Wormhole solutions in Beyond Horndeski theories

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Outline

In collaboration with:

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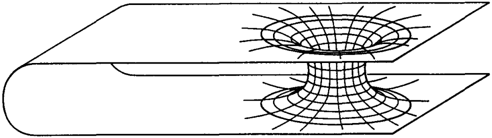
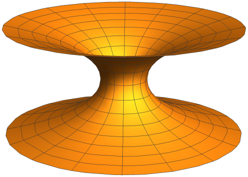
Wormholes in General Relativity

A wormhole is a solution of the Einstein's field equations which has the property to connect two distant regions in spacetime.





A wormhole may connect:

- Two distant regions of our universe (intra-universe wormholes)[1].
- Two different universes (inter-universe wormhole).

The difference between the two kind of wormholes is topological. An observer who makes measurements near the wormhole cannot identify the class of the wormhole.

[1] The figure of the intra-universe wormhole is from the following book: C. W. Misner, K. S. Thorne and J. A. Wheeler, "*Gravitation*", San Francisco, 1973

The Einstein-Rosen Bridge

The first wormhole solution, known as Einstein-Rosen Bridge, was found by Einstein and Rosen at 1935[2].

The Einstein-Rosen Bridge was a geometrical model for elementary particles

The wormhole solutions may be constructed from the Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

By making the coordinate change $u^2 = r - 2M$ (with $u \in (-\infty, +\infty)$) the Schwarzschild solution takes the form:

$$ds^2 = -\frac{u^2}{u^2 + 2M} dt^2 + 4\left(u^2 + 2M\right) du^2 + \left(u^2 + 2M\right)^2 d\Omega^2.$$

The circumferential radius $R_c = \left(u^2 + 2M\right)$ has a minimum at $u = 0$ with $R_c(0) = 2M$ which is the definition of the throat.

[2] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).

The coordinate u is a bad coordinate which covers only a part of the black hole geometry.

The coordinate transformation discards the interior region of the black hole $r \in [0, 2M)$ and double covers the exterior region $r \in [2M, +\infty)$.

The Einstein-Rosen Bridge is not an actual wormhole, it is a "disguised" black hole.

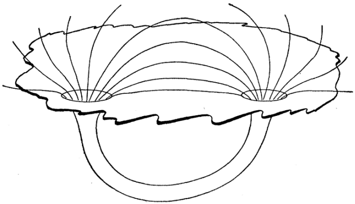
There are some big problems with the Einstein-Rosen wormholes:

- Einstein-Rosen bridges are one way wormholes. The horizon is an one way surface. The wormholes have an entrance (a black hole horizon) and an exit (a white hole horizon).
- The huge Tidal forces near the wormhole would rip apart a traveler before he reaches the throat.
- The interior of the black hole is dynamic. The throat of the Einstein-Rosen bridge opens and closes so fast that even a photon cannot cross it.

The Geons

The geons (gravitational-electromagnetic entities) are solutions of the Einstein-Maxwell coupled system found by Wheeler at 1955[3].

The metric for a geon is everywhere flat except of two widely separated regions which are connected with a wormhole-like structure.



The geon is an unstable gravitational-electromagnetic quasisoliton. They are constructed as models for electrical charges and particle like entities.

The main problem with the wheeler wormholes, except of their stability, is their size. A typical size for a geon is of the order of the Plank length.

[3] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
C. W. Misner and J. A. Wheeler, Annals Phys. **2**, 525 (1957).

Traversable Wormholes

The renaissance of wormhole physics was performed by Morris and Thorne at 1988[4].

- Morris and Thorne suggest that we may construct traversable wormholes using an "*engineering-like*" technique.
- We start with a metric which describes a traversable wormhole and by solving the Einstein field equation in the reverse direction we find the associate energy-momentum tensor.
- If we know the form of the matter distribution which supports the solution we theoretically may build the wormhole

[4] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
M. Visser, "*Lorentzian wormholes: From Einstein to Hawking*", Woodbury, USA: AIP (1995)

How to "cook" a traversable wormhole

- Start with a static spherically symmetric line-element of the form

$$ds^2 = -e^{2\phi(r)}dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

- The metric must obey everywhere the Einstein's equations.

- The wormhole must have a throat i.e. $\exists\, r_0 : b(r_0) = r_0$.

- There must be no horizons i.e. $\phi(r)$ must be everywhere finite.

- The radial coordinate covers two asymptotically flat regions $r_+ \in [r_0, +\infty)$ and $r_- \in (-\infty, -r_0]$.

