Taking charge of O(N): from d=3 to d=6 via the large-charge expansion

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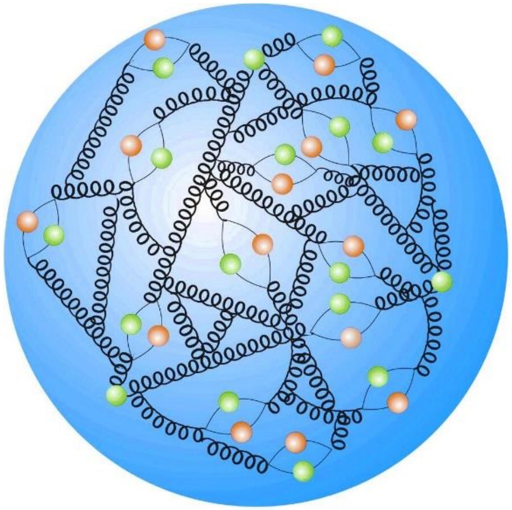
O. Antipin, JB, F. Sannino, Z. Wang and C. Zhang, *Phys. Rev. D* 102 (2020) 4, 045011.

O. Antipin, JB, F. Sannino, Z. Wang and C. Zhang, arXiv:2107.02528[hep-th]

Hrvatska zaklada za znanost

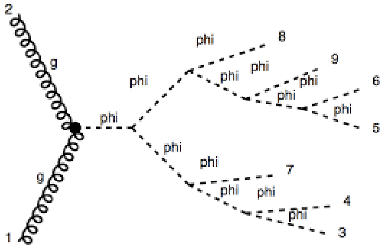
Solve QFT



Strongly-coupled QFTs
Non-perturbative physics

Weakly-coupled QFTs
Perturbative expansion: { **Many-loops** / **Many-legs** }

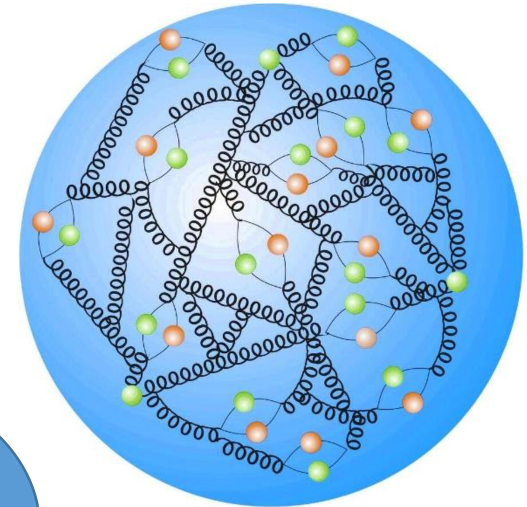
Rapid growth of the number of Feynman diagrams, with the number of loops/ external legs



The perturbative expansion diverges.

Evidence of violation of perturbative unitarity in multi-boson (H, W, Z) production processes at E≈100 TeV.

Solve QFT



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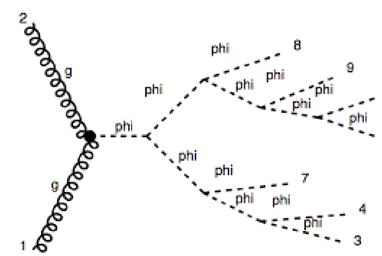

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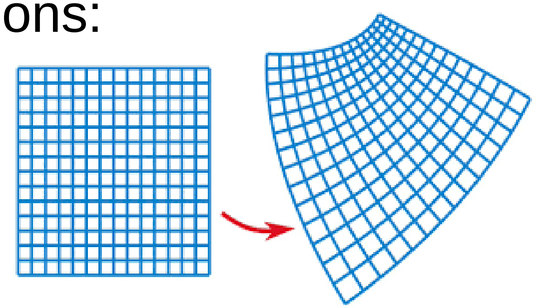
Conformal field theories

CFT = QFT invariant under conformal transformations: transformations which locally preserve angles.



