Tools for higher loops*

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FCC Physics Workshop, Liverpool

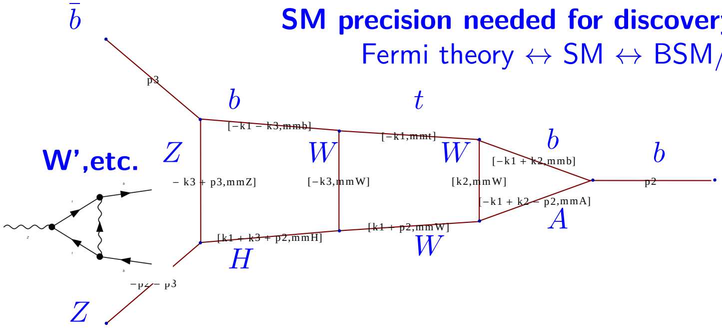
Feb 9, 2022 (on-line)

* New directions in science are launched by new tools much more often than by new concepts, F. Dyson

	analytic	numerical
pole cancellation	exact	with numerical uncertainty
control of integrable singularities	analytic continuation	less straightforward
fast evaluation	yes	depends
extension to more scales/loops	difficult	*promising*
automation	difficult	*less difficult*

SM precision needed for discovery studies (indirect effects)

Fermi theory $\leftrightarrow$ SM $\leftrightarrow$ BSM/EFT/...



Four scales :

$$\left\{ \frac{M_H^2}{M_Z^2}, \frac{M_W^2}{M_Z^2}, \frac{m_t^2}{M_Z^2}, \frac{s+i\varepsilon}{M_Z^2} \right\}$$

TOOLS and methods

Many groups present rapid progress:

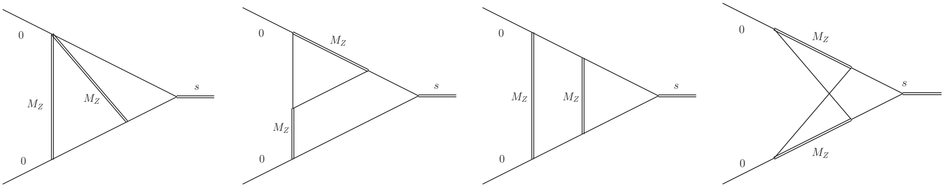
- Analytical/**numerical** solutions for Master Integrals (`MIs`) by *differential equations* **`DEs`**;
- Sector decomposition *(SD)*;
- Mellin-Barnes representations *(MB)*;
- Reductions at the integrand level;
- Expansions by regions; Taylor expansion in Feynman parameters;
- Loop-tree duality (G. Rodrigo et al, Weinzierl et al);
- Multi-loop amplitudes with numerical unitarity (Abreu et al.);
- Four-dimensional unsubtraction; Direct numerical evaluation of multi-loop integrals without contour deformation (R. Pittau et al.);
- Feynman parameters and dispersion relations (Song, Freitas);
- ...

- Sector decomposition (SD)
  - FIESTA 3 [2014], [A.V.Smirnov]
  - FIESTA 4 [2016], [A.V.Smirnov]
  - SecDec 3 [2015], pySecDec [2017], [S. Borowka, G. Heinrich, et. al.]
  - pySecDec [2019], A GPU compatible quasi-Monte Carlo integrator interfaced to pySecDec]
  - pySecDec [2022], Expansion by regions with pySecDec], https://doi.org/10.1016/j.cpc.2021.108267
- The Mellin-Barnes (MB) method:
  - PlanarityTest [I.Dubovyk, K.Bielas, 2013]
  - AMBRE 2 [J.Gluza, et. al., 2011], AMBRE 3 [I.Dubovyk, et. al., 2015]
  - MB [M.Czakon, 2006], MBresolve [A.V.Smirnov, V.A.Smirnov, 2009]
  - MBnumerics [J.Usovitsch, I.Dubovyk, T.Riemann, 2015] – Minkowskian kinematics
  - QMB (MB+quasiMC) [I.Dubovyk, JG, T.Riemann, 2019] – Minkowskian kinematics
  - MBConicHulls (MB+series) [B. Ananthanarayan et al, 2021] – Multiple Series Representations of N-fold Mellin-Barnes Integrals, https://doi.org/10.1103/PhysRevLett.127.151601

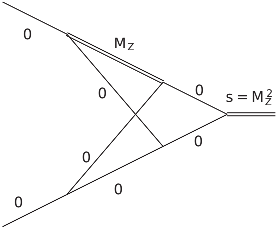
Mellin-Barnes and Sector Decomposition methods are very much complementary

- MB works well for hard threshold, on-shell cases, not many internal masses (more IR); SD more useful for integrals with many internal masses

10^{-8} accuracy achieved for **any** self-energy and vertex Feynman integral with one of the methods - in **Minkowskian region**.