  We may use a different coordinate $r^2 = l^2 + r_0^2$ which covers both asymptotic regions $l \in (-\infty, +\infty)$:

$$ds^2 = -e^{2\mathrm{v}(l)}dt^2 + f(l)dl^2 + (l^2 + r_0^2)d\Omega^2.$$

  In this coordinate system $\mathrm{v}(r)$ and $f(r)$ are everywhere finite.

How to "cook" a traversable wormhole

- Adjust the parameters in order for a traveler to feel acceleration compared to Earth's gravity acceleration and small tidal forces.

- A traveler should be able to cross the wormhole in a finite and small proper time (Morris and Thorne demand less than a year).

- The solutions should be stable under perturbations.

- Find the energy-momentum tensor and build the wormhole!

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So, where is the trap?

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- Find the energy-momentum tensor and build the wormhole!

A traversable wormhole violates the Energy Conditions

We assume a spherically symmetric form energy-momentum tensor $T_{\mu\nu} = \mathrm{diag}\,(\rho, -\tau, p, p)$ where ρ is the energy density, τ is the tension per unit area measured in the radial direction and p is the pressure measured in the other two directions.

Near the throat of a traversable wormhole

$$\rho - \tau < 0.$$

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Near the throat of a traversable wormhole

$$\rho - \tau < 0.$$

We need Exotic Matter in order to keep the throat open!

Wormholes in Scalar-Tensor theories

- Scalar-Tensor theories allow the derivation of wormhole solutions.
- Usually we need phantom scalar fields.

The most characteristic example is the Ellis-Bronikov wormhole [5]:

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \left(R + \frac{1}{2} \nabla_\mu \phi \, \nabla^\mu \phi \right)$$

$$ds^2 = -dt^2 + dl^2 + (l^2 + r_0^2) d\Omega^2$$

$$\phi = \frac{\pi}{2} - tan^{-1}\left(\frac{l}{r_0}\right)$$

[5] H. G. Ellis, J. Math. Phys. **14** (1973) 104–118.
K. A. Bronnikov, Acta Phys. Polon. B **4** (1973) 251–266.

The Einstein Scalar Gauss-Bonnet Theory

$$S = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}\nabla_\mu\phi\,\nabla^\mu\phi + F(\phi)R^2{}_{GB}\right],$$

where:

$$R^2{}_{GB} \;=\; R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \;-\; 4\,R_{\mu\nu}R^{\mu\nu} \;+\; R^2.$$

The equations of motion are:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}, \qquad \nabla^2\phi + \frac{d\,F(\phi)}{d\phi}\,R^2{}_{GB} = 0,$$

with:

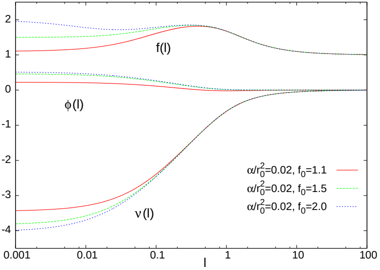
$$T_{\mu\nu} = -\frac{1}{4}g_{\mu\nu}\,\nabla_\rho\phi\,\nabla^\rho\phi + \frac{1}{2}\nabla_\mu\phi\,\nabla_\nu\phi + \frac{1}{2}\,\left(g_{\rho\mu}g_{\lambda\nu} + g_{\lambda\mu}g_{\rho\nu}\right)\,\eta^{\kappa\lambda\alpha\beta}\,\tilde{R}^{\rho\gamma}{}_{\alpha\beta}\,\nabla_\gamma\nabla_\kappa F(\phi)$$

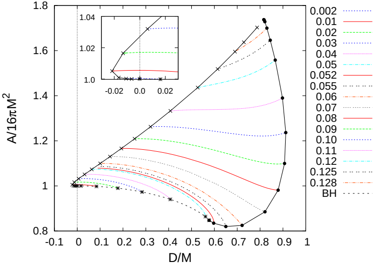
The first realistic wormhole solutions in 4D

The first realistic wormholes in 4D were found by Kanti, Kleihaus and Kunz in EdGB theory ($F(\phi) = \alpha e^{-\gamma\phi}$)[6].

They used a spherically symmetric line-element

$$ds^2 = -e^{2\nu(l)}dt^2 + f(l)dl^2 + (l^2 + r_0^2)d\Omega^2.$$





They found regular, traversable wormholes which **do not need exotic mater**.

[6] P. Kanti, B. Kleihaus and J. Kunz, Phys.Rev.Lett. **107** (2011) 271101.
P. Kanti, B. Kleihaus and J. Kunz, Phys.Rev. D**85** (2012) 044007.

