Three-loop elliptic integrals for the Standard Model ρ parameter

with M.Becchetti, C.Duhr, and R.Marzucca

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QCD corrections to the SM ρ parameter

- The SM ρ parameter:

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1 + \delta\rho \qquad \delta\rho = \frac{\Sigma_Z(0)}{M_Z^2} - \frac{\Sigma_W(0)}{M_W^2}$$

$$\delta\rho = \frac{3\, G_F\, m_t^2}{16\,\pi^2\sqrt{2}}\left(\delta^{(0)} + \alpha_s\,\delta^{(1)} + \alpha_s^2\,\delta^{(2)} + \ldots\right)$$

- Massless b quarks: known at four loops, i.e. $\delta^{(3)}$

[Schröder,Steinhauser, Phys. Lett. B622, '05]
[Chetyrkin, Faisst, Kühn, Maierhöfer, Sturm, PRL97, '06]
[Boughezal, Czakon, Nucl.Phys. B755, '06]

- Massive b quarks: known at three loops, i.e. $\delta^{(2)}$

[Grigo, Hoff, Marquard, Steinhauser, Nucl.Phys. B864, '12]
[Blumlein, de Freitas, Imamoglu, Marquard, Schneider, van Hoeij, PoS LL2018, '18]
[Abreu, Becchetti, Duhr, Marzucca, JHEP 2002, '19]

$$\delta^{(2)} \equiv \delta^{(2)}(t) \qquad t = \frac{m_b^2}{m_t^2} \qquad \delta^{(2)}(t_{phys} \sim 5.10^{-4}) = -9.03594\ldots$$

[Grigo, Hoff, Marquard, Steinhauser, Nucl.Phys. B864, '12]

This talk: analytic expression for $\delta^{(2)}$

[Abreu, Becchetti, Duhr, Marzucca, JHEP 2002, '19]

Anatomy of a multi-loop calculation

Step 1: Find expression for $\delta^{(2)}$ using Feynman rules

- QGRAF/Form/FeynArts

Step 2: Reduction to master integrals

- Integration-by-parts (IBP) relations [LiteRed, FIRE, Kira, ...]

$$\delta^{(2)} = \sum_i c_i(m_t^2, t; D)\, I_i(m_t^2, t; D)$$

Easy for $\delta^{(2)}$
(with current tech)

Step 3: Compute master integrals

- Differential equations, direct integration, ...
- Numerical integration [pySecDec, Fiesta, diffexp, ...]

Find expression for master integrals in terms of functions we understand:

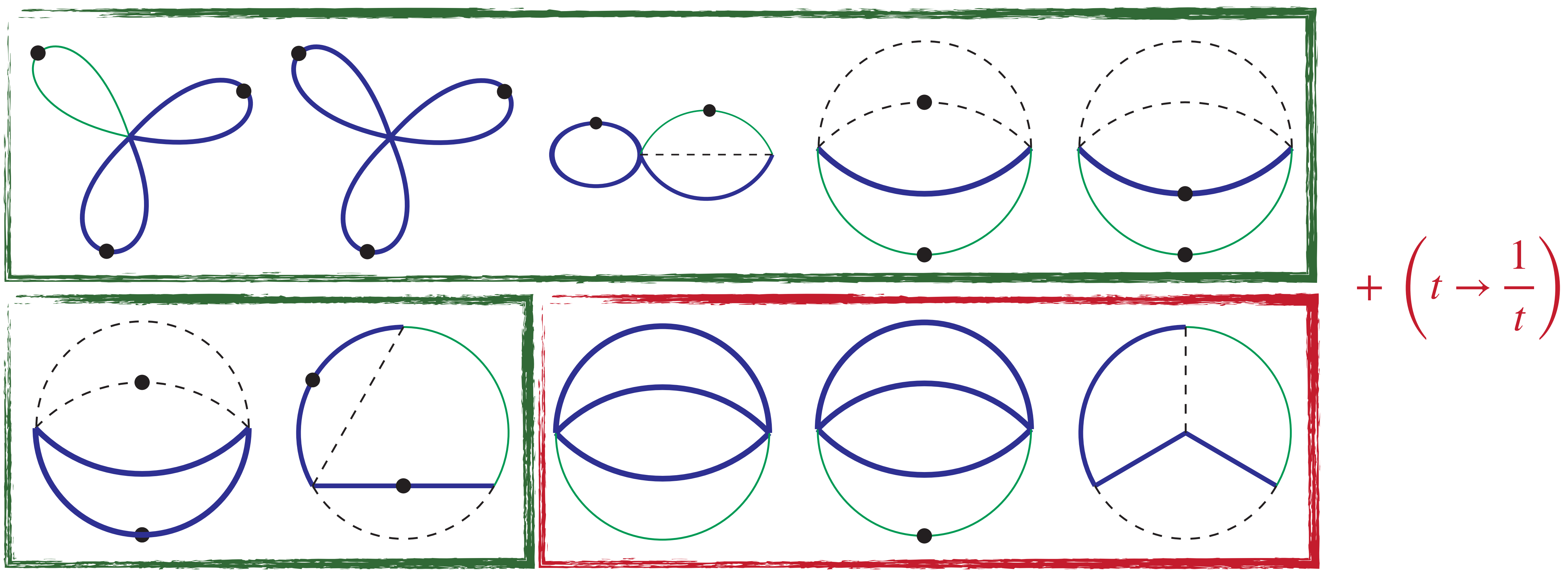
✓ Relations between functions, analytic continuation: are $f(t^2)$, $f(1/t)$ and $f(t)$ related?
$\Rightarrow$ simplify expressions and better convergence

✓ Efficient numerical evaluation

Master Integrals for Three-loop ρ Parameter

$$\delta^{(2)} = \sum_i c_i(m_t^2, t; D)\, I_i(m_t^2, t; D)$$

[Grigo, Hoff, Marquard, Steinhauser, Nucl.Phys. B864, '12]



- First seven integrals evaluate to ***Multiple Polylogarithms*** (MPLs)

  [Grigo, Hoff, Marquard, Steinhauser, Nucl.Phys. B864, '12]

  - ✓ Trivial with today's technology

- Last three integrals evaluate to ***Elliptic Multiple Polylogarithms*** (eMPLs)

  [Abreu, Becchetti, Duhr, Marzucca, JHEP 2002, '19]

  - ✓ New type of functions, state-of-the-art for analytic expressions

MPLs and eMPLs

- **MPLs**: punctures on Riemann sphere (includes $\log^n(t)$, $\mathrm{Li}_n(t)$, …)

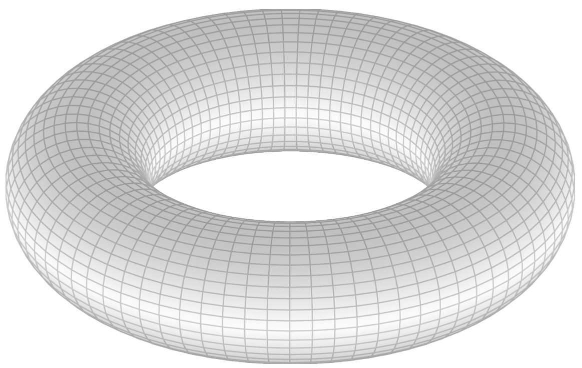
$$G(a_1, \cdots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \cdots, a_n; t)$$

  - ✓ All relations can be generated algorithmically
  - ✓ Efficient and precise numerical evaluation

- **eMPLs**: punctures on torus – elliptic curve $y^2 = (x - a_1)(x - a_2)(x - a_3)$

$$\mathcal{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt\, \psi_{n_1}(c_1, t, \vec{a})\, \mathcal{E}_3\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

