t-channel singularities

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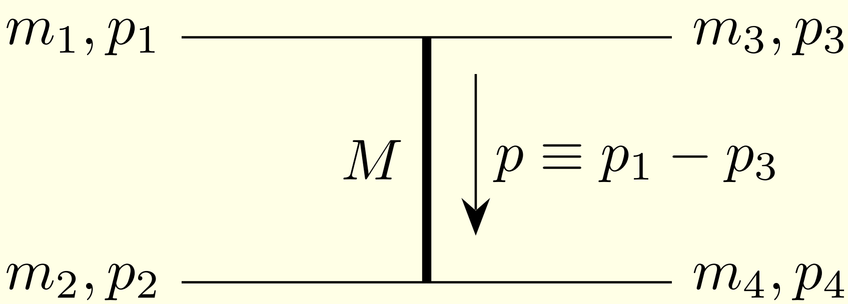
- Conditions
- SM examples
- Earlier regularization attempts
- Cosmology - medium regularization
- Summary

◇ BG, Michał Iglicki, Stanisław Mrówczyński, "t-channel singularities in cosmology and particle physics", e-Print: 2108.01757

Conditions



$$\mathcal{M} \sim \frac{1}{t - M^2}, \quad t \equiv p^2 = (p_1 - p_3)^2$$

$$t_{\rm min}(s) < M^2 < t_{\rm max}(s) \quad \Longrightarrow \quad \sigma_{\rm tot} \sim \int_{t_{\rm min}(s)}^{t_{\rm max}(s)} \frac{dt}{(t - M^2)^2} \to \infty$$

$$t = M^2 \iff \cos(\theta_{13}) = \frac{2E_1E_3 - m_1^2 - m_3^2 + M^2}{2\sqrt{E_1^2 - m_1^2}\sqrt{E_3^2 - m_3^2}} \, .$$

$$|\cos(\theta_{13})| \leq 1 \iff \sqrt{s_1} \leq \sqrt{s} \leq \sqrt{s_2}$$

$$s_{1,2} \equiv \frac{(-\beta \mp \sqrt{\Delta})}{2\alpha}, \quad \alpha \equiv M^2 \quad \text{and} \quad \Delta \equiv \beta^2 - 4\alpha\gamma \geq 0$$

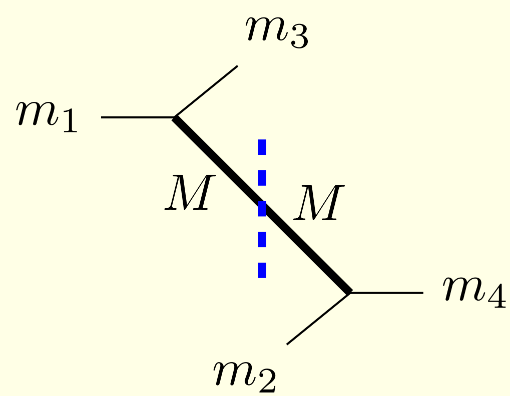
$$\beta \equiv M^4 - M^2(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2),$$

$$\gamma \equiv M^2(m_1^2 - m_2^2)(m_3^2 - m_4^2) + (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 - m_2^2 - m_3^2 + m_4^2).$$

- For an elastic scattering $m_1 = m_3$ and $m_2 = m_4$ and $t < 0$, so there is t-channel singularity.

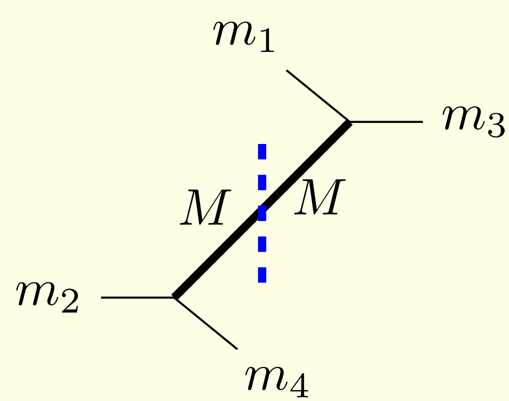
- t-channel singularity appears iff exactly one particle in the initial and in the final state is unstable and the other is stable*.