Double throat Wormhole solutions in EsGB theory[7]

We use the following line element:

$$ds^2 = -e^{f_0(\eta)}dt^2 + e^{f_1(\eta)}\left\{d\eta^2 + \left(\eta^2 + \eta_0^2\right)d\Omega^2\right\} .$$

The circumferential radius $R_c(\eta) = e^{f_1/2}\sqrt{\eta^2 + \eta_0^2}$ may have:

- A minimum $\longrightarrow$ Single throat wormholes.
- A local maximum and a minimum $\longrightarrow$ Double throat wormholes.

For a spherical symmetric spacetime the null energy condition (NEC) has the form

$$T_{\mu\nu}n^\mu n^\nu \geq 0 \implies \quad -T_t^t + T_\eta^\eta \geq 0, \quad \text{and} \quad -T_t^t + T_\theta^\theta \geq 0.$$

Using our metric we find

$$\left[-T^t_{\ t} + T^\eta_{\ \eta}\right]_{\eta=\eta_{\rm th}} = \left[\frac{-2e^{-f_1}R_c''}{R_c}\right]_{\eta=\eta_{\rm th}} .$$

Since there is a minimum at the throat ($R_c'' > 0$), the NEC is always violated there.

[7] G. Antoniou, A. B., P. Kanti, B. Kleihaus, and J. Kunz, Phys. Rev. D **101**, (2020) 024033.

Asymptotic solutions

We assume that there is an extremum at $\eta = 0$ (throat or equator). By taking the expansion near the throat/equator we find that our solutions have a polynomial form

$$e^{f_0} = a_0(1 + a_1\eta + a_2\eta^2 + ...), \qquad e^{f_1} = b_0(1 + b_1\eta + b_2\eta^2 + ...)$$
$$\phi = \phi_0 + \phi_1\eta + \phi_2\eta^2 + ...$$

The condition for an extremum at $\eta = 0$ is $f_1'(0) = 0$ which implies $b_1 = 0$.

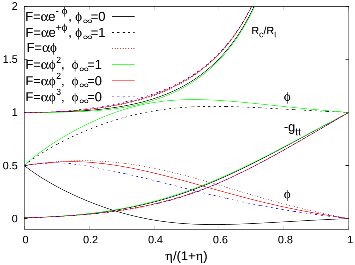
In the limit $\eta \to \infty$ we demand that the space-time is flat and the scalar field assumes a constant value. We find:

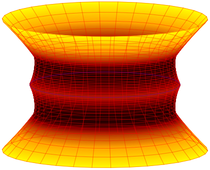
$$e^{f_0(\eta)} = 1 - \frac{2M}{\eta} + \mathcal{O}\left(1/\eta^3\right), \qquad e^{f_1(\eta)} = 1 + \frac{2M}{\eta} + \mathcal{O}\left(1/\eta^2\right),$$

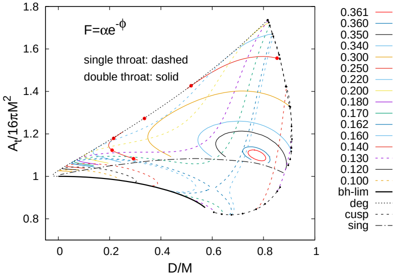
$$\phi\left(\eta\right) = \phi_\infty + \frac{D}{\eta} + \mathcal{O}\left(1/\eta^2\right).$$

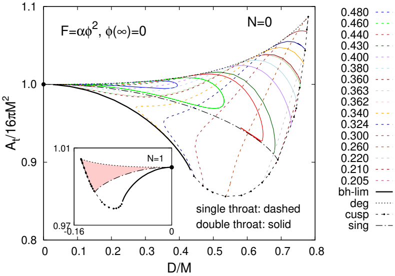
For any form of the coupling function we may construct both a regular throat/equator and an asymptotically flat region.

Numerical Solutions









Junction Conditions

If we demand that the solutions will be symmetric under the change $\eta \to -\eta$, the derivatives of the metric and scalar field functions are not continuous at $\eta = 0$.

These discontinuities may be attributed to the presence of a thin-shell of matter located at $\eta = 0$.

The embedding of this matter distribution is determined through the junction conditions.

