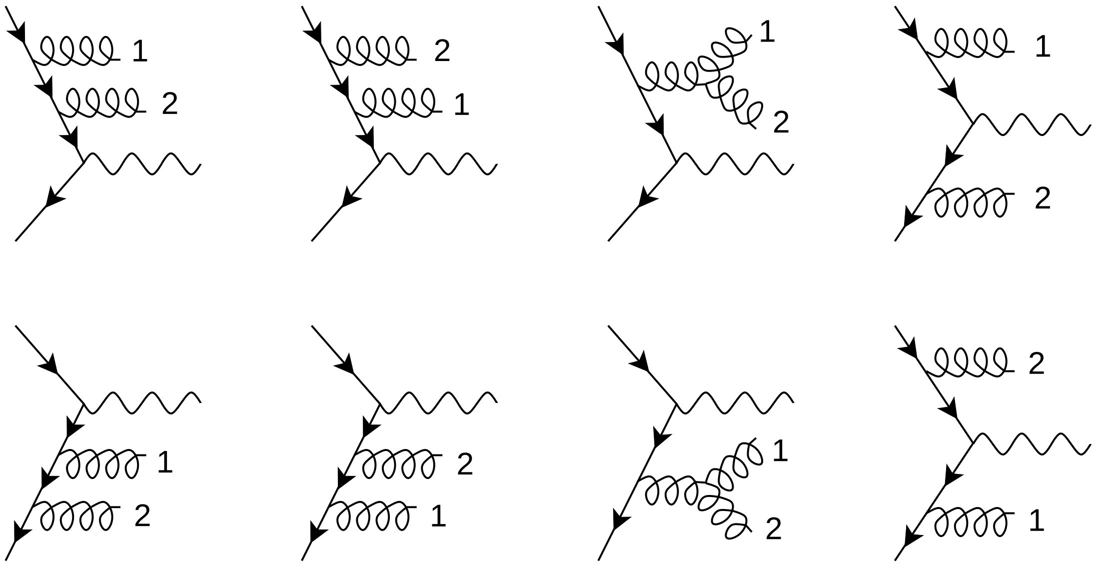


Figure 4: Feynman graphs for the process $q\bar{q} \to \mathcal{J} gg$.

The amplitude reads

$$\mathcal{M}^{a,b} = \frac{1}{2}\,\bar{v}(p)\Big(T^a T^b I^{(1,2)} + T^b T^a I^{(2,1)}\Big)u(q)\,. \tag{11}$$

where For the T^aT^b-part, we find

$$I^{(1,2)} = \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q\cdot k_2} + \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right) \quad (12)$$

$$+ \frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\not{J} \quad (13)$$

$$+ \not{J}\frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2}\right) \quad (14)$$

$$+ \not{J}\left(1 - \frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2} - \frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\right)\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right) \quad (15)$$

$$- \frac{1}{4}\frac{1}{p\cdot k_1 + p\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2 - \not{e}_2\not{k}_2\not{e}_1\not{k}_1}{k_1\cdot k_2}\right)\not{J} \quad (16)$$

$$- \frac{1}{4}\not{J}\frac{1}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2 - \not{k}_2\not{e}_2\not{k}_1\not{e}_1}{k_1\cdot k_2}\right). \quad (17)$$

The part proportional to T^bT^a is obtained by a permutation of the momenta and polarization vectors of the gluons. Lot of similarities with QED. One can group these terms, and identify spin amplitude properties necessary for BFKL (DGLAP, CCFM). Of course that is not the whole story.

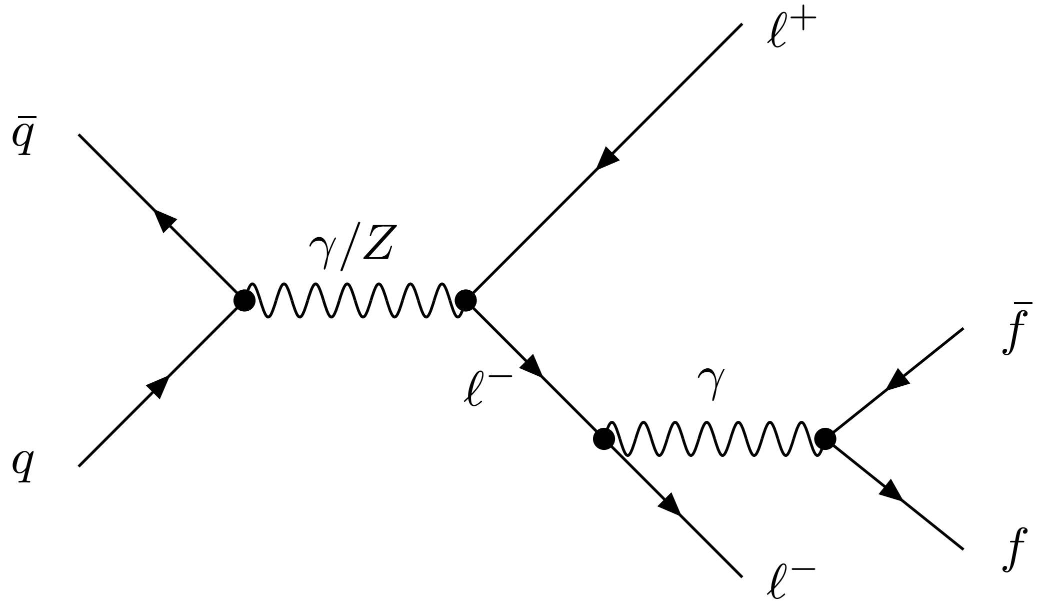
Spin amplitudes for double bremsstrahlung in QCD

