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Time evolution of Lepton Numbers of Majorana neutrinos in the Schrödinger versus Heisenberg pictures

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1 Introduction and Motivation

- Lepton family number violation of neutrinos is of great interest. (eg. neutrino's flavor oscillation, neutrino-less double beta decay.)

- We are interested in the time evolution of the lepton numbers by turning on the neutrino mass term (Majorana (this work) and Dirac).

- An example of the time evolution is the Cosmic Neutrinos Background (CNB) which comes from the era of the decoupling from weak interaction ($T \simeq 1$(MeV)) to the present $T_\nu \simeq 10^{-4}$(ev).

- The neutrinos with the rest mass $m_\nu \sim 10^{-3}$(eV) have experienced from relativistic regime to non-relativistic regime under the time evolution and expansion of the universe.

- In the previous works, (Apriadi Salim Adam, et.al. Arxiv. 2101.07751 (PTEP 2021) , 2106.02783, 2105.04306) we studied the time evolution of lepton number $L(t)$ by constructing the Heisenberg operator for it. For one flavor case;

$$L(t=0_-) = \int d^3x : \overline{\nu_L(\mathbf{x})}\gamma^0\nu_L(\mathbf{x}) : u_L(\mathbf{p}) = -v_L(\mathbf{p}) = \begin{pmatrix} 0 \\ \phi_-(\mathbf{n}) \end{pmatrix}$$

$$\nu_L(\mathbf{x}, t=0_-) = \int' \frac{d^3\mathbf{p}}{(2\pi)^3 2|\mathbf{p}|} \left(a(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}u_L(\mathbf{p}) + b^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}v_L(\mathbf{p})\right),$$

$\int'$ implies that zero momentum is excluded from the integration.
$\sigma \cdot \mathbf{n}\phi_-(\mathbf{n}) = -\phi_-(\mathbf{n})$

- The underlying assumption of the derivation is the Majorana mass; m is turned on at $t=0$ and continues to be non-zero for $t>0$.

$$\mathcal{L}_m = -\theta(t)\frac{m}{2}\left[\overline{(\nu_L)^c}\nu_L + h.c.\right].$$

The time evolution of the Heisenberg operators ;
$a(\mathbf{p},t) = e^{iHt}a(\mathbf{p})e^{-iHt}$ where H is a Hamiltonian for massive Majorana field.

$$a(\mathbf{p},t) = \left(\cos(E(\mathbf{p})t) - \frac{i|\mathbf{p}|\sin(E(\mathbf{p})t)}{E(\mathbf{p})}\right)a(\mathbf{p}) - \frac{m\sin(E(\mathbf{p})t)}{E(\mathbf{p})}a^\dagger(-\mathbf{p}),$$

$$a^\dagger(-\mathbf{p},t) = \left(\cos(E(\mathbf{p})t) + \frac{i|\mathbf{p}|\sin(E(\mathbf{p})t)}{E(\mathbf{p})}\right)a^\dagger(-\mathbf{p}) + \frac{m\sin(E(\mathbf{p})t)}{E(\mathbf{p})}a(\mathbf{p}),$$

Similar relations hold by replacing $(a(\pm\mathbf{p},t),)$ to $(b(\pm\mathbf{p},t))$ and $a(\mathbf{p})$ to $b(\mathbf{p})$.
Heisenberg operator for lepton number:

$$L(t) = \int_{\mathbf{p}\in A} \frac{d^3p}{(2\pi)^3 2|\mathbf{p}|}\big(a^\dagger(\mathbf{p},t)a(\mathbf{p},t) + a^\dagger(-\mathbf{p},t)a(-\mathbf{p},t)$$
$$-b^\dagger(\mathbf{p},t)b(\mathbf{p},t) - b^\dagger(-\mathbf{p},t)b(-\mathbf{p},t)\big).$$

$\mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, The momentum integration region A corresponds to the hemisphere $0 \leq \phi < \pi$.

$$L(t) = \int' \frac{d^3\mathbf{p}}{|2\mathbf{p}|(2\pi)^3}\left[\frac{|\mathbf{p}|^2 + m^2\cos(2E(\mathbf{p})t)}{E(\mathbf{p})^2}\right]\left(a^\dagger(\mathbf{p})a(\mathbf{p}) - b^\dagger(\mathbf{p})b(\mathbf{p})\right)$$
$$- \int_{\mathbf{p}\in A} \frac{d^3\mathbf{p}}{|2\mathbf{p}|(2\pi)^3}\frac{m\sin(2E(\mathbf{p})t)}{E(\mathbf{p})}\left(a(-\mathbf{p})a(\mathbf{p}) + a^\dagger(\mathbf{p})a^\dagger(-\mathbf{p}) - (a\to b)\right)$$
$$+ \int_{\mathbf{p}\in A} \frac{d^3\mathbf{p}}{|\mathbf{p}|(2\pi)^3}\frac{im|\mathbf{p}|\sin^2(E(\mathbf{p})t)}{E(\mathbf{p})^2}\left(a(-\mathbf{p})a(\mathbf{p}) - a^\dagger(\mathbf{p})a^\dagger(-\mathbf{p}) - (a\to b)\right).$$