$$m_1 > M + m_3 \quad \text{and} \quad m_4 > M + m_2$$



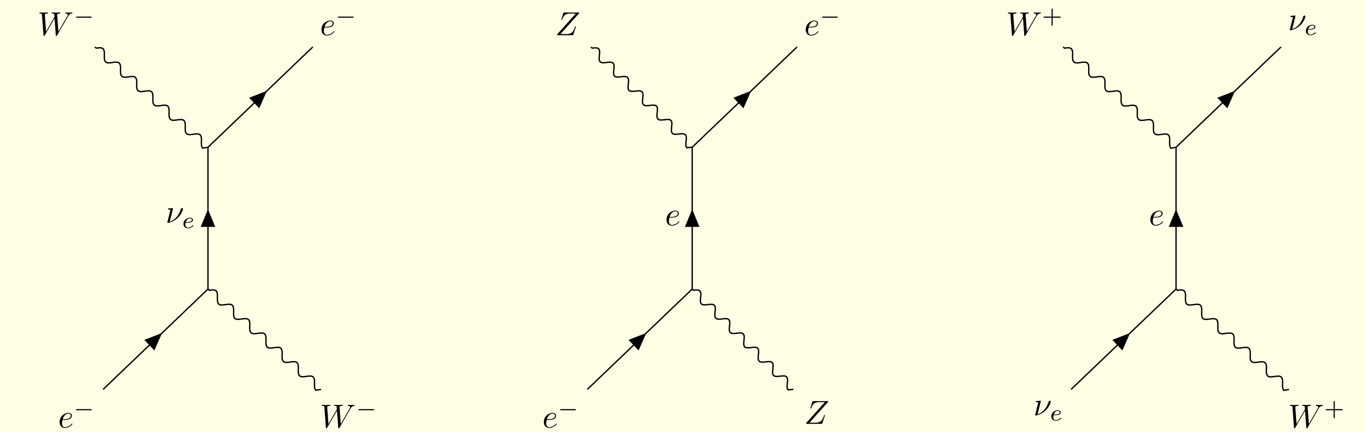
or

$$m_2 > M + m_4 \quad \text{and} \quad m_3 > M + m_1$$

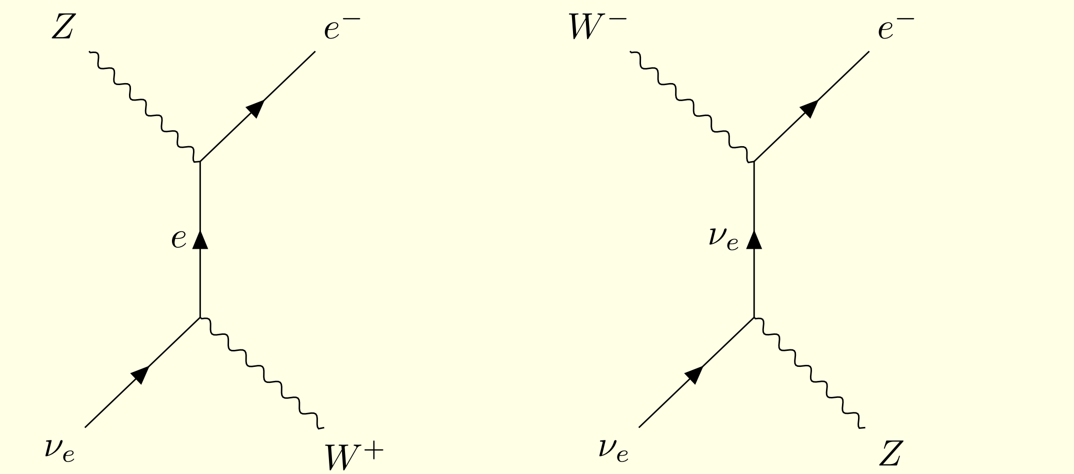


$\star$ A special case of the Coleman-Norton theorem, "Singularities in the Physical Region", Il Nuovo Cimento vol. XXXVIII, no 1, 1965, p. 438.