CFTs are defined by their CFT data:

$$\{\mathcal{O}_i \ , \{\Delta_i, C_{ij}^k\}\}$$

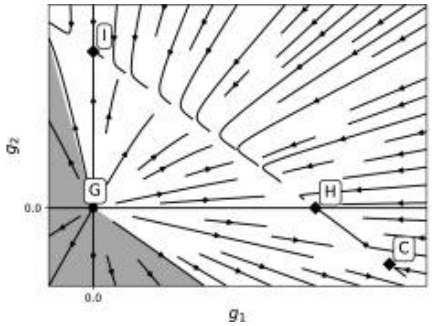
Primary operators

Scaling dimensions

OPE coefficients

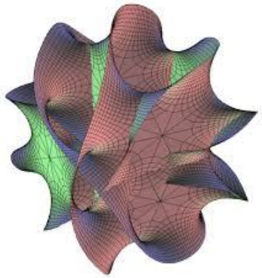
$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle = \frac{C_{ij}^k}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|y-z|^{\Delta_j+\Delta_k-\Delta_i}|z-x|^{\Delta_k+\Delta_i-\Delta_j}}$$

Extrema of the RG flow

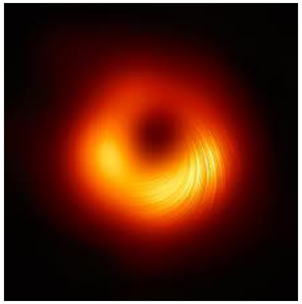


Critical phenomena

String theory



Quantum gravity



Solve CFT

- CFT (QFT) simplifies in certain limits when a small/large parameter exists. (Perturbative expansion)

- Our large parameter(s): conserved charge(s) of the internal symmetry group of the CFT:

LARGE-CHARGE EXPANSION FOR CFT DATA

Quantum VS Classical

Quantum physics *"classicalizes"* in the presence of large quantum numbers.

Hydrogen atom with infinite mass of the proton
at fixed magnetic quantum number m:

QUANTUM ground state energy:

$$E_0^{\mathrm{QM}}(m) = -\frac{M_e \alpha^2}{2(m+1)^2}$$

CLASSICAL ground state energy:

$$E_0^{cl}(m) = -\frac{M_e \alpha^2}{2m^2}$$

$$\boxed{\lim_{m\to\infty} \left(E_0^{\mathrm{QM}}(m) - E_0^{\mathrm{cl}}(m)\right) = 0}$$

LARGE-CHARGE EXPANSION =
SEMICLASSICAL EXPANSION

Charging the O(N) model

- We study the O(N) scalar theory in d= 4-ε dimensions where it features an infrared Wilson-Fisher fixed point

$$\mathcal{S} = \int d^d x \left(\frac{(\partial \phi_i)^2}{2} + \frac{(4\pi)^2 g_0}{4!} (\phi_i \phi_i)^2 \right) \qquad g^*(\epsilon) = \frac{3\epsilon}{8+N} + \mathcal{O}\left(\epsilon^2\right)$$

$\epsilon = \mathbf{0}$ and $\mathbf{N} = \mathbf{4}$

Standard Model Higgs

$\epsilon \to \mathbf{1}$

Superfluid He^4, Magnets,

Superconductors, ..

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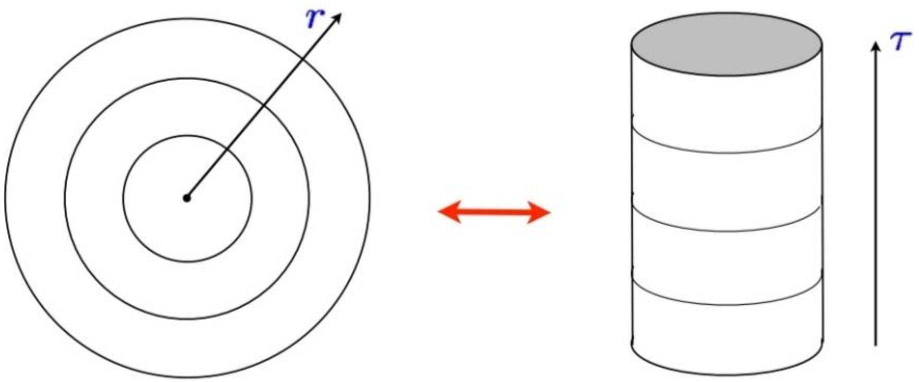
- Consider even N: we can fix up to N/2 charges, which is the rank of the O(N) group. We fix them all.

