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Collinear factorisation for e^+e^- collisions

Based on: 1909.03886 (SF), 1911.12040 (Bertone, Cacciari, SF, Stagnitto)
2105.06688 (SF), and work in progress within
MadGraph5_aMC@NLO (2108.10261, SF, Mattelaer, Zaro, Zhao)

5th FCC Physics Workshop, Liverpool, 9/2/2022

- Cross sections stemming from e^+e^- collisions are plagued by large logs that must be resummed
- One way to do that is by means of collinear factorisation; another, with YFS
- If you run on the peak of a narrow resonance, YFS is probably the way to go. In all of the other cases, I'd employ collinear factorisation

- There is no precision physics without the ability of assessing uncertainties

The problem with collinear factorisation pre-2019:

- Collinear formulae used thus far are leading-log accurate: computation of uncertainties is not well defined

  α is literally an arbitrary parameter

This is what has motivated the work I'm talking about

The problem is now solved

Consider the production of a system X at an e^+e^- collider:

$$e^+(P_{e^+}) + e^-(P_{e^-}) \longrightarrow X$$

Its cross section is written as follows:

$$d\Sigma_{e^+e^-}(P_{e^+}, P_{e^-}) = \sum_{kl=e^+e^-\gamma} \int dy_+ dy_- \, \mathcal{B}_{kl}(y_+, y_-) \, d\sigma_{kl}(y_+ P_{e^+}, y_- P_{e^-})$$

Here:

- $d\Sigma_{e^+e^-}$: the collider-level cross section
- $d\sigma_{kl}$: the particle-level cross section
- $\mathcal{B}_{kl}(y_+, y_-)$: describes beam dynamics (including beamstrahlung)
- e^+, e^- on the lhs: the beams
- e^+, e^-, γ on the rhs: the particles

I'll mostly be concerned with computing $d\sigma_{kl}$ in the rest of the talk

The particle-level cross section $d\sigma$ embeds all that is not beam dynamics

It is perturbatively computable, but plagued by $\log(m/E)$ terms to all orders. Fortunately, the dominant classes of these are factorisable:

$$d\sigma\left(\log(m/E), m/E\right) = \mathcal{K}\left(\log(m/E)\right) \otimes d\hat{\sigma}\left(m/E\right)$$

The idea is to compute $d\hat{\sigma}$ to some fixed order in perturbation theory, and $\mathcal{K}$ to all orders (so that logs are resummed)

The definitions of $\mathcal{K}$ and of the convolution $(\otimes)$ determine unambiguously how the logs are resummed.

Therefore, two things to be done:

1. Compute $d\hat{\sigma}$

2. Compute $\mathcal{K}$ to all orders within a definite convolution scheme

Therefore, two things to be done:

1. Compute $d\hat{\sigma}$

   This is a process-by-process operation. Note that at the NLO *automation* has completely solved the problem for arbitrary processes – see e.g. the e^+e^- results of MadGraph5_aMC@NLO (1405.0301, 1804.10017, 2108.10261)

Therefore, two things to be done:

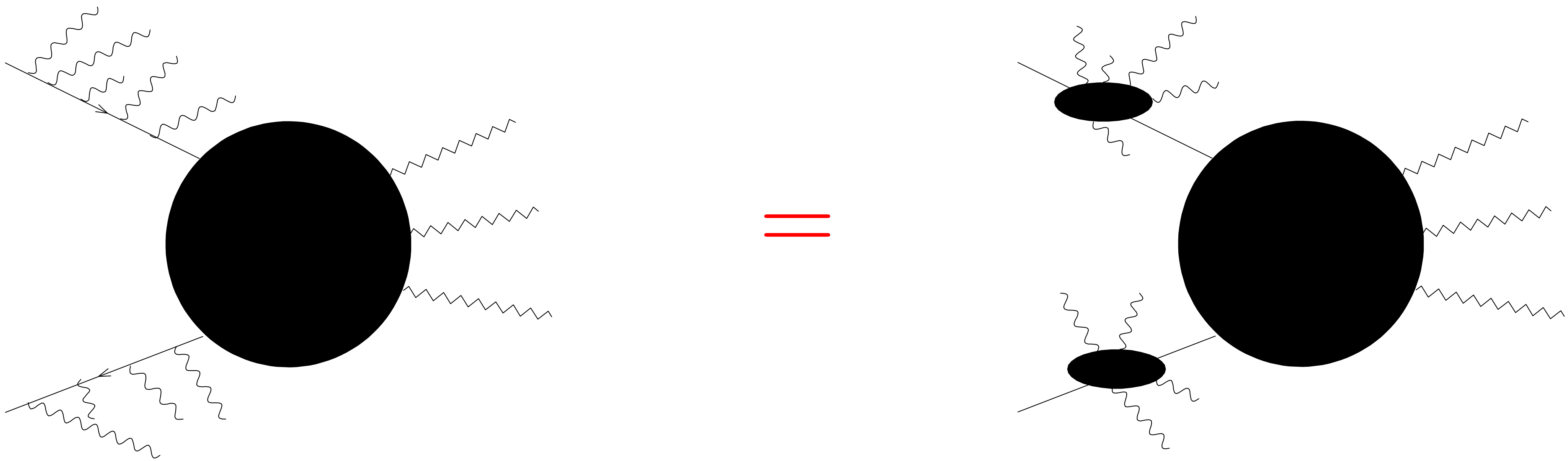
1. Compute $d\hat{\sigma}$

2. Compute $\mathcal{K}$ to all orders within a definite convolution scheme

   This is a universal (i.e. process independent) operation, which we carry out by means of a collinear-factorisation approach.

   (I'll concentrate here on ISR. Analogous formulae hold for FSR)

Collinear factorisation



$$d\sigma = \text{PDF} \star \text{PDF} \star d\hat{\sigma}$$

PDFs collect (universal) small-angle dynamics

$$
\begin{aligned}
d\sigma_{kl}(p_k, p_l) \;\; &= \sum_{ij=e^+,e^-,\gamma} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2) \, \Gamma_{j/l}(z_-, \mu^2, m^2) \\
&\quad \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2) + \mathcal{O}\left(\left(\frac{m^2}{s}\right)^p\right)
\end{aligned}
$$

where one calculates Γ and $d\hat{\sigma}$ to predict $d\sigma$

- $k\,, l = e^+\,, e^-\,, \gamma$ on the lhs: the particles that emerge from beamstrahlung
- $i\,, j = e^+\,, e^-\,, \gamma$ on the rhs: the partons
- $d\sigma_{kl}$: the particle-level (ie observable) cross section
- $d\hat{\sigma}_{ij}$: the subtracted parton-level cross section.
  Generally with $m = 0$ $\Longrightarrow$ power-suppressed terms in $d\sigma$ discarded
- $\Gamma_{i/k}$: the PDF of parton i inside particle k
- μ: the hard scale, $m^2 \ll \mu^2 \sim s$

Very similar to QCD, with some notable differences:

- PDFs and power-suppressed terms can be computed perturbatively
- An object (e.g. e^-) may play the role of both particle and parton

As in QCD, a particle is a physical object, a parton is not

z-space LO+LL PDFs $(\alpha \log(E/m))^k$:

~ 1992

- $0 \leq k \leq \infty$ for $z \simeq 1$ (Gribov, Lipatov)
- $0 \leq k \leq 3$ for $z < 1$ (Skrzypek, Jadach; Cacciari, Deandrea, Montagna, Nicrosini; Skrzypek)
- matching between these two regimes

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z-space NLO+NLL PDFs $(\alpha \log(E/m))^k + \alpha\,(\alpha \log(E/m))^{k-1}$:

$\longrightarrow$ 1909.03886, 1911.12040, 2105.06688

- $0 \leq k \leq \infty$ for $z \simeq 1$
- $0 \leq k \leq 3$ for $z < 1 \quad \Longleftrightarrow \quad \mathcal{O}(\alpha^3)$
- matching between these two regimes
- for e^+, e^-, and γ
- both numerical and analytical

Main tool: the solution of PDFs evolution equations

Henceforth, I consider the dominant production mechanism at an e^+e^- collider, namely that associated with partons inside an electron*

Simplified notation:

$$\Gamma_i(z,\mu^2) \;\equiv\; \Gamma_{i/e^-}(z,\mu^2)$$

*The case of the positron is identical, at least in QED, and will be understood

NLO initial conditions (1909.03886)

Conventions for the perturbative coefficients:

$$\Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi}\Gamma_i^{[1]} + \mathcal{O}(\alpha^2)$$

Results:

$$\begin{aligned}
\Gamma_i^{[0]}(z,\mu_0^2) &= \delta_{ie^-}\delta(1-z) \\
\Gamma_{e^-}^{[1]}(z,\mu_0^2) &= \left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right]_+ + K_{ee}(z) \\
\Gamma_{\gamma}^{[1]}(z,\mu_0^2) &= \frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) + K_{\gamma e}(z) \\
\Gamma_{e^+}^{[1]}(z,\mu_0^2) &= 0
\end{aligned}$$

Note:

- Meaningful only if $\mu_0 \sim m$
- In $\overline{\mathrm{MS}}$, $K_{ij}(z) = 0$; in general, these functions *define* a factorisation scheme

NLL evolution (1911.12040, 2105.06688)

General idea: solve the evolution equations starting from the initial conditions computed previously

$$\frac{\partial \Gamma_i(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} \left[P_{ij} \otimes \Gamma_j\right](z,\mu^2) \quad \Longleftrightarrow \quad \frac{\partial \Gamma(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} \left[\mathbb{P} \otimes \Gamma\right](z,\mu^2)\,,$$

Done conveniently in terms of non-singlet, singlet, and photon

Two ways:

- Mellin space: suited to both numerical solution and all-order, large-z analytical solution (called *asymptotic solution*). Dominant
- Directly in z space in an integrated form: suited to fixed-order, all-z analytical solution (called *recursive solution*). Subleading

Bear in mind that PDFs are fully defined only after adopting a definite *factorisation scheme*, which is the choice of the finite terms associated with the subtraction of the collinear poles

(done by means of the $K_{ij}(z)$ functions)

- 1911.12040 $\longrightarrow$ $\overline{\rm MS}$
- 2105.06688 $\longrightarrow$ a DIS-like scheme (called Δ)

At variance with the QCD case, there is also an interesting *renormalisation-scheme* dependence of QED PDFs

(not discussed in this talk)

Asymptotic $\overline{\mathrm{MS}}$ solution

Non-singlet $\equiv$ singlet; photon is more complicated

$$\Gamma_{\rm NLL}(z,\mu^2) \stackrel{z\to 1}{\longrightarrow} \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1}$$
$$\times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\left(L_0 - 1\right) \left(A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} \right.\right.$$
$$\left.\left. + \left(L_0 - 1 - 2A(\xi_1) \right) \log(1-z) - \log^2(1-z) \right] \right\}$$

where $L_0 = \log \mu_0^2/m^2$, and:

$$A(\kappa) = -\gamma_{\rm E} - \psi_0(\kappa)$$
$$B(\kappa) = \frac{1}{2}\gamma_{\rm E}^2 + \frac{\pi^2}{12} + \gamma_{\rm E}\psi_0(\kappa) + \frac{1}{2}\psi_0(\kappa)^2 - \frac{1}{2}\psi_1(\kappa)$$

with:

$$
\begin{aligned}
\xi_1 &= 2t - \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\frac{20}{9}n_F + \frac{4\pi b_1}{b_0}\right) \\
&= 2t + \mathcal{O}(\alpha t) = \eta_0 + \ldots \\
\hat{\xi}_1 &= \frac{3}{2}\,t + \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\lambda_1 - \frac{3\pi b_1}{b_0}\right) \\
&= \frac{3}{2}\,t + \mathcal{O}(\alpha t) = \lambda_0 \eta_0 + \ldots \\
\lambda_1 &= \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 - \frac{n_F}{18}(3 + 4\pi^2)
\end{aligned}
$$

and:

$$
\begin{aligned}
t &= \frac{1}{2\pi b_0}\log\frac{\alpha(\mu)}{\alpha(\mu_0)} \\
&= \frac{\alpha(\mu)}{2\pi}\,L - \frac{\alpha^2(\mu)}{4\pi}\left(b_0 L^2 - \frac{2b_1}{b_0}L\right) + \mathcal{O}(\alpha^3)\,, \qquad L = \log\frac{\mu^2}{\mu_0^2}\,.
\end{aligned}
$$

Asymptotic Δ solution

Non-singlet $\equiv$ singlet

$$\Gamma_{\rm NLL}(z,\mu^2) \quad \xrightarrow{z\to 1} \quad \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\,\xi_1(1-z)^{-1+\xi_1} \times \left[\left(1+\frac{3\alpha(\mu_0)}{4\pi}L_0\right)\sum_{p=0}^{\infty}\mathcal{S}_{1,p}(z) - \frac{\alpha(\mu_0)}{\pi}L_0\sum_{p=0}^{\infty}\mathcal{S}_{2,p}(z)\right]$$

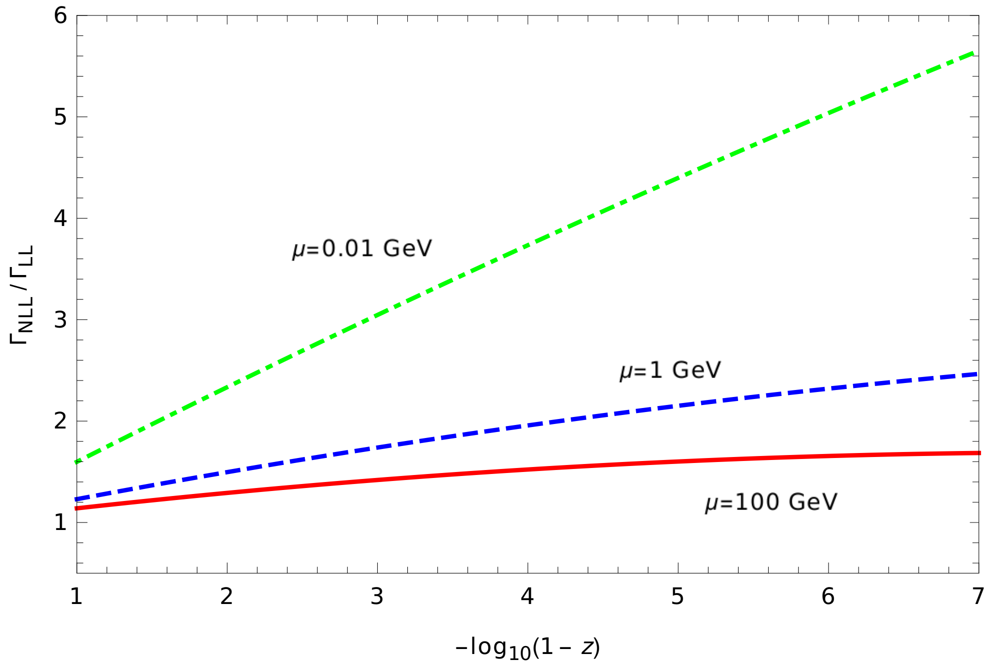
The $\mathcal{S}_{i,p}(z)$ functions are increasingly suppressed at $z\to 1$ with growing p. The dominant behaviour is:

$$\Gamma_{\rm NLL}(z,\mu^2) \quad \xrightarrow{z\to 1} \quad \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\,\xi_1(1-z)^{-1+\xi_1} \times \left[\frac{\alpha(\mu)}{\alpha(\mu_0)} + \frac{\alpha(\mu)}{\pi}L_0\left(A(\xi_1)+\log(1-z)+\frac{3}{4}\right)\right]$$