SM examples



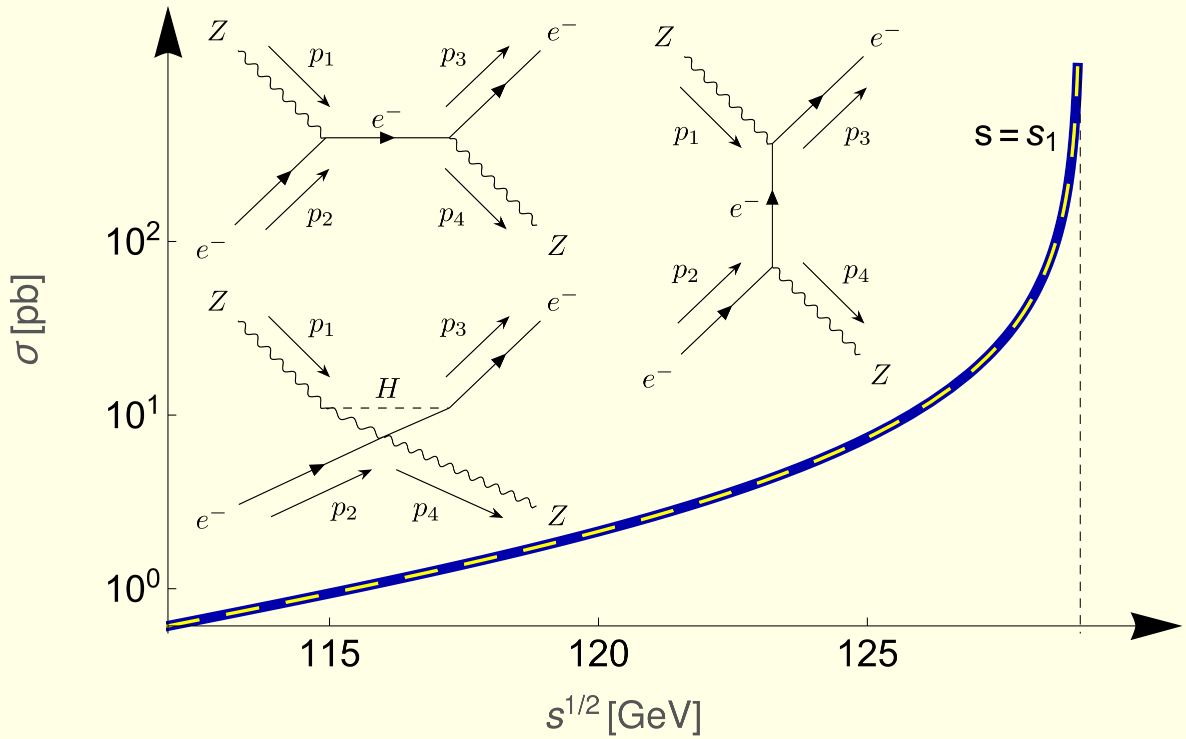
elastic scattering



inelastic scattering

† Hadron binary processes e.g $\pi N^\star \to N^\star \pi$: R. F. Peierls, "Possible Mechanism for the Pion-Nucleon Second Resonance", Phys. Rev. Lett. 6, 641 (1961).

$*$ $\nu_e W^+ \to W^+ \nu_e$: M. Nowakowski and A. Pilaftsis, "On Gauge Invariance of Breit-Wigner Propagators", Z. Phys. C 60, 121 (1993).



$$\sqrt{s_1} \le \sqrt{s} \le \sqrt{s_2},$$

$$\sqrt{s_1} \simeq \sqrt{2} m_Z$$
$$\sqrt{s_2} \simeq m_Z^2/m_e \sim 1.6 \cdot 10^7 \text{ GeV}$$

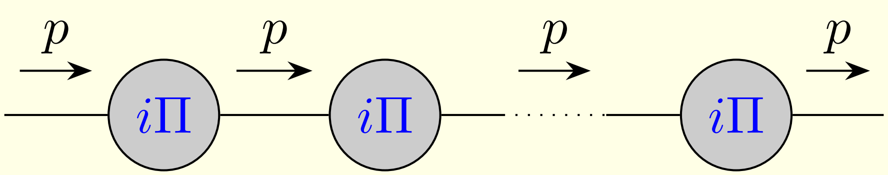
Figure 1: The total cross section $\sigma_{Ze\to eZ}$.

The analog of the Klein-Nishina formula for the weak Compton scattering for $m_e = 0$ for $m_Z^2 \lesssim s \lesssim 2m_Z^2$:

$$\frac{d\sigma_{Ze\to eZ}}{dt} = g^4 \, \frac{g_v^4 + 6 g_a^2 g_v^2 + g_a^4}{12\,\pi\, c_w^4} \frac{(m_Z^4 - s\,t)}{(s - m_Z^2)^2} \times \frac{m_Z^4\, s^2 + 4 m_Z^2 (2 m_Z^2 - s)\, st + (m_Z^4 - 4 m_Z^2 s + s^2)\, t^2}{m_Z^4\, s^2\, t^2},$$

Earlier regularization attempts

The Breit-Wigner propagator



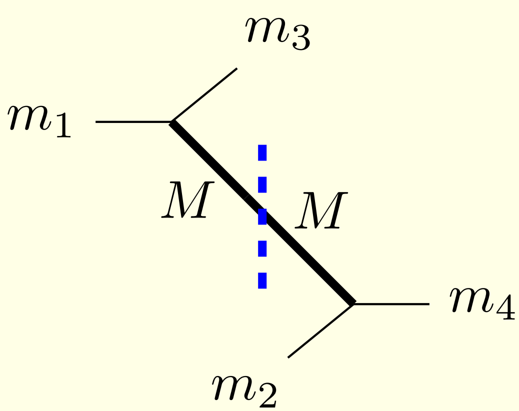
$$\mathcal{M} \sim \frac{i}{p^2 - M^2} \sum_{k=0}^{\infty} \left(i\Pi(p^2) \frac{i}{p^2 - M^2} \right)^k$$

$$= \frac{i}{p^2 - M^2 + \Pi(p^2)}, \quad \mathrm{Im}\,\Pi(M^2) = M\Gamma$$

Problem: for a stable mediator $\Gamma = 0$

t-channel singularities emerge when some initial or final states are unstable, therefore indeed one can expect some difficulties since the corresponding asymptotic states do not exist and the perturbative expansion is jeopardized.

Complex mass of unstable particles



in the rest-frame of particle 1:

$$e^{im_1 t} \to e^{im_1 t - \Gamma_1 t} = e^{i\tilde{m}_1 t}, \quad \tilde{m}_1 \equiv m_1 \left(1 + i\frac{\Gamma_1}{m_1}\right)$$

after Lorentz boost:

$$p_1 \to \tilde{p}_1 \equiv p_1 \left(1 + i\frac{\Gamma_1}{m_1}\right)$$

- Conclusion: t is complex*, so $t \neq m^2$, however, it turns out that the complex momentum is not conserved.

- Quantized free unstable particles† $\Longrightarrow$ incorrect relativistic elongation of lifetime

★ Ilia Ginzburg, "Initial Particle Instability in Muon Collisions", Nucl. Phys. B Proc. Suppl. 51, 85 (1996).