Weyl map to the cylinder

$$\mathbb{R}^d \to \mathbb{R}\times S^{d-1}$$

$$r = Re^{\tau/R}$$

R is the radius of the sphere.



The eigenvalues of the dilation charge (the scaling dimensions) become the energy spectrum on the cylinder.

$$\mathbf{E} = \mathbf{\Delta}/\mathbf{R}$$

Minimal scaling dimension

- We compute the scaling dimension $\Delta_{\vec{Q}}$ of the operators carrying the charges Q_i and the minimal scaling dimension.
- i.e. we compute the ground state energy on the cylinder.

Hamiltonian

$$T = \tau_f - \tau_i$$

Normalization

$$< \vec{Q}|\underbrace{e^{-HT}}_{\text{Evolution operator}}|\vec{Q} > \underset{T\to\infty}{=} \tilde{\mathcal{N}} e^{-E_{\vec{Q}} T} = \tilde{\mathcal{N}} e^{-\frac{\Delta_{\vec{Q}}}{R} T}$$

Projection onto the ground state

Arbitrary state with charge

$$\vec{Q} \equiv (Q_1, Q_2, \dots)$$

Minimal scaling dimension

- We compute the scaling dimension $\Delta_{\vec{Q}}$ of the operators carrying the charges Q_i and the minimal scaling dimension.

These operators have classical scaling dimension Q and transform in the Q-indices traceless symmetric O(N) representations where

$$\boxed{Q \equiv \sum_{i=1}^{N/2} Q_i}$$

$$\underbrace{\square\square\square\square\square\cdots\square}_{Q}$$

is the total charge.

$$Q = 1 \rightarrow \phi_i \qquad\qquad Q = 2 \rightarrow \phi_i\phi_j - \frac{\delta_{ij}}{N}\phi^2$$

These operators represent anisotropic perturbations in O(N)-invariant systems. $\Delta_{\vec{Q}}$ define a set of crossover (critical) exponents measuring the stability of the system (e.g. magnets) against anisotropic perturbations (e.g. crystal structure).

Ground state

The classical solution with the minimal energy is spatially homogeneous.

$$\varphi_i = \frac{1}{\sqrt{2}}(\phi_{2i-1} + i\phi_{2i}) = \frac{1}{\sqrt{2}}\sigma_i e^{i\chi_i}$$

$$\sigma_i = A_i\,, \qquad \chi_i = -i\mu\tau$$

VEV

Chemical potential

Then we can rotate the charges as

$$\vec{Q} = \big(Q_1, Q_2, \ldots, Q_{N/2}\big) \to \big(Q, 0, 0, \ldots\big)$$

The sum of the charges acts as a single charge.

$$< Q|e^{-HT}|Q >= \frac{1}{\mathcal{Z}}\int D^{N/2}\sigma\, D^{N/2}\chi\, e^{-\mathcal{S}_{eff}[\sigma_i,\chi_i]} \qquad \mathcal{S}_{eff} = \mathcal{S} + \mu Q + \frac{1}{2}\left(\frac{d-2}{2R}\right)^2 \sigma_i\sigma_i$$

Charge-fixing

Conformal coupling

Computing the path integral semiclassically, we have

$$\mathbf{\Delta_Q = \sum_{k=-1}^{\infty} \frac{1}{Q^k}\Delta_k(\mathcal{A})} \qquad \mathcal{A} \equiv \mathbf{gQ}$$

Every Δ_k resums an infinite series of Feynman diagrams.