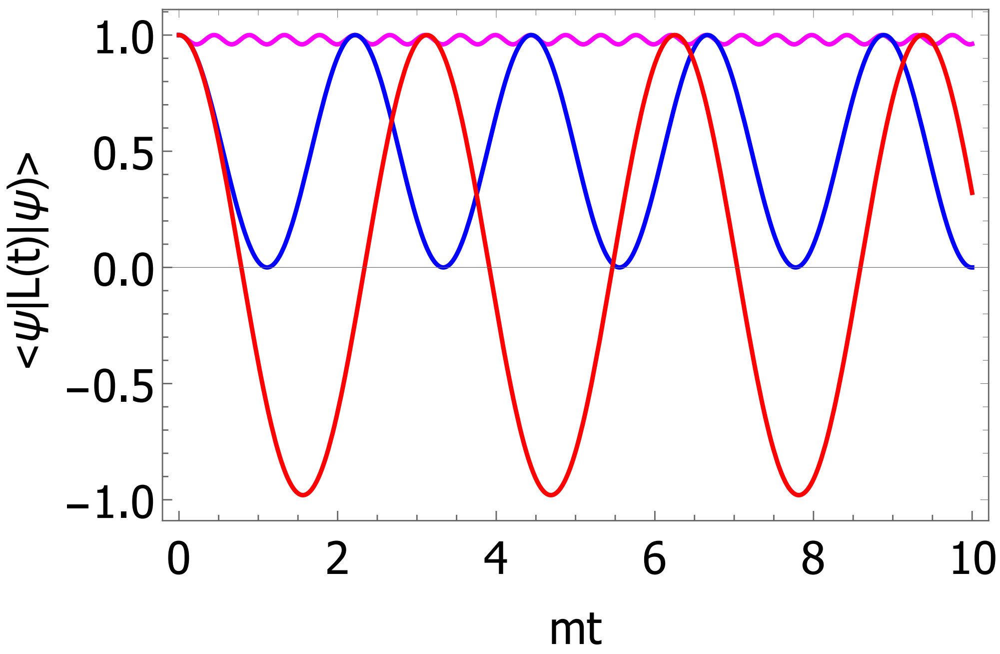
The expectation value of the lepton number in the Heisenberg representation. The vaccum annihilated as $a(q)|0\rangle = 0$ and $b(q)|0\rangle = 0$.

$$|\psi\rangle = \frac{a^\dagger(q)|0\rangle}{\sqrt{\langle 0|a(q)a^\dagger(q)|0\rangle}}, \quad \langle\psi|\psi\rangle = 1.$$

$$\langle\psi|L(t)|\psi\rangle = \frac{|\mathbf{q}|^2 + m^2\cos(2E(\mathbf{q})t)}{E(\mathbf{q})^2}$$

2 Time dependence of the expectation value for lepton number:

$$\langle\psi|L(t)|\psi\rangle = v_q^2 + (1 - v_q^2)\cos\left(\frac{2mt}{\sqrt{1-v_q^2}}\right),\ v_q = \frac{|q|}{E_q}$$



For relativistic case: $v_q = 0.99$, the lepton number vibrates small and fast around 1. ($\langle L(t)\rangle \simeq 1$)

For $v_q = \frac{1}{\sqrt{2}}$, $0 \leq \langle L(t)\rangle \leq 1$.

For non-relativistic case: $v_q = 0.1$, it oscillates slowly with the large amplitude. ($-1 < \langle L(t)\rangle \leq 1$)

3 Expectation value of the Lepton number in the Schrödinger picture

Our goal is to show the expectation value is the same in both picture.

$$\langle\psi|L_H(t)|\psi\rangle = \langle\psi(t)|L_s|\psi(t)\rangle$$

This should be shown by knowing the time evolution of initial state $\sim a^\dagger(q)|0\rangle$

$$|\psi(t)\rangle = \exp[-iHt]|\psi\rangle = n_q \exp[-iHt]a^\dagger(q)|0\rangle$$

In order to find the time evolution of the initial state $a^\dagger(q)|0\rangle$, we need to rewrite the state with operators for mass eigenstates $a_M(q, \lambda = \pm)$ (λ helicities) and the vacuum $|0_M\rangle$ which is defined by,

$$a_M(q, \lambda = \pm)|0_M\rangle = 0.$$

Relation between massive Majorana operator $a_M(p,\lambda)$ ($\lambda = \pm$, helicities) and operators for massless (anti-) neutrino states $a(p), b(p)$: $\cos 2\phi_p = v_p = \frac{|p|}{E_p}$. $(\cos\phi_p, \sin\phi_p) = (\sqrt{\frac{1+v_p}{2}}, \sqrt{\frac{1-v_p}{2}})$.