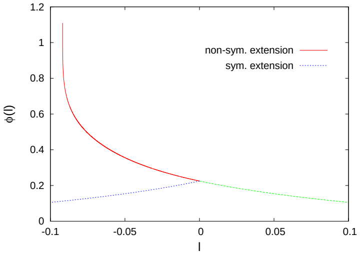
$$\langle G^\mu_\nu - T^\mu_\nu \rangle = s^\mu_\nu, \quad \left\langle \nabla^2 \phi + \dot{F} R^2_{GB} \right\rangle = s_{\rm sc}, \quad S_\Sigma = \int \left(\lambda_1 + 2\lambda_0 F(\phi_0) \bar{R} \right) \sqrt{-\bar{h}} d^3x,$$

If we substitute the metric we find

$$8\dot{F}\phi' e^{-\frac{3f_1}{2}} = \lambda_1 \eta_0^2 + 4\lambda_0 F e^{-f_1} - \rho \eta_0^2 \ ,$$

$$e^{-\frac{f_1}{2}} f_0' = \lambda_1 + p,$$

$$e^{-f_1}\phi' - 4\frac{\dot{F}}{\eta_0^2} f_0' e^{-2f_1} = -4\lambda_0 \frac{\dot{F}}{\eta_0^2} e^{-\frac{3f_1}{2}} + \frac{\rho_{\rm scal}}{2} \ .$$



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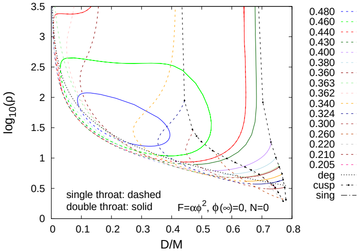
For $p = 0$ (dust) and $\lambda_1 = \lambda_0$ we find:

$$\lambda_1 = e^{-f1/2} f_0'$$

$$\rho = \frac{e^{-\frac{3f_1}{2}}}{\eta_0} \left[\left(4F + \eta_0^2 e^{f_1} \right) f_0' - 8\dot{F}\phi' \right],$$

$$\rho_{\rm sc} = 2e^{-f_1}\phi'.$$



Stability

- There are indications that the EsGB wormholes are unstable[8].
- No-Go theorems in Horndeski theory[9].
- Wormholes may be stable in beyond Horndeski[10].

[8] M. A. Cuyubamba, R. A. Konoplya and A. Zhidenko, Phys. Rev. D **98** (2018) no.4, 044040.
[9] V. A. Rubakov, Theor. Math. Phys. **188** (2016) no.2, 1253-1258.
O. A. Evseev and O. I. Melichev, Phys. Rev. D **97** (2018) no.12, 124040.
[10] S. Mironov, V. Rubakov and V. Volkova, Class. Quant. Grav. **36** (2019) no.13, 135008.
G. Franciolini, L. Hui, R. Penco, L. Santoni and E. Trincherini, JHEP **01** (2019), 221.

Beyond Horndeski theory

$$S = \int \mathrm{d}^4x\sqrt{-g}\,\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_4^{\mathrm{bH}} + \mathcal{L}_5^{\mathrm{bH}}\right),$$

with

$$X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi,$$

$$\mathcal{L}_2 = G_2(X), \qquad \mathcal{L}_3 = -G_3(X)\,\Box\phi,$$

$$\mathcal{L}_4 = G_4(X)R + G_{4X}\left[(\Box\phi)^2 - \nabla_\mu\partial_\nu\phi\,\nabla^\mu\partial^\nu\phi\right],$$

$$\mathcal{L}_5 = G_5(X)G_{\mu\nu}\nabla^\mu\partial^\nu\phi - \frac{1}{6}\,G_{5X}\Big[(\Box\phi)^3 - 3\,\Box\phi\,\nabla_\mu\partial_\nu\phi\,\nabla^\mu\partial^\nu\phi + 2\,\nabla_\mu\partial_\nu\phi\,\nabla^\nu\partial^\rho\phi\,\nabla_\rho\partial^\mu\phi\Big],$$

$$\mathcal{L}_4^{\mathrm{bH}} = F_4(X)\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon^{\alpha\beta\gamma}{}_\sigma\,\partial_\mu\phi\,\partial_\alpha\phi\,\nabla_\nu\partial_\beta\phi\,\nabla_\rho\partial_\gamma\phi,$$

$$\mathcal{L}_5^{\mathrm{bH}} = F_5(X)\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon^{\alpha\beta\gamma\delta}\,\partial_\mu\phi\,\partial_\alpha\phi\,\nabla_\nu\partial_\beta\phi\,\nabla_\rho\partial_\gamma\phi\,\nabla_\sigma\partial_\delta\phi,$$

and

$$XG_{5X}F_4 = 3F_5(G_4 - 2XG_{4X}).$$

Wormhole Solution

We use a spherically symmetric line-element

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2.$$

We find

$$h(r) = 1 + \frac{r^2}{2a} - \frac{r}{2}\sqrt{\frac{8aM + r^3}{a^2 r}}, \qquad f(r) = h - \frac{b_1 h}{r}\left(1 - \sqrt{h}\right),$$

$$\phi'(r) = \frac{\sqrt{h} - 1}{r\sqrt{h}}.$$

The nature of the compact object is determined from the roots of the metric functions:

$$f(r_0) = 0 \Longrightarrow h(r_0) = (1-\lambda)^2 \begin{cases} \lambda = 1 \longrightarrow \text{Black Hole} \\ 0 < \lambda < 1 \longrightarrow \text{Wormhole} \end{cases}$$

where $\lambda = \dfrac{r_0}{b_1}$.