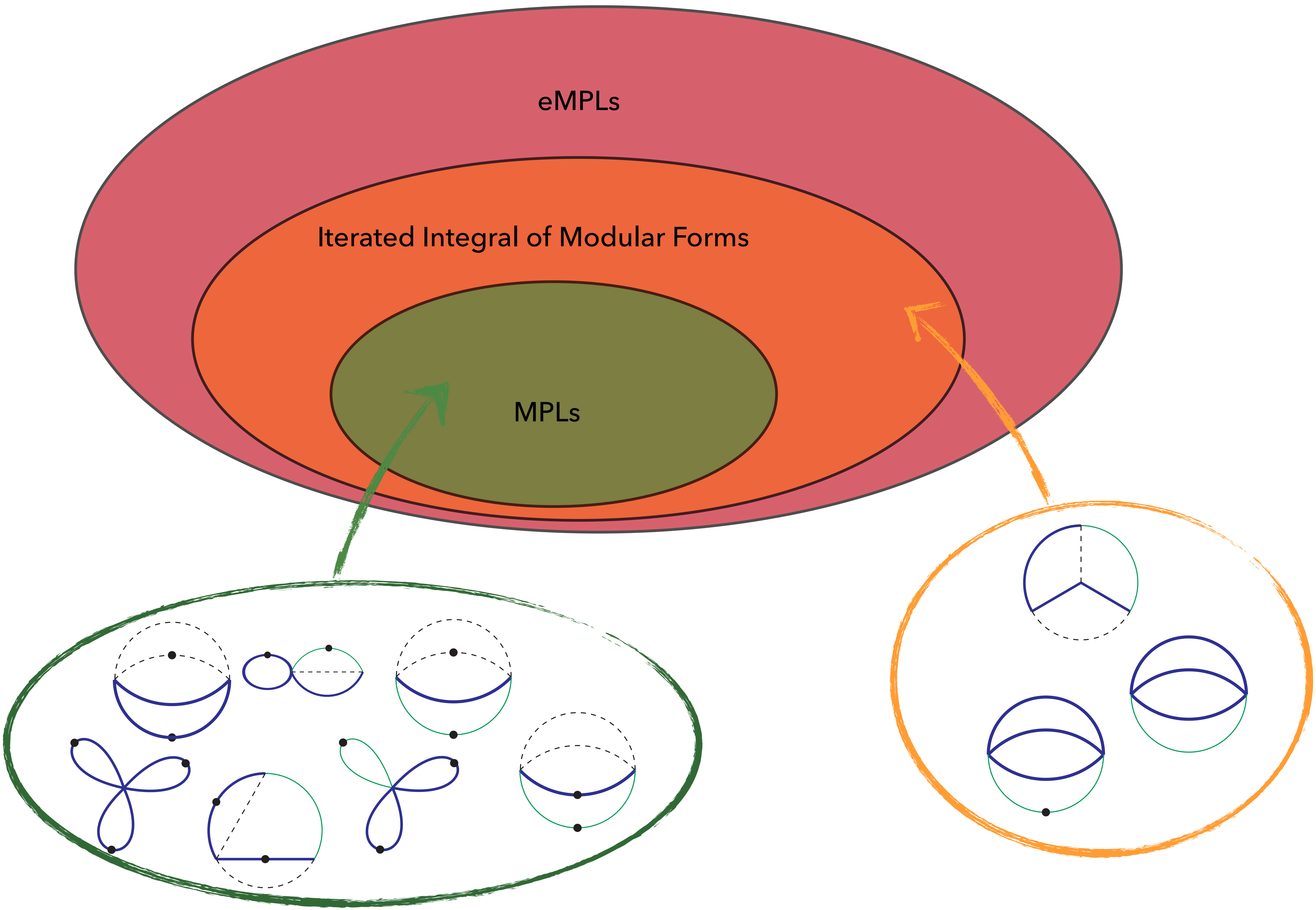
$$\psi_0(0, x, \vec{a}) = \frac{c_3}{y\,\omega_1}, \quad \psi_1(c, x, \vec{a}) = \frac{1}{x - c}, \ \ldots$$



  - ✓ Well defined class of functions
  - Relations not yet fully understood
  - Slow evaluation

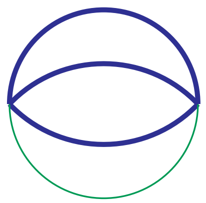
  - ✓ Rational points on torus: iterated integrals of modular forms
    - ✓ Analytic continuation, efficient and precise numerics