Leading order: Δ_{-1}

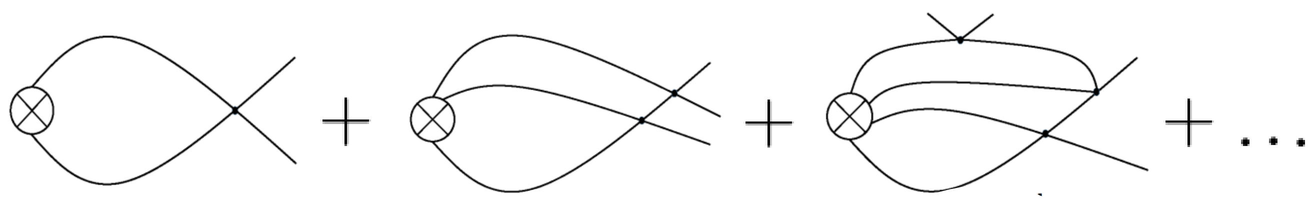
$$\mathcal{S} = \mathcal{S}(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 \mathcal{S}''(\phi_0) + \ldots$$

Given by the effective action evaluated on the classical solution of the EOM

$$\frac{4\Delta_{-1}(\mathcal{A})}{\mathcal{A}} = \frac{3^{\frac{2}{3}} x^{\frac{1}{3}}}{3^{\frac{1}{3}} + x^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}\left(3^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}{x^{\frac{1}{3}}} \qquad x \equiv 6\mathcal{A} + \sqrt{-3 + 36\mathcal{A}^2}$$

This classical result resums an infinite number of Feynman diagrams!

$$Q\Delta_{-1} =$$



$$\sim gQ^2 \qquad \sim g^2 Q^3 \qquad \sim g^3 Q^4$$

Q counts the number of external legs.

g counts the number of vertices.

Many-loops – Many-legs

Next-to-leading order: Δ_0

$$\mathcal{S} = \mathcal{S}(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 \mathcal{S}''(\phi_0) + \ldots$$

Δ_0 is given by the fluctuation determinant around the classical trajectory

$$\Delta_0 = \frac{R}{2} \sum_{\ell=0}^{\infty} n_\ell \left[\omega_+(\ell) + \omega_-(\ell) + \left(\frac{N}{2} - 1\right) (\omega_{++}(\ell) + \omega_{--}(\ell)) \right]$$

ℓ labels the eigenvalues of the momentum which have degeneracy n_ℓ.

$\omega_+ \; , \omega_-, \omega_{++}, \omega_{--}$ $\Longrightarrow$ Dispersion relations of the spectrum.

Boosting perturbation theory

By expanding the Δ_k's in the limit of small 't Hooft coupling A=gQ, we obtain the conventional perturbative expansion

Red terms: Δ_{-1}

Blue terms: Δ_0

$$
\begin{aligned}
\Delta_Q =\ & Q + \left(\frac{Q^2}{8+N} - \frac{(N+10)}{2(8+N)}Q\right)\epsilon \\
& - \left[\frac{2}{(8+N)^2}Q^3 + \frac{(N-22)(N+6)}{2(8+N)^3}Q^2 + \frac{184+N(14-3N)}{4(8+N)^3}Q\right]\epsilon^2 \\
& + \left[\frac{8}{(8+N)^3}Q^4 + \frac{-456-64N+N^2+2(8+N)(14+N)\zeta(3)}{(8+N)^4}Q^3\right. \\
& - \frac{-31136-8272N-276N^2+56N^3+N^4+24(N+6)(N+8)(N+26)\zeta(3)}{4(N+8)^5}Q^2 \\
& \left. + \frac{-65664-8064N+4912N^2+1116N^3+48N^4-N^5+64(N+8)(178+N(37+N))\zeta(3)}{16(N+8)^5}Q\right]\epsilon^3 \\
& + \left[c_5Q^5 + c_4Q^4 + c_3Q^3 + c_2Q^2 + c_1Q\right]\epsilon^4 + \left[e_6Q^6 + e_5Q^5 + e_4Q^4 + e_3Q^3 + e_2Q^2 + e_1Q\right]\epsilon^5 +
\end{aligned}
$$

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By expanding the Δ_k's in the limit of small 't Hooft coupling, we obtain the conventional perturbative expansion

Red terms: Δ_{-1}

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$$\Delta_Q = Q + \left(\frac{Q^2}{8+N} - \frac{(N+10)}{2(8+N)}Q\right)\epsilon$$
$$-\left[\frac{2}{(8+N)^2}Q^3 + \frac{(N-22)(N+6)}{2(8+N)^3}Q^2 + \frac{184+N(14-3N)}{4(8+N)^3}Q\right]\epsilon^2$$
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Complete 4-loop ($O(\varepsilon^4)$) scaling dimension obtained by combining our results with the known perturbative results for Q= 1, 2, 4.