† Hideji Kita and Yoko Kawai, "Quantized Free Unstable Particle Field", Progress of Theoretical Physics, Vol. 31, No. 2, 1964.

Finite beam width*

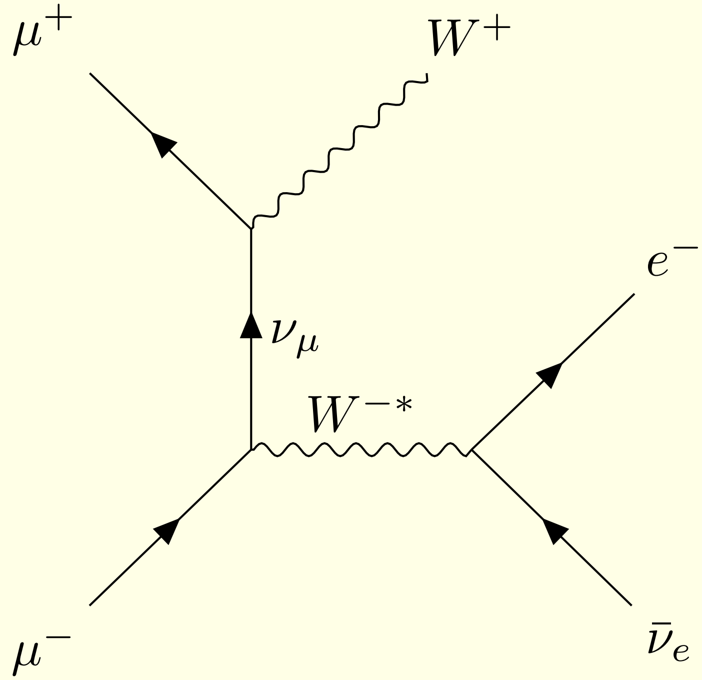


Figure 2: t-channel singularity in $\mu^+\mu^- \to W^+W^{-\,\star} \to W^+e^-\bar{\nu}_e$

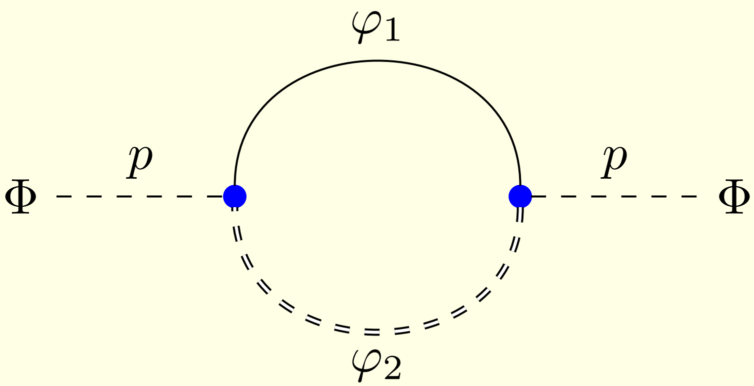
$$\int \frac{dt}{|t - M^2 + i\varepsilon|^2} \to \int \frac{dt}{(t - \Pi - M^2 + i\varepsilon)(t + \Pi - M^2 - i\varepsilon)}$$

where $\Pi \propto 1/a$ where $a^{-1} \sim$ beam width.

- Conclusion: the strategy is not applicable in cosmological context.

⋆ K. Melnikov and V. G. Serbo, "New type of beam size effect and the W boson production at $\mu^+\mu^-$ colliders", Phys. Rev. Lett. 76, 3263 (1996), C. Dams and R. Kleiss, "Muon colliders, Monte Carlo and gauge invariance", Eur. Phys. J. C 29, 11 (2003)

Cosmology - medium regularization



$$\mathcal{L} = \frac{1}{2}\left[(\partial^\mu\varphi_1)(\partial_\mu\varphi_1) - m_1^2\varphi_1^2\right] + \frac{1}{2}\left[(\partial^\mu\varphi_2)(\partial_\mu\varphi_2) - m_2^2\varphi_2^2\right] + \frac{1}{2}\left[(\partial^\mu\Phi)(\partial_\mu\Phi) - M^2\Phi^2\right] + \mu\,\varphi_1\varphi_2\Phi\ ,$$

- Non-zero imaginary part of Φ-self-energy emerges as a result of thermal interactions with medium of particles in thermal equilibrium (Keldysh-Schwinger formalism)

$$\Pi^+(p,T) = \frac{i}{2}\mu^2 \int \frac{d^4k}{(2\pi)^4}\Big[\Delta_1^+(k+p)\,\Delta_2^{\rm sym}(k,T) + \Delta_1^{\rm sym}(k,T)\,\Delta_2^-(k-p)\Big],$$

$$\Delta^\pm(p) = \frac{1}{p^2 - m^2 \pm i\,{\rm sgn}(p_0)\,\varepsilon},$$

$$\Delta^{\rm sym}(p,T) = -\frac{i\pi}{E_p}\left[\delta(E_p - p_0) + \delta(E_p + p_0)\right] \times \left[2f(E_p,T) + 1\right]$$

where $E_p \equiv \sqrt{\vec{p}^2 + m^2}$ and $f(E_p,T)$ is the Bose-Einstein distribution function $f(E_p,T) = (e^{\beta E_p} - 1)^{-1}$ with $\beta \equiv 1/T$.