At infinity we find:

$$h(r) = 1 - \frac{2M}{r} + \frac{4\alpha M^2}{r^4} + \mathcal{O}\left(r^{-5}\right), \qquad f(r) = 1 + \frac{2M}{r} + \frac{b_1 M + 4M^2}{r^2} + \mathcal{O}\left(r^{-3}\right),$$

$$\phi'(r) = -\frac{M}{r^2} + \mathcal{O}\left(r^{-3}\right).$$

The coordinate r covers only a half of the wormhole spacetime.

We may describe the solutions in both asymptotically flat regions using the coordinate transformation $r^2 = l^2 + r_0^2$ with $l \in (-\infty, +\infty)$.

$$ds^2 = -H(l)dt^2 + \frac{1}{F(l)}dl^2 + (l^2 + r_0^2)d\Omega^2,$$

where

$$H(l) = h(r(l)), \quad \text{and} \quad F(l) = \frac{f(r(l))\,(l^2 + r_0^2)}{l^2}.$$

The Theory

$$G_2 = \frac{4\ 2^{3/4}\alpha y^4}{\sqrt{\frac{1}{\sqrt{2}-2b_1 y}}}, \qquad G_{3X} = \frac{16\sqrt[4]{2}\alpha\sqrt{\frac{1}{\sqrt{2}-2b_1 y}}}{3\sqrt{2}b_1 y - 2},$$

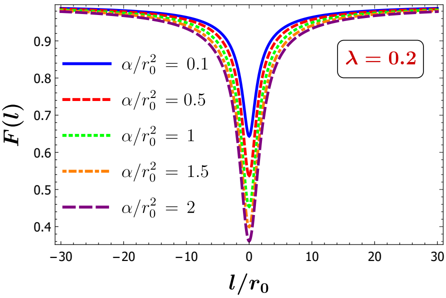
$$G_4 = \frac{1 - 4\alpha y^2}{\sqrt[4]{2}\sqrt{\frac{1}{\sqrt{2}-2b_1 y}}}, \qquad G_{5X} = -\frac{8\sqrt[4]{2}\alpha\sqrt{\frac{1}{\sqrt{2}-2b_1 y}}}{y^2\left(3\sqrt{2}b_1 y - 2\right)},$$

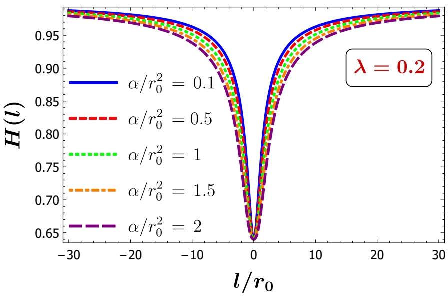
$$F_4 = \frac{b_1\left(\sqrt{2} - 4b_1 y\right)\left(\frac{1}{\sqrt{2}-2b_1 y}\right)^{5/2}\left(4\alpha y^2 + 1\right)}{2^{3/4}y^3\left(3\sqrt{2}b_1 y - 2\right)},$$

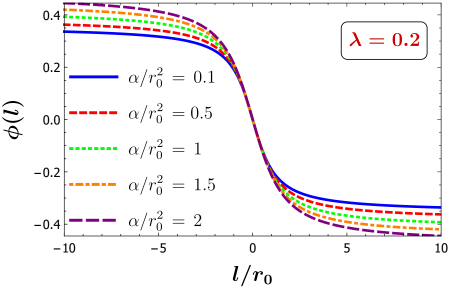
$$F_5 = \frac{2\ 2^{3/4}\alpha b_1\left(\sqrt{2} - 4b_1 y\right)\left(\frac{1}{\sqrt{2}-2b_1 y}\right)^{7/2}}{3y^3\left(2 - 3\sqrt{2}b_1 y\right)},$$