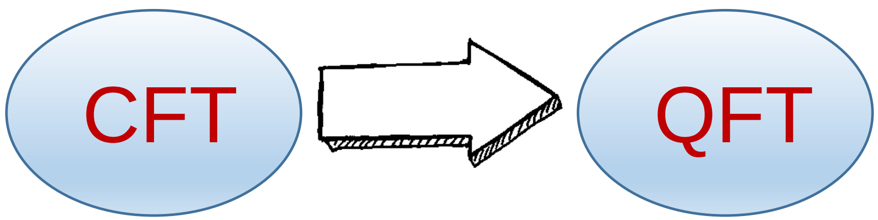
Infinite number of checks for future diagrammatic computations

d=4

We can rewrite the result in terms of the coupling and take $\epsilon = 0$

$$\Delta_Q = Q + \frac{1}{3} g Q (Q-1) - \frac{1}{18} g^2 \left(4Q^3 + (N-6)Q^2 + \frac{1}{2}(2-3N)Q \right) + \mathcal{O}\left(g^3\right)$$

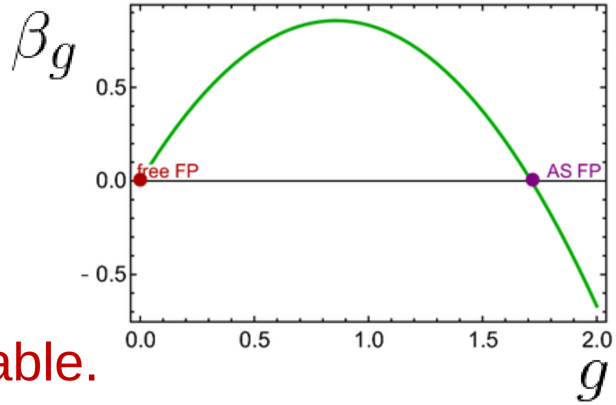
This result is correct for arbitrary values of the coupling (not only the fixed point one).



We are computing RG functions (anomalous dimensions) of Higgs operators. For N=4 and Q=1 we have the Higgs field itself.

The O(N) model in 4<d<6

$$\mathcal{S} = \int d^d x \left(\frac{(\partial\phi_i)^2}{2} + \frac{(4\pi)^2 g_0}{4!} (\phi_i\phi_i)^2 \right)$$



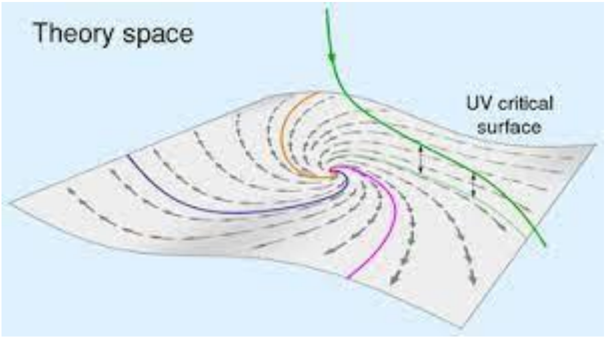
In d > 4 the quartic interaction is non-renormalizable.

In this range, the model is studied in the large N expansion.

At large N the model flows to an interacting fixed point in the UV.

Non-perturbative renormalizability

Asymptotic safety



Non-SUSY 5D CFT

AdS/CFT

O(N) model in d dimension/higher-spin gauge theories on AdS_{d+1}

The cubic O(N) model

Conjecture: in 4<d<6 the UV fixed point of the quartic O(N) model is equivalent to the IR fixed point of the cubic O(N) model below

$$\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 + \frac{1}{2}(\partial\eta)^2 + \frac{g_0}{2}\eta(\phi_a)^2 + \frac{h_0}{6}\eta^3$$

N+1 fields and cubic interactions. Weakly coupled near d=6

A priori these are two different universality classes (CFTs).