$$\Pi^+(p,T) = \frac{i}{2}\mu^2 \int \frac{d^4k}{(2\pi)^4}\Big[\Delta_1^+(k+p)\,\Delta_2^{\rm sym}(k,T) + \Delta_1^{\rm sym}(k,T)\,\Delta_2^-(k-p)\Big],$$

$$\Sigma(|\vec{p}|,T) \equiv {\rm Im}\,\Pi^+(|\vec{p}|,T) = -\frac{\mu^2}{16\pi}\frac{1}{\beta|\vec{p}|}\left[\ln\frac{e^{\beta(A+C)}-1}{e^{\beta A}-1} - \ln\frac{e^{\beta(A+B+C)}-1}{e^{\beta(A+B)}-1}\right],$$

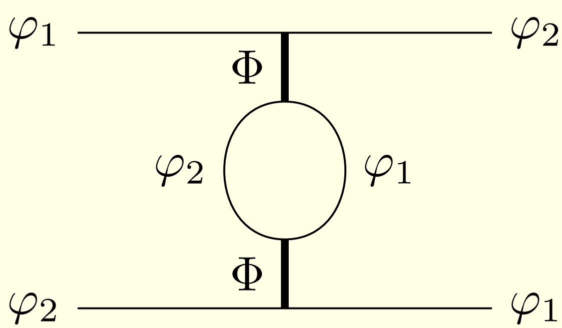
where

$$A \equiv \frac{(m_1^2 - m_2^2 - M^2)E_p - 2Mk_*\,|\vec{p}|}{2M^2}, \qquad B \equiv E_p, \qquad C \equiv \frac{2\,k_*\,|\vec{p}|}{M}$$

$$E_p \equiv \sqrt{\vec{p}^2 + M^2}, \qquad k_* \equiv \frac{\sqrt{\lambda(m_1^2, m_2^2, M^2)}}{2M}, \qquad \lambda - \text{Källén function}$$

- Effective thermal width emerges: $\Gamma_{\rm eff}(|\vec{p}|,T) \equiv M^{-1}\Sigma(|\vec{p}|,T)$.

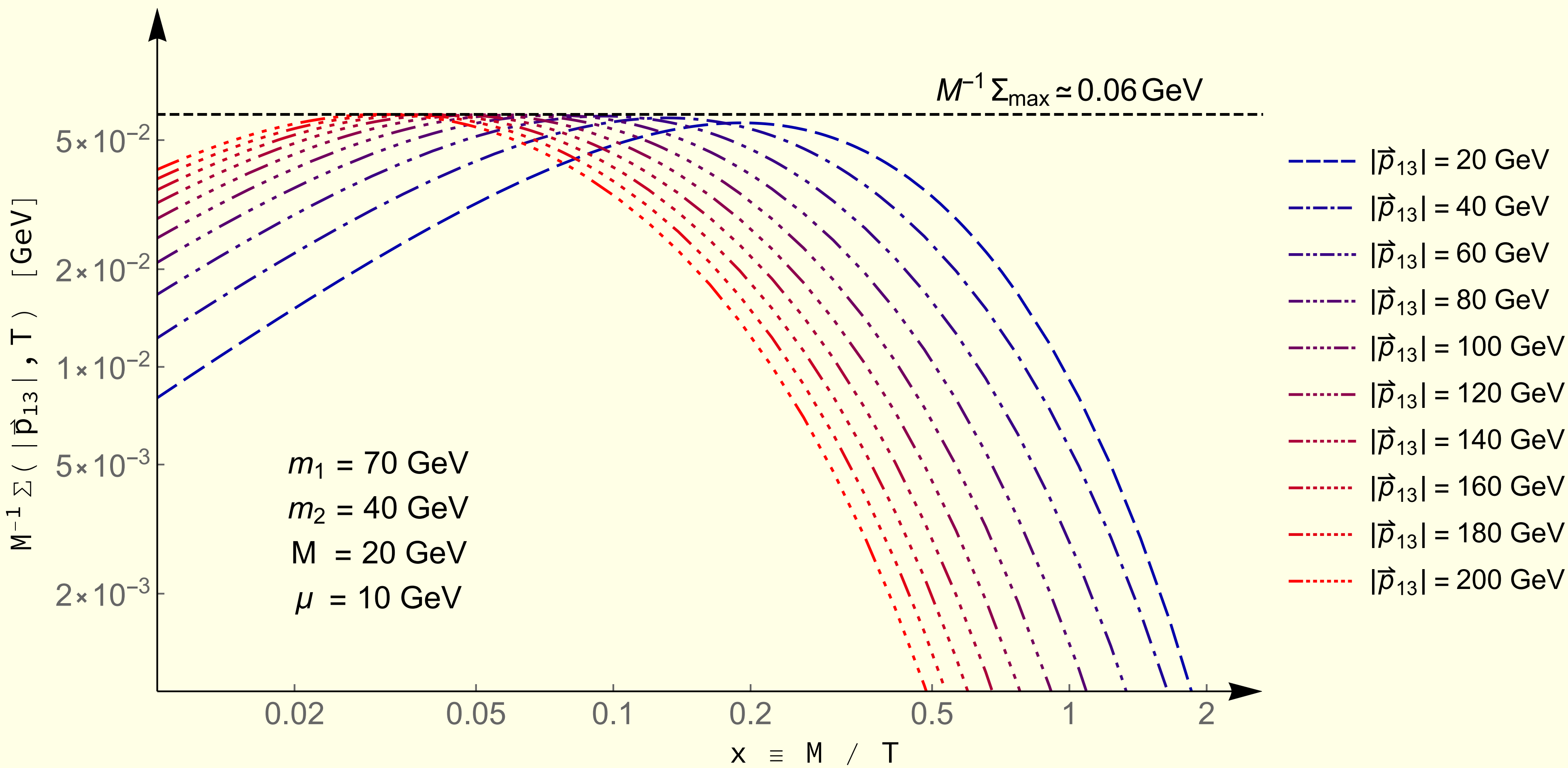
$$\Sigma(|\vec{p}|,T) \overset{\beta A \gtrsim 3}{\approx} -\frac{\mu^2}{16\pi}\frac{e^{-\beta A}}{\beta|\vec{p}|} \times \left(1 - e^{-\beta B} - e^{-\beta C} + e^{-\beta(B+C)}\right)$$