- Surprisingly, amplitudes parts are closer to QED than that of scalar QED.
- One can identify parts corresponding for DGLAP etc. For that, partial phase space integration is not necessary. Searching for gauge invariant parts encapsulating most singular terms (remaining from previous steps).
- One get precision improvement but without guarantee of universality for all observables.
- Nonetheless with full phase space cover.
- A. van Hameren, Z. Was Eur.Phys.J.C 61 (2009) 33

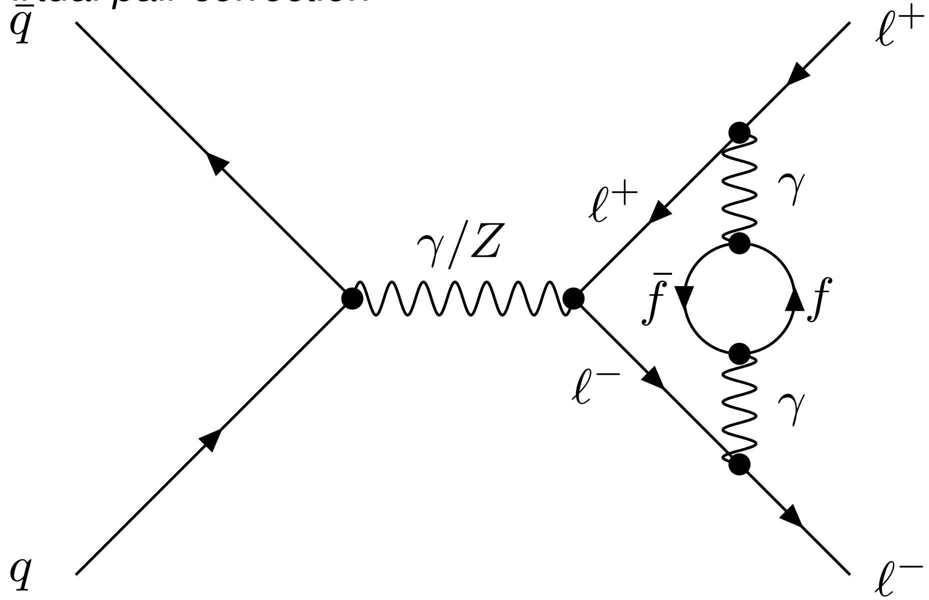
Approximate methods

Approximate methods

Real pair emission



Virtual pair correction



more difficult: two new lines. What represent correction, what main process. Distinct kinematical regimes possible: collinear emissions of pairs, lepton evolution to photon and back.

Complicated and not necessary for MC. Results are encouraging, even if still not systematized.

Approximate methods

- In this case exact separation into parts was not productive.
- Too many parts, often not following the required pattern of singularites but for some other processes.
- One can use eikonal part and previous results for inspiraton. Emission factor to multiply Born amplitude:

$$F(p,p',q,q_1,q_2,a) \quad \sim \left(\frac{2p-aq}{aq^2-2pq} - \frac{2p'-aq}{aq^2-2p'q}\right)_\mu \left(\frac{2p-aq}{aq^2-2pq} - \frac{2p'-aq}{aq^2-2p'q}\right)_\nu \frac{4q_1^\mu q_2^\nu - q^2 g^{\mu\nu}}{2q^4}$$

- No guarrantee to work well in all corners of phase space.
- Nonetheless with full phase space cover. Amplitudes can be applied everywhere.
- S. Antropov, A. Arbuzov, R. Sadykov, Z. Was Acta Phys.Polon.B 48 (2017) 1469, S. Antropov, Sw. Banerjee, Z. Was, J. Zaremba 1912.11376
- Project is going on slowly, and we evaluate efficiency of emission factor improvements, comparing with matrix element simulations.
- Precision requirements are not pressing.
- Comparison programs (also $F(\ldots)$) not much insight into Matrix Element structures.
- Better, for that perspective, algebraic manipulation programs/experts would help.

Summary

- Symmetries helped separation of spin amplitudes into parts....
- ... useful for fully differential and valid all over the phase space predictions.
- That is important for Monte Carlo and calculations of multidimensional distributions.
- It exposes parts of GSW predictions from sub-theories like QED (eikonal QED).
- Those are parts which need to be taken to higher orders.
- Calculation of negligible terms may be then avoided.
- I talked litte about loop corrections, they are important, but less of technical importance for Monte Carlo construction, that is for me.
- I was not talking about amplitudes parts similarities between different processes, like there were supersymmetric connected or something.
- Disadvantage: pressure on automated methods and higer orders validity proofs.
- At LEP separation was indispensable, at LHC too.
- What future will decide? Keep these methods in tool-box.
  Possible help in new symmetries search. But: $\pi^{\pm}$ and $e^{\pm}$ are not partners.
- **Do not drop the topic out.** Even as its use bring sometimes pain, but some fun too.