where,

$$X = y^2(-1 + \sqrt{2}b_1 y).$$



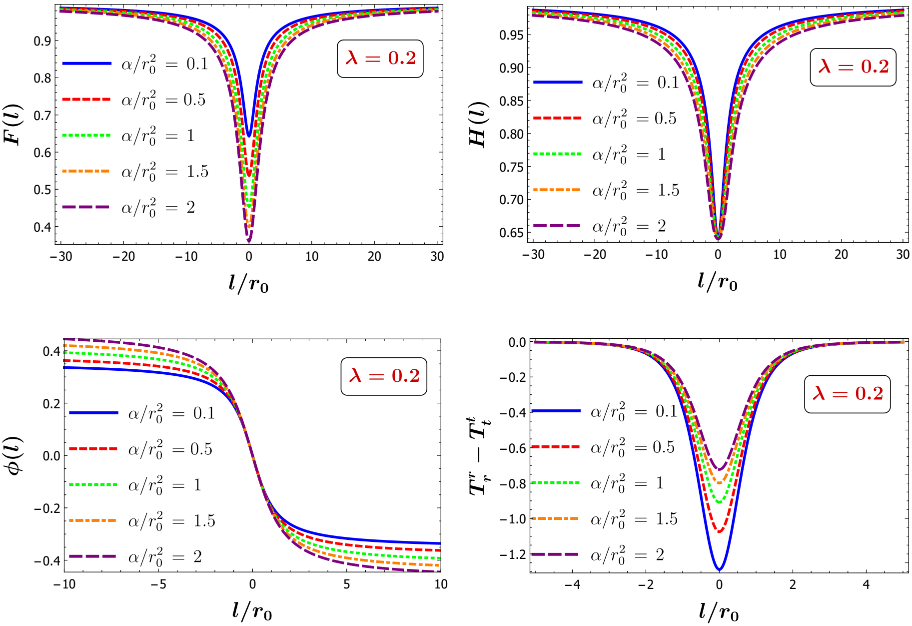




For a spherical symmetric spacetime the null energy condition (NEC) has the form

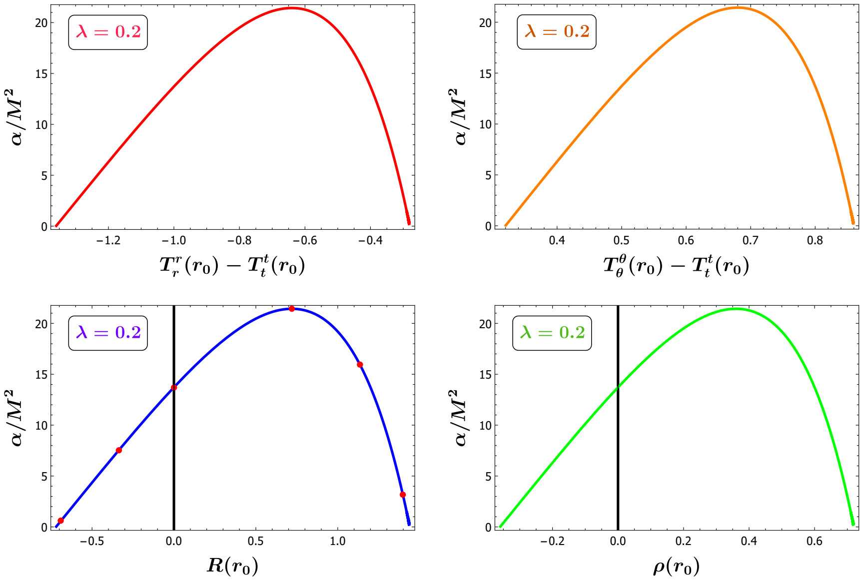
$$-T^t_t + T^r_r \geq 0, \quad \text{and} \quad -T^t_t + T^\theta_\theta \geq 0,$$

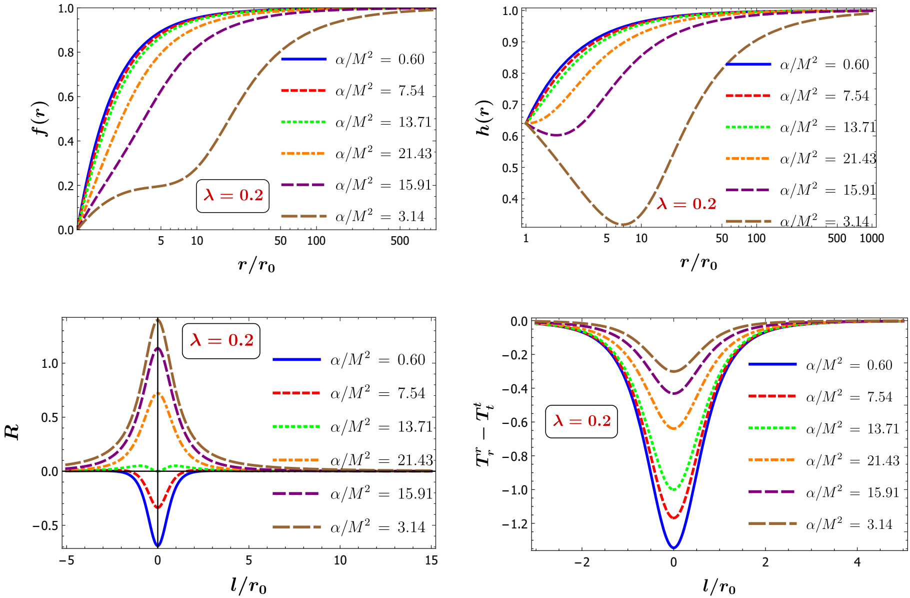
where $T_{\mu\nu}$ is the effective energy momentum tensor due to the scalar field defined from the equation $G_{\mu\nu} = T_{\mu\nu}$.