$$\Longrightarrow \quad \frac{1}{(t-M^2)^2} \to \frac{1}{(t-M_{\text{ren}}^2)^2+\Sigma^2(|\vec{p}|,T)}$$

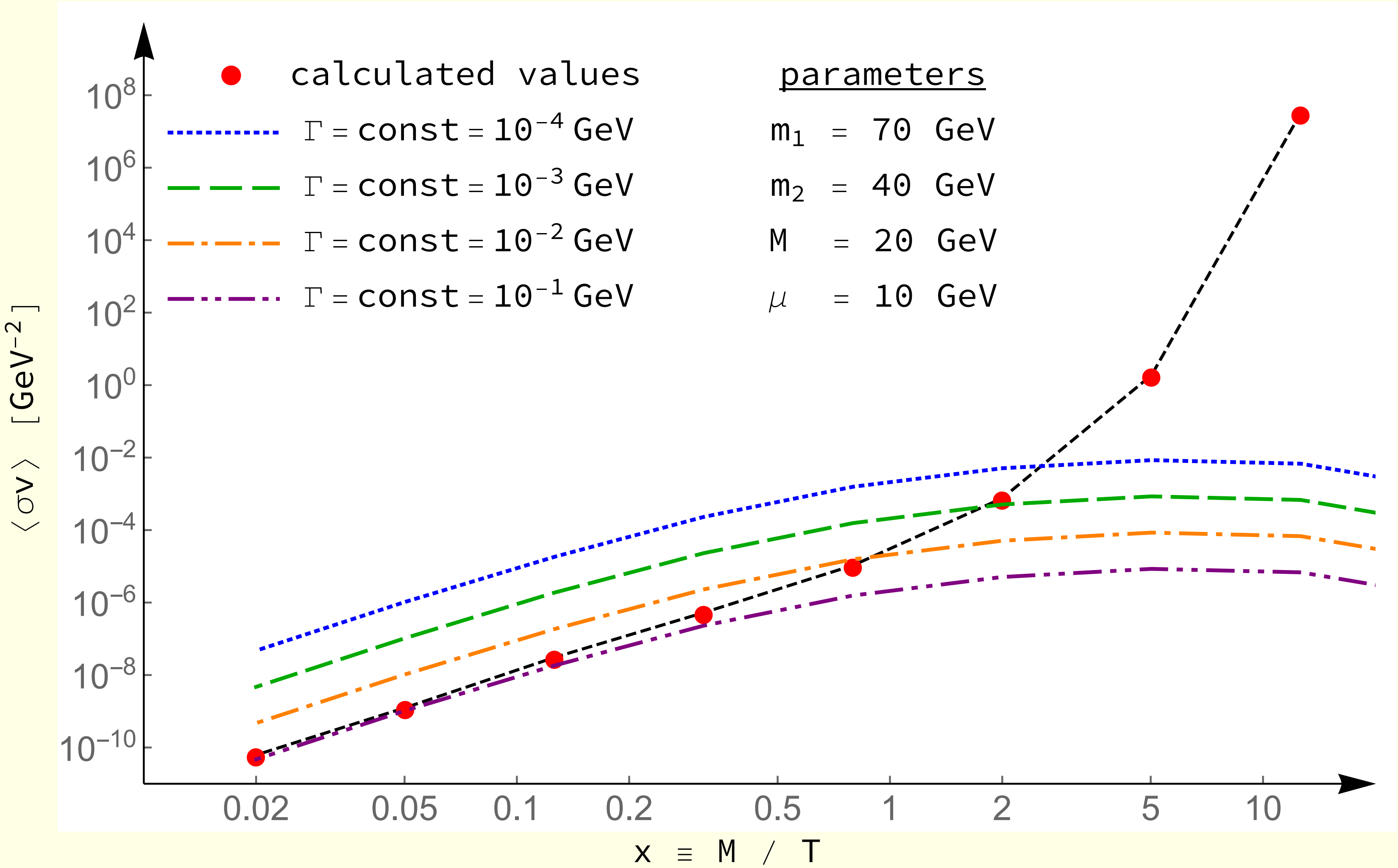
$$\Gamma_{\text{eff}}(|\vec{p}|,T) \equiv M^{-1}\Sigma(|\vec{p}|,T)$$