$$a(p) = \sqrt{\cos 2\phi_p}\left(\cos\phi_p a_M(p,-) + i\sin\phi_p a_M^\dagger(-p,-)\right),$$
$$b(p) = \sqrt{\cos 2\phi_p}\left(\cos\phi_p a_M(p,+) + i\sin\phi_p a_M^\dagger(-p,+)\right).$$

3.1 Relation between two vacua

Two vacua are related to each other as,

$$|0\rangle = \prod_{p\in A} |0\rangle_p = \prod_{p\in A} (\cos^2\phi_p|0_M\rangle_p - \sin^2\phi_p|4_M\rangle_p + i\sin\phi_p\cos\phi_p|2_M\rangle_p)$$

where the subscript p of the vacuum implies only the creation and annihilation operators with the momentum $\pm p$ act on the vacuum.

$$a(\pm p)|0\rangle_p = b(\pm p)|0\rangle_p = 0, \quad a_M(\pm p, \lambda = \pm 1)|0_M\rangle_p = 0.$$

$|4_M\rangle_p$ and $|2_M\rangle_p$ correspond to the two pairs and one pair of Majorana neutrinos with opposite momentum. With the creation operator of a pair $B^\dagger_{M\lambda}(p) = \alpha^\dagger_M(-p,\lambda)\alpha^\dagger_M(p,\lambda)$, they are defined by,

$$|4_M\rangle_p = B^\dagger_{M+}(p)B^\dagger_{M-}(p)|0_M\rangle_p,$$
$$|2_M\rangle_p = \sum_{\lambda=\pm} B^\dagger_{M\lambda}(p)|0_M\rangle_p.$$

3.2 Time evolution of the state with a definite lepton number

$$|\Psi(t)\rangle = |q,t\rangle \prod_{(\mathbf{p}\neq\mathbf{q})\in A} |0(t)\rangle_p .$$

$$|q,0\rangle \equiv a^\dagger(q)|0\rangle_q = n_q\sqrt{\cos 2\phi_q}\{\cos\phi_q + i\sin\phi_q B^\dagger_{M+}(q)\}a^\dagger_M(q,-)|0_M\rangle_q,$$

$$\downarrow$$

$$|q,t\rangle = n_q e^{-iE_q t}\sqrt{\cos 2\phi_q}\{\cos\phi_q + ie^{-2iE_q t}\sin\phi_q B^\dagger_{M+}(q)\}a^\dagger_M(q,-)|0_M\rangle_q,$$

$$|0\rangle_p$$

$$\downarrow$$

$$|0(t)\rangle_p = \cos^2\phi_p|0_M\rangle_p - \sin^2\phi_p e^{-4iE_p t}|4_M\rangle_p + i\sin\phi_p\cos\phi_p e^{-2iE_p t}|2_M\rangle_p .$$

3.3 Final step: Expectation value of the lepton number operator in the Schrodinger picture: $\langle\Psi(t)|L_s|\Psi(t)\rangle$

$$L_s = \int_{\mathbf{p}\in A} \frac{V d^3p}{(2\pi)^3} \quad \hat{l}(p); \quad N_{M\lambda}(p) = \alpha_M^\dagger(p,\lambda)\alpha_M(p,\lambda)$$

$$\hat{l}(p) = \frac{p}{E_p}\left(N_{M-}(p) + N_{M-}(-p) - N_{M+}(p) - N_{M+}(-p)\right)$$
$$- \frac{im}{E_p}\left(B^\dagger_{M-}(p) - B_{M-}(p) - B^\dagger_{M+}(p) + B_{M+}(p)\right)$$

$$\langle\Psi(t)|L_s|\Psi(t)\rangle = \langle q,t|\hat{l}(q)|q,t\rangle = \frac{q^2}{E_q^2} + \frac{m^2}{E_q^2}\cos 2E_q t$$

$\rightarrow$ the same as Heisenberg picture

3.4 Conclusion

- Time evolution of lepton number is investigated in Schrödinger picture.
- The expectation value is the same as that obtained by Heisenberg picture. (The large amplitude and slow oscillation for non-relativistic case. ($\langle \Delta L \rangle = \pm 1$) The small amplitude and fast oscillation for relativistic case. ($\langle L \rangle \simeq 1$)
- The vacuum with null lepton number ($|0\rangle_p$) is a superposition of the vacuum for mass eigenstgate ($|0_M\rangle_p$), a pair of Majorana particles ($|2_M\rangle_p$), and two pairs ($|4_M\rangle_p$). One vacuum is not the other vacuum.
- Similar to the vacuum, the one particle state ($a^\dagger(q)|0\rangle_q$) with $\langle \hat{l}(q) \rangle = 1$ is a superposition of a mass eigenstgate ($a^\dagger_M(q,-)|0_M\rangle_q$) and a state with an additional Majorana pair. ($B^\dagger_{M+}(q) a^\dagger_M(q,-)|0_M\rangle_q$)
- These non-trivial superposition of states with different energies give rise to the oscillating behavior for the expectation value of lepton number.