$F(l)$
$\lambda = 0.2$
$\alpha/r_0^2 = 0.1$
$\alpha/r_0^2 = 0.5$
$\alpha/r_0^2 = 1$
$\alpha/r_0^2 = 1.5$
$\alpha/r_0^2 = 2$
l/r_0
$H(l)$
$\lambda = 0.2$
$\alpha/r_0^2 = 0.1$
$\alpha/r_0^2 = 0.5$
$\alpha/r_0^2 = 1$
$\alpha/r_0^2 = 1.5$
$\alpha/r_0^2 = 2$
l/r_0
$\phi(l)$
$\lambda = 0.2$
$\alpha/r_0^2 = 0.1$
$\alpha/r_0^2 = 0.5$
$\alpha/r_0^2 = 1$
$\alpha/r_0^2 = 1.5$
$\alpha/r_0^2 = 2$
l/r_0
$T^r_r - T^t_t$
$\lambda = 0.2$
$\alpha/r_0^2 = 0.1$
$\alpha/r_0^2 = 0.5$
$\alpha/r_0^2 = 1$
$\alpha/r_0^2 = 1.5$
$\alpha/r_0^2 = 2$
l/r_0

The domain of existence




$f(r)$
r/r_0
$\alpha/M^2 = 0.60$
$\alpha/M^2 = 7.54$
$\alpha/M^2 = 13.71$
$\alpha/M^2 = 21.43$
$\alpha/M^2 = 15.91$
$\alpha/M^2 = 3.14$
$\lambda = 0.2$
$h(r)$
r/r_0
R
l/r_0
$T^r_r - T^t_t$
l/r_0

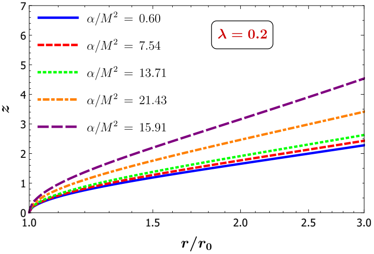
The Embedding Diagram

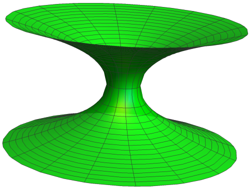
A very useful way to visualize the geometry of a given manifold is the construction of the corresponding embedding diagram.

Since our solutions are static and spherically symmetric, we may simplify the analysis by choosing $t = const$ and $\theta = \pi/2$.

$$\frac{dr^2}{f} + r^2 d\varphi^2 = dz^2 + d\rho^2 + \rho^2 d\varphi^2,$$

$$\text{with} \qquad \rho = r, \qquad \left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 = \frac{1}{f}.$$





Conclusions

- Regular analytic wormhole solutions were found for a class of the beyond Horndeski theories.
- The profile for the scalar field, in all of the cases considered, was found to be regular over the entire radial domain.
- The solutions are traversable and **do not need phantom field or exotic matter**.
- The solutions are continuous at the throat and they do not need a thin-shell of matter.
- For every wormhole solution the domain of existence was found.

Thank You!

The field equations

For a spherically symmetric line-element

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

The field equations of the beyond Horndeski theory are:

The (tt) equation:

$$G_2 + f\phi' X' G_{3X} + \frac{2}{r}\left(\frac{1-f}{r} - f'\right)G_4 + \frac{4}{r}f\left(\frac{1}{r} + \frac{X'}{X} + \frac{f'}{f}\right)XG_{4X}$$
$$+\frac{8}{r}fXX'G_{4XX} + \frac{1}{r^2}f\phi'\left[(1-3f)\frac{X'}{X} - 2f'\right]XG_{5X} - \frac{2}{r^2}f^2\phi' XX'G_{5XX}$$
$$+\frac{16}{r}fX^2X'F_{4X} + \frac{8}{r}f\left(4\frac{X'}{X} + \frac{f'}{f} + \frac{1}{r}\right)X^2F_4$$
$$+\frac{12}{r^2}f^2\phi' X^2\left(\frac{2f'}{f} + \frac{5X'}{X}\right)F_5 + \frac{24}{r^2}f^2\phi' X^2X'F_{5X} = 0.$$