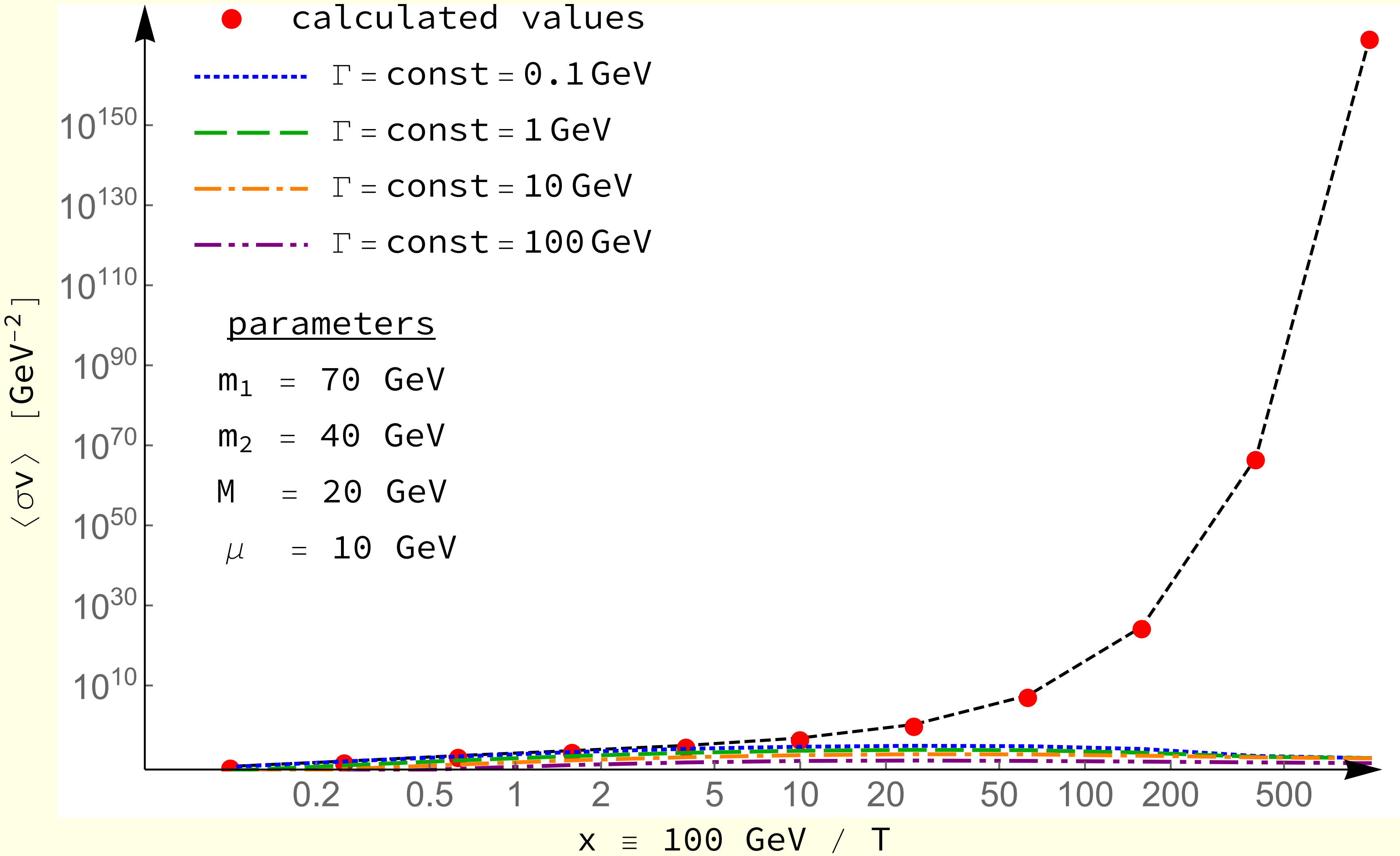
$$p_0 = E_p = \sqrt{\vec{p}^2+M^2}$$



$$\langle\sigma v\rangle(T) = \int d\Pi_1\, d\Pi_2\, \frac{f_1(E_1,T)}{n_1(T)}\,\frac{f_2(E_2,T)}{n_2(T)}$$

$$\times \int d\Pi_3\, d\Pi_4\, \frac{\mu^4}{(t-M^2)^2+\Sigma^2(|\vec{p}_{13}|,T)}(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4)$$




calculated values
Γ = const = 0.1 GeV
Γ = const = 1 GeV
Γ = const = 10 GeV
Γ = const = 100 GeV
parameters
m1 = 70 GeV
m2 = 40 GeV
M = 20 GeV
μ = 10 GeV
⟨σv⟩ [GeV−2]
10150
10130
10110
1090
1070
1050
1030
1010
0.2
0.5
1
2
5
10
20
50
100
200
500
x ≡ 100 GeV / T

Summary

- Conditions for the t-channel singularity to appear have been formulated.
- The singular Standard Model process $Ze^- \to e^- Z$ has been discussed in detail.
- It has been argued that the medium naturally regulates the singularity as particles become quasi-particles with a finite width.
- The singularities appear naturally in multicomponent dark matter scenarios, e.g., in the vector-fermion model, so they could be adopted in models of strongly interacting dark matter.