We test the conjecture using the large-charge expansion

We compute Δ_Q in the cubic model in d=6-ε and compare with large N results in the quartic model

Cubic

$$\Delta_Q = \sum_{k=-1} \frac{1}{Q^k}\Delta_k(\mathcal{A})$$

$$\mathcal{A} \equiv Q\epsilon$$

Quartic

$$\Delta_Q = \sum_{k=-1} \frac{1}{N^k}F_k(J)$$

$$J \equiv Q/N$$

Comparison

Cubic

$$\Delta_Q = \sum_{k=-1} \frac{1}{Q^k} \Delta_k(\mathcal{A})$$

$$\mathcal{A} \equiv Q\epsilon$$

We computed Δ_{-1} and Δ_0.

O. Antipin, JB, F. Sannino, Z. Wang and C. Zhang, arXiv:2107.02528[hep-th]

Quartic

$$\Delta_Q = \sum_{k=-1} \frac{1}{N^k} F_k(J)$$

$$J \equiv Q/N$$

F_{-1} is known.

Simone Giombi and Jonah Hyman, arXiv:2011.11622[hep-th]

The overlapping terms between the two expansions are

- $\alpha_j Q \left(\frac{Q\epsilon}{N}\right)^j, \qquad j \geq 0$ — Match between Δ_{-1} and F_{-1}
- $\beta_j \left(\frac{Q\epsilon}{N}\right)^j, \qquad j \geq 0$ — Match between Δ_0 and F_{-1}

The convex charge conjecture

Convex charge conjecture: every unitary CFT satisfies

$$\Delta_{Q_1+Q_2} \geq \Delta_{Q_1} + \Delta_{Q_2}$$

O. Aharony and E. Palti 2108.04594

Holographic version of the weak gravity conjecture:

"exists at least one charged particle in the theory which has a non-negative self-binding energy"

O(N) model in 2<d<4: satisfies the conjecture

O(N) model in 4<d<6: violates the conjecture

Multi-boson production

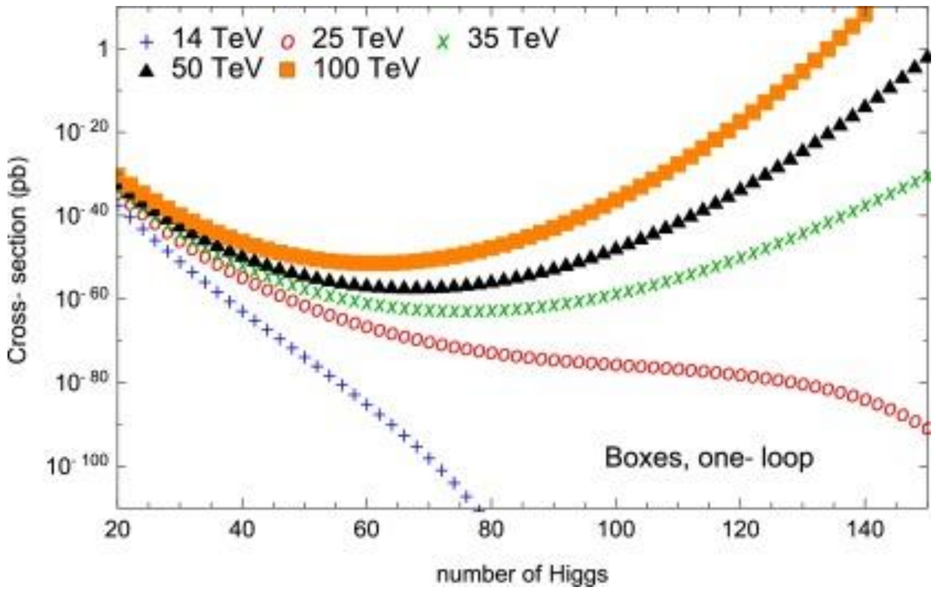
$$\boxed{\lambda\phi^4}$$

Consider the $1 \to n$ amplitude

$$A^{tree} = n!\lambda^{\frac{n-1}{2}} e^{-\frac{5}{6}En}$$

$$A = A^{tree} e^{B\lambda n}$$

$$\sigma(1 \to n) = e^{F(\lambda n, E)}$$



[Degrande, Khoze, Mattelaer, 2016]

$$\boxed{n \approx \sqrt{s}/m}$$