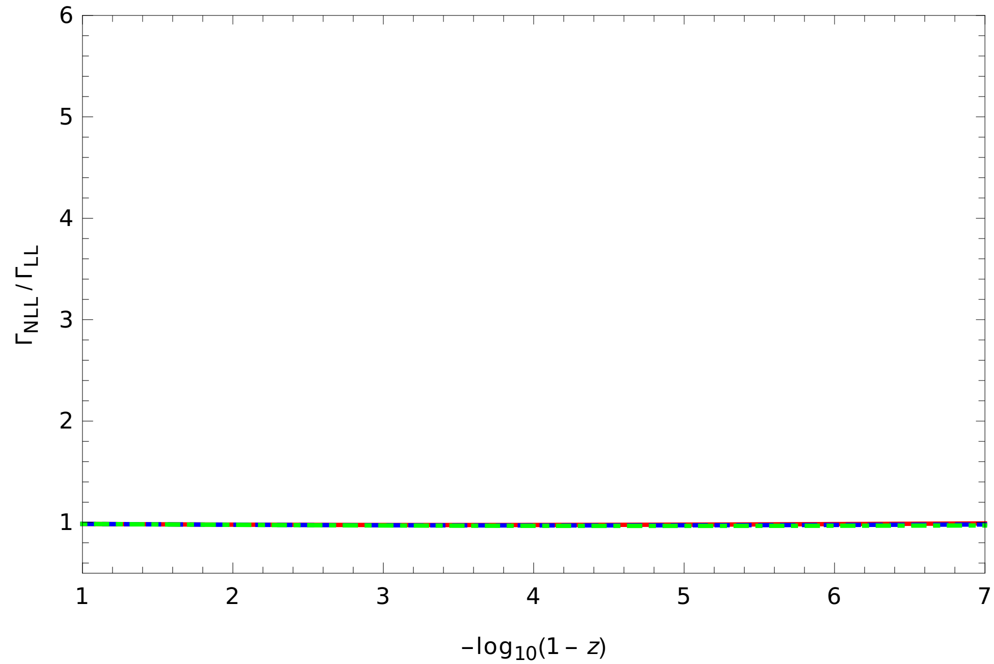
- A vastly different logarithmic behaviour w.r.t. the $\overline{\rm MS}$ case

  However, $\Gamma^{(\overline{\rm MS})}_{\rm NLL} - \Gamma^{(\Delta)}_{\rm NLL} = \mathcal{O}(\alpha^2)$