Backup slides

$Ze^- \to e^- Z$: the total cross-section

Note that in the limit $m_e \to 0$ the t-channel divergence appears at $t = 0$, $s_1 = 2m_Z^2$ and $s_2 = \infty$.

$$\sigma_{Ze\to eZ}(s) = \int_{2m_Z^2-s}^{m_Z^4/s} \frac{d\sigma}{dt}\, dt = g^4\, \frac{g_v^4 + 6g_a^2 g_v^2 + g_a^4}{24\,\pi\, c_w^4}$$

$$\times \left[\frac{2\, m_Z^{10} - 13\, m_Z^8\, s - 6\, m_Z^6\, s^2 + 12\, m_Z^4\, s^3 - s^5}{m_Z^4\, s^3\, (2m_Z^2 - s)} + \right.$$

$$\left. -2\, \frac{16\, m_Z^6 - 16\, m_Z^4 s + 2\, m_Z^2 s^2 + s^3}{s\,(2m_Z^2 - s)}\, \frac{\ln \frac{s(2m_Z^2-s)}{m_Z^4}}{(s - m_Z^2)^2}\right].$$

$\mu^+\mu^-$ collisions

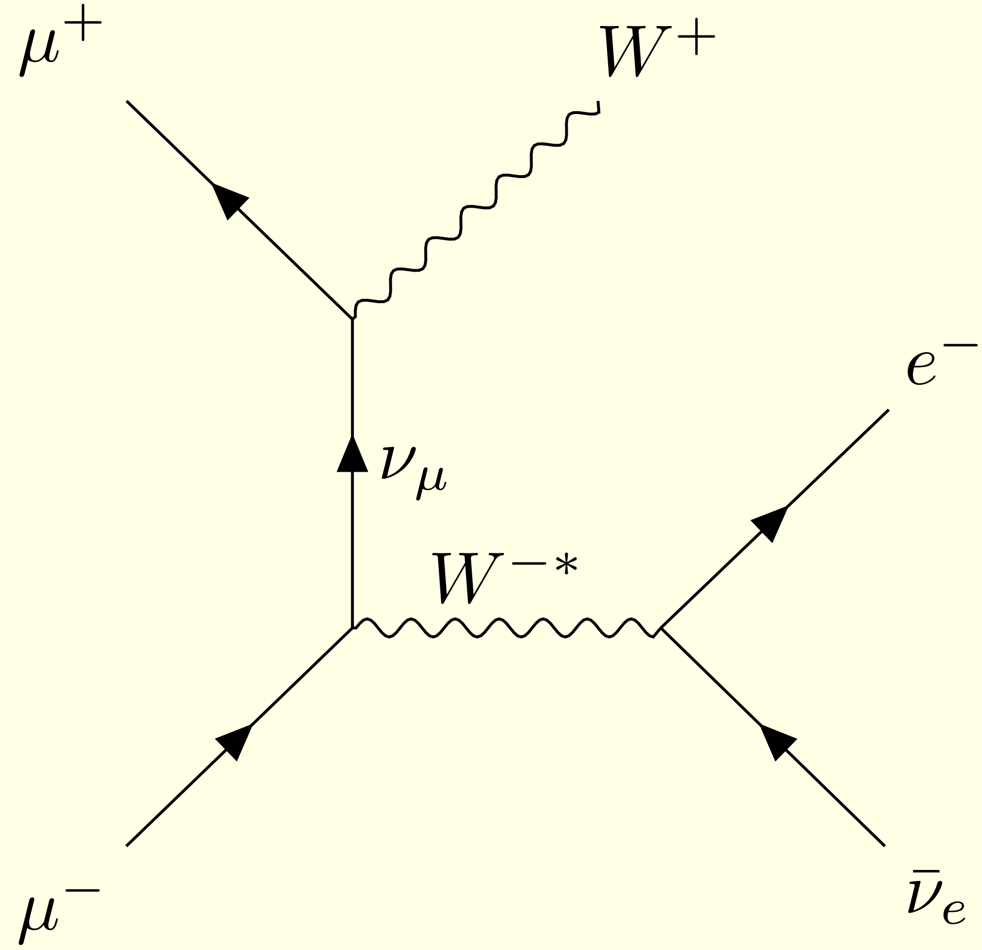


Figure 3: t-channel singularity in $\mu^+\mu^- \to W^+W^{-\star} \to W^+e^-\bar{\nu}_e$