Master Integrals for Three-loop ρ Parameter



Master Integrals for Three-loop ρ Parameter

- ✓ From parametric integration



$$= \frac{1}{2\sqrt{1-t}} \int_0^1 dx \, \frac{\Omega(x;t)}{y}$$

$$\frac{\partial \Omega(x;t)}{\partial x} = \left[\frac{1}{\sqrt{1-t}(1-x)y} - \frac{\sqrt{1-t}}{y} - \frac{t}{\sqrt{1-t}xy} \right] \times \left[G(0;t) - G(0;x) - G(1;x) + G\left(\frac{t}{t-1};x\right) \right]$$

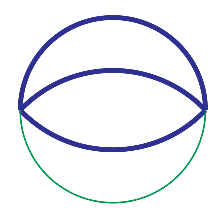
$$\Omega(0;t) = \frac{\pi^2}{3} + \log^2 t$$

- ✓ Elliptic curve $\quad y^2 = (x-a_1)(x-a_2)(x-a_3)$

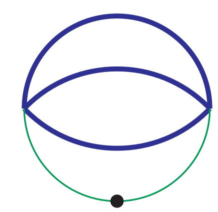
$$a_1 = \frac{t}{t-1}, \quad a_2 = \frac{1}{8}\left(t+3-\sqrt{(t-1)(t-9)}\right), \quad a_3 = \frac{1}{8}\left(t+3+\sqrt{(t-1)(t-9)}\right)$$

- ✓ Second elliptic master as derivative

$$\frac{\partial}{\partial t}$$

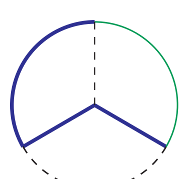


$$= -$$

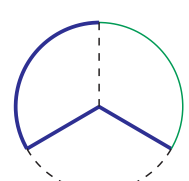


- ✓ Final master from differential equation

$$\frac{\partial}{\partial t}$$

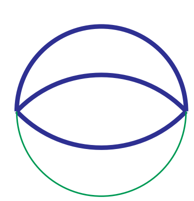


$$= \epsilon \frac{2}{1-t}$$



$$+\epsilon^4 \left[\frac{2}{t}\left(\zeta_3 - \mathrm{Li}_3(1-t)\right) + \frac{1}{1-t}\left(2\mathrm{Li}_3(t) - \mathrm{Li}_2(t)\log t - \frac{\pi^2}{6}\log t + \frac{10}{3}\zeta_3 \right) \right.$$

$$-\frac{1}{3}\left(\frac{4}{1-t} + \frac{1}{2}\right)$$



$$\left. \vphantom{\frac{1}{1}} \right] + \mathcal{O}(\epsilon^5)$$

[Abreu, Becchetti, Duhr, Marzucca, JHEP 2002, '19]