$\Gamma_{\rm NLL}/\Gamma_{\rm LL}$ at large z $(\mu_0 = m)$



$\overline{\rm MS}$ scheme

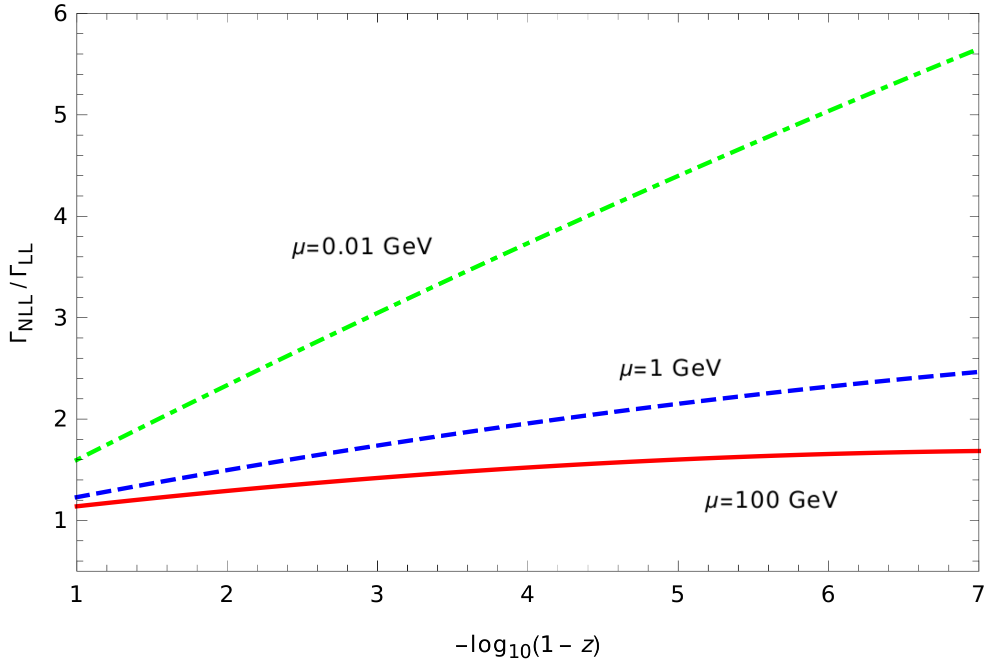


Δ scheme

In $\overline{\rm MS}$, significant scale dependence, and significant differences w.r.t. LL results. This doesn't happen in Δ (note the y ranges in the plots)

This *does not mean* NLO and LO cross sections will differ by large factors: PDFs are unphysical, and there are huge cancellations with partonic cross sections. Also, bear in mind that $\Gamma_{\rm NLL}^{(\overline{\rm MS})} - \Gamma_{\rm NLL}^{(\Delta)} = \mathcal{O}(\alpha^2)$

$\Gamma_{\rm NLL}/\Gamma_{\rm LL}$ at large z $(\mu_0 = m)$



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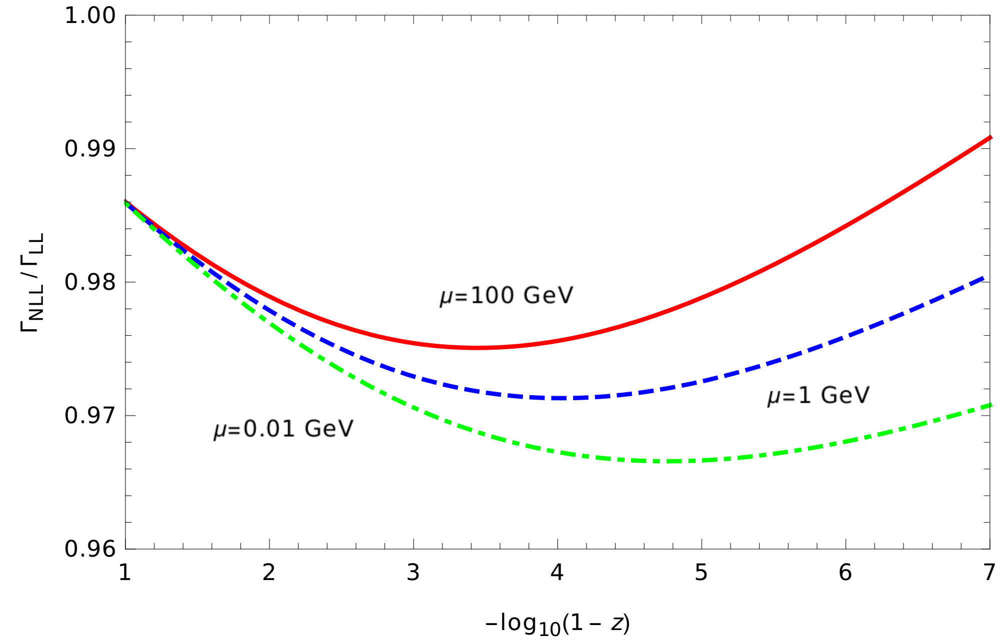