The radial component of the current J^r associated to the shift symmetry:

$$\begin{aligned} J^r = &-f\phi' G_{2X} - f\frac{rh'+4h}{rh} X G_{3X} + 2f\phi'\frac{fh-h+rfh'}{r^2h} G_{4X} \\ &+ 4f^2\phi'\frac{h+rh'}{r^2h} X G_{4XX} - fh'\frac{1-3f}{r^2h} X G_{5X} + 2\frac{h'f^2}{r^2h} X^2 G_{5XX} \\ &+ 8f^2\phi\frac{h+rh'}{r^2h} X(2F_4 + XF_{4X}) - 12\frac{f^2h'}{r^2h} X^2(5F_5 + 2XF_{5X}) = 0. \end{aligned}$$

Rather than the (rr) equation itself we use the combination $(rr) - J^r\partial^r\phi$:

$$\begin{aligned} &G_2 - \frac{2}{r^2h}(frh' + fh - h)G_4 + \frac{4f}{r^2h}(rh'+h)XG_{4X} - \frac{2}{r^2h}f^2h'\phi' X G_{5X} \\ &\frac{8f}{r^2h}(rh'+h)X^2F_4 + \frac{24}{r^2h}f^2h'\phi' X^2 F_5 = 0. \end{aligned}$$

GB near throat solutions

We assume that there is an extremum at $\eta = 0$ (throat or equator). By taking the expansion near the throat/equator we find that our solutions have a polynomial form

$$e^{f_0} = a_0(1 + a_1\eta + a_2\eta^2 + ...), \qquad e^{f_1} = b_0(1 + b_1\eta + b_2\eta^2 + ...)$$
$$\phi = \phi_0 + \phi_1\eta + \phi_2\eta^2 + ...$$

The condition for an extremum at $\eta = 0$ is $f_1'(0) = 0$ which implies $b_1 = 0$.

$$a_1 = -\frac{b_0\left(\eta_0^2\phi_1^2 + 4\right)}{8\dot{F}_0\phi_1}$$

$$a_2 = \frac{b_0\left[4b_0\dot{F}_0^2\phi_1^2\left(\eta_0^2\phi_1^2 + 4\right)^2 + b_0^3\eta_0^2\left(\eta_0^2\phi_1^2 + 4\right)^2 - 128\dot{F}_0^2\eta_0^2\phi_1^6\ddot{F}_0\right]}{256\dot{F}_0^2\phi_1^2\left(b_0^2\eta_0^2 + 4\dot{F}_0^2\phi_1^2\right)},$$

$$b_2 = -\frac{2b_0\phi_1^2\ddot{F}_0}{b_0^2\eta_0^2 + 4\dot{F}_0^2\phi_1^2},$$

$$\phi_2 = -\frac{4b_0\dot{F}_0^2\phi_1^2\left(\eta_0^2\phi_1^2 + 4\right) + b_0^3\eta_0^2\left(\eta_0^2\phi_1^2 + 4\right) + 64\dot{F}_0^2\phi_1^4\ddot{F}_0}{32\left(b_0^2\dot{F}_0\eta_0^2 + 4\dot{F}_0^3\phi_1^2\right)}.$$

GB Solutions at infinity

In the limit $\eta \to \infty$ we demand that the space-time is flat and the scalar field assumes a constant value. We find:

$$e^{f_0(\eta)} = 1 - \frac{2M}{\eta} + \frac{MD^2}{12\eta^3} + \frac{24MD\dot{F} + M^2D^2}{6\eta^4} + \mathcal{O}\left(1/\eta^5\right),$$

$$e^{f_1(\eta)} = 1 + \frac{2M}{\eta} + \frac{16M^2 - D^2}{4\eta^2} + \frac{32M^3 - 5MD^2}{4\eta^3} + \mathcal{O}\left(1/\eta^4\right),$$

$$\phi(\eta) = \phi_\infty + \frac{D}{\eta} + \frac{MD}{\eta^2} + \frac{32M^2D - D^3}{24\eta^3} + \frac{12M^3D - 24M^2\dot{F} - MD^3}{6\eta^4} + \mathcal{O}\left(1/\eta^5\right).$$

For any form of the coupling function we may construct both a regular throat/equator and an asymptotically flat region.