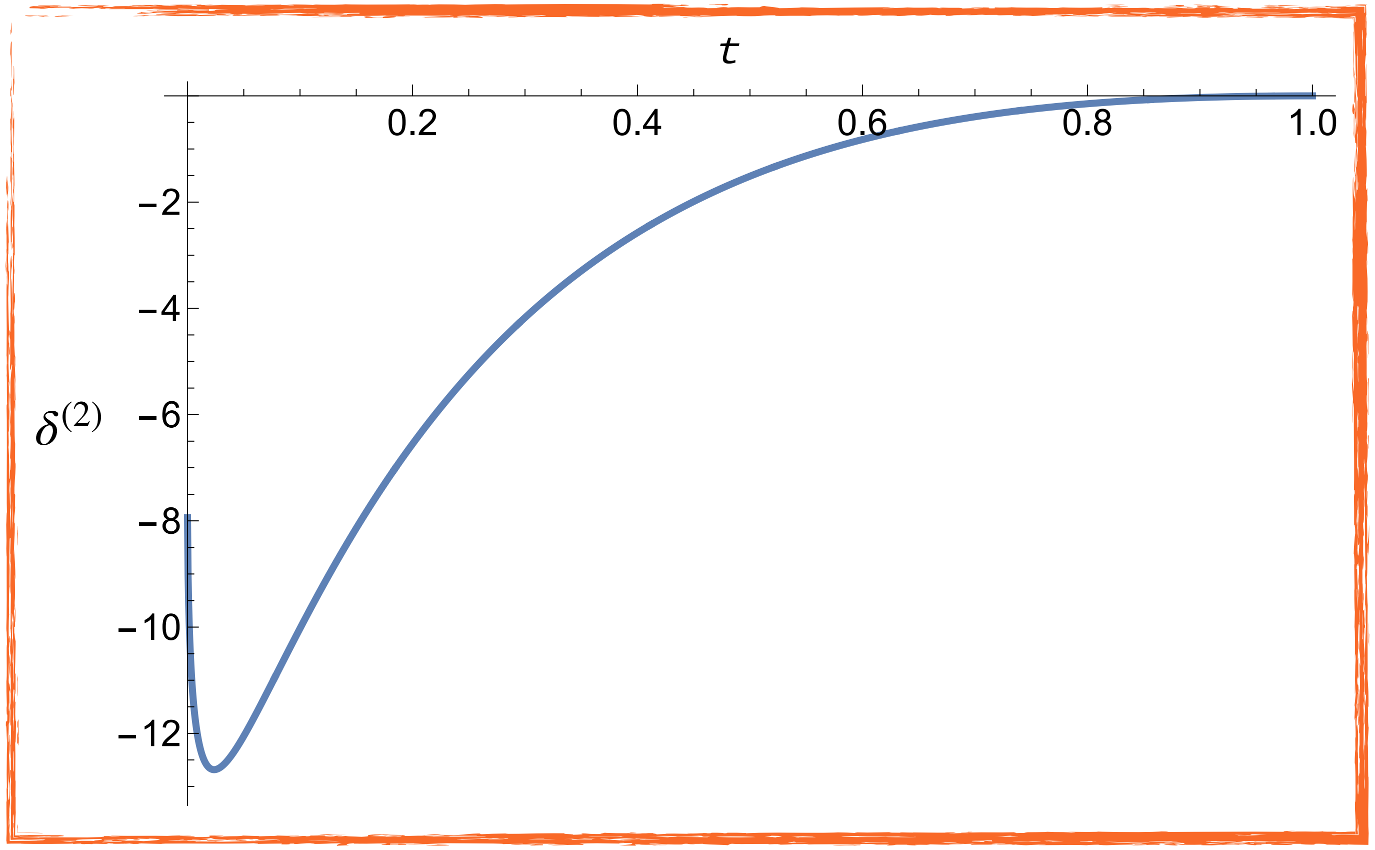
Three-Loop QCD Corrections to ρ parameter

[Abreu, Becchetti, Duhr, Marzucca, JHEP 2002, '19]

- For $\delta^{(2)}$: need master integrals at t and $1/t$
  - ✓ Iterated integrals of modular forms ⇒ analytically continue expressions

- Compact analytic solution

- Piecewise solution for all values of t: fast-converging numerical evaluation



$$\delta^{(2)}(t_{phys} \sim 5.10^{-4}) = -9.03594\ldots$$

Full agreement with:
[Grigo, Hoff, Marquard, Steinhauser, Nucl.Phys. B864, '12]
[Blumlein, de Freitas, Imamoglu, Marquard, Schneider, van Hoeij, PoS LL2018, '18]

Summary and Outlook

- Proof-of-principle calculation: physical observable involving elliptic integrals

- Elliptic integrals ubiquitous in multi-loop EW calculations

- Learn from example of MPLs in (massless) QCD calculations:
  - ✓ Simplify expressions: crucial for multi-scale processes
  - ✓ Better representations for numerical evaluation: control large cancelations
  - ✓ Also for purely numerical calculations: what is a `good basis' of master integrals?

- A lot of technology developed for LHC calculation, ready to be used for EW

- The community is currently working on this, a lot of progress is being made!
  - ✓ See the other talks of this session!

THANK YOU!