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Take-home message

Many theoretical techniques have been developed with LHC physics in mind, that can be ported to an e^+e^- environment

A convenient way to do so is to exploit the similarities of collinear-factorisation formulae in QCD and QED

In a recent series of papers we have upgraded the accuracy of those formulae from the LL to the NLL, thus paving the way for phenomenology studies (now ongoing) characterised by sensible uncertainties

Together with automated (NLO) methods, the above will give the community powerful tools that will help enlarge the physics scope of e^+e^- simulations

EXTRA SLIDES

A look at the photon:

$$\Gamma_\gamma^{(\overline{\rm MS})}(z,\mu^2) \quad \xrightarrow{z\to 1} \quad \frac{t\alpha(\mu_0)^2}{\alpha(\mu)}\,\frac{3}{2\pi\xi_1}\,\log(1-z) - \frac{t\alpha(\mu_0)^3}{\alpha(\mu)}\,\frac{1}{2\pi^2\xi_1}\,\log^3(1-z)$$

$$\Gamma_\gamma^{(\Delta)}(z,\mu^2) \quad \xrightarrow{z\to 1} \quad \frac{1}{2\pi}\,\frac{\alpha^2(\mu_0)}{\alpha(\mu)}\,\frac{1+(1-z)^2}{z}\,L_0 + \frac{1}{2\pi\xi_1}\,\frac{t\,\alpha^2(\mu_0)}{\alpha(\mu)}\,L_0 - \frac{t\,\alpha(\mu)}{2\pi\xi_1}\,\frac{e^{-\gamma_{\rm E}\xi_1}e^{\hat{\xi}_1}}{\Gamma\left(1+\xi_1\right)}\,(1-z)^{\xi_1}\,L_0\,.$$

- $\overline{\rm MS}$ vs Δ exhibits the same pattern as for (non-)singlet: logarithmic terms dominate at $z\to 1$ in $\overline{\rm MS}$, but are absent in Δ

Recursive solution

Too involved to be reported here. For the record, the (previously unknown) recursive NLL equations are:

$$
\begin{aligned}
\mathcal{J}_k^{\mathrm{LL}} &= \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1}^{\mathrm{LL}} \\
\mathcal{J}_k^{\mathrm{NLL}} &= (-)^k (2\pi b_0)^k \mathcal{F}^{[1]}(\mu_0^2) \\
&\quad + \sum_{p=0}^{k-1} (-)^p (2\pi b_0)^p \Bigg(\mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{NLL}} + \mathbb{P}^{[1]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} \\
&\qquad - \frac{2\pi b_1}{b_0} \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} \Bigg)
\end{aligned}
$$

Integrated PDFs expanded on the basis of the $\mathcal{J}^{\mathrm{LL}}$ and $\mathcal{J}^{\mathrm{NLL}}$ functions with known coefficients

We have computed these for $k \leq 3$ ($\mathcal{J}^{\mathrm{LL}}$) and $k \leq 2$ ($\mathcal{J}^{\mathrm{NLL}}$), ie to $\mathcal{O}(\alpha^3)$

Results in 1911.12040 and its ancillary files

A remarkable fact

Our asymptotic solutions, expanded in α, feature *all* of the terms:

$$\frac{\log^q(1-z)}{1-z} \qquad \mathrm{singlet,\ non-singlet}$$

$$\log^q(1-z) \qquad \mathrm{photon}$$

of our recursive solutions. This ensures a smooth matching

Non-trivial; stems from keeping subleading terms (at $z \to 1$) in the AP kernels

The above applies to the $\overline{\mathrm{MS}}$ case: in the Δ scheme, logarithms are absent