Substantial progress for critical cases



Minkowskian results (constant part, $-(p_1+p_2)^2 = m^2 = 1$:):

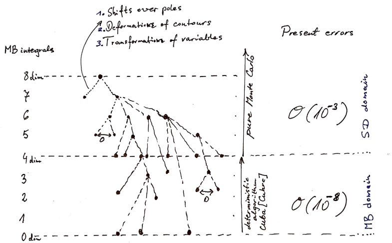
$$\begin{aligned}
\text{Analytical}: &\quad -0.778599608979684 - 4.123512593396311 \cdot i \\
\text{MBnumerics}: &\quad -0.778599608324769 - 4.123512600516016 \cdot i \\
\text{MB} + \text{thresholds}: &\quad -0.7785242512636401 - 4.123512600516016 \cdot i \\
\text{SecDec}: &\quad \text{big error } [\mathbf{2016}], -0.77 - \mathrm{i} \cdot 4.1\ [\mathbf{2017}], -0.778 - \mathrm{i} \cdot 4.123\ [\mathbf{2019}] \\
\text{pySecDec} + \text{rescaling}: &\quad -0.778598 - i \cdot 4.123512\ [\mathbf{2020}]
\end{aligned}$$

Related and auxiliary Software

MBnumerics

Project: I. Dubovyk, T. Riemann, J. Usovitsch (jusovitsch,googlemail.com)
Software: Johann Usovitsch
Publications: https://doi.org/10.18452/19484 , https://doi.org/10.1016/j.cpc.2006.07.002, https://doi.org/10.1016/j.physletb.2016.09.012
To be cited by users in publications, for details see README_copyright in the downloaded tarball.
Features: MBnumerics is a software for evaluation of MB integrals in the Minkowski kinematics
Download: http://us.edu.pl/~gluza/ambre/packages/mbnumerics.tgz

- gives high accuracy results
  up to certain dimensionality of MB integrals
- can produce huge cascade of
  lower-dimensional integrals



The package (available since 2020)

```
gluza@gluza-x1:~/calculations/MBnumerics/MBnumericsv2/MBnumerics_git$ ls
libcuba4.a          README               res_zbb_fig1c_mink
libkernlib.a        README_copyright     run_script_1loop_QED_vertex_euclid
libmathlib.a        res_1loop_QED_eucl   run_script_1loop_QED_vertex_mink
MB.m                res_1loop_QED_mink   run_script_zbb_fig1a_examples_euclid
MBnumericsv2.m      res_zbb_fig1a_eucl   run_script_zbb_fig1a_examples_mink
MBsplits.m          res_zbb_fig1a_mink   run_script_zbb_fig1c_examples_euclid
plb16_examples.nb   res_zbb_fig1c_eucl   run_script_zbb_fig1c_examples_mink
```

Needs:

1. MB.m
2. Cuba/Cuhre library
3. CERNlib

Complete examples: MBnumerics section at http://prac.us.edu.pl/~gluza/ambre

Solving MIs by differential equations; canonical basis, uniform weight

$$\frac{d}{dx}M(x) = A(x)M(x) + B(x), \quad x\text{- some parameter involved (kinematic).}$$

$$\frac{d}{dx}H(x) = A(x)H(x),$$

$$\mathrm{M(x)} = \mathrm{H(x)} \left[Const + \int^x \frac{dx'}{H(x')} B(x')\right].$$

$$\mathrm{M}{=}\sum_{i=-\alpha}^{\beta} m^i \epsilon^i, A = \sum_{i=0}^{\alpha+\beta} a^i \epsilon^i, \quad B = \sum_{i=-\alpha}^{\beta} b^i \epsilon^i.$$

1. $M(x) \longrightarrow \vec{M}(x)$; Coupled eqs.
2. $A(x,\epsilon) \longrightarrow \epsilon A(x)$, J. Henn, 2013
3. Boundary conditions.

$$\frac{d}{dx}m^i(x) = \sum_{j=0}^{\alpha-i} a^j(x)\; m^{i-j}(x) + b^i(x).$$

$$m^i(x) = H(x)\left(Const + \int^x \frac{dx'}{H(x')} \left[\sum_{j=1}^{\alpha-i} a^j(x) m^{i-j}(x) + b^i(x)\right]\right).$$

Many works, public programs available:

Lee, 1411.0911, https://doi.org/10.1007/JHEP04(2015)108
Prausa ('epsilon'), 1701.00725, https://doi.org/10.1016/j.cpc.2017.05.026
Gituliar, Magerya ('Fuchsia') , 1701.04269, https://doi.org/10.1016/j.cpc.2017.05.004
Meyer ('Canonica'), 1705.06252, https://doi.org/10.1016/j.cpc.2017.09.014
Dlapa, Henn, Yan, ('INITIAL') 2002.02340, https://doi.org/10.1007/JHEP05(2020)025
...

Set up differential equations in analytic form, **solving analytically** (with MPL) Henn 2013, Chen 1997 https://inspirehep.net/literature/1235976, Vollinga, Weinzierl 2005, https://doi.org/10.1016/j.cpc.2004.12.009, (beyond MPL) Brown, Duhr, 2006.09413, https://inspirehep.net/literature/1801706, see Samuel Abreu's talk and eMPLs, ...

$$\partial \vec{f}(\vec{x},\epsilon) = \epsilon \sum_i A_i(\vec{x}) \vec{f}(\vec{x},\epsilon) \partial x_i, \;\; A_i(\vec{x}) = \partial_{x_i} \tilde{A}(\vec{x}),$$

$$\tilde{A}(\vec{x}) = \sum_i c_i \log(\alpha_i(\vec{x})),$$

c_i - matrices of rational numbers; $\alpha_i(\vec{x})$ - algebraic functions of the kinematic invariants (letters). The set of letters - alphabet.

Some applications:

e.g. 1307.4083, https://doi.org/10.1007/JHEP11(2013)041, 2-loop Bhabha massive QED, planar case. $-\frac{s}{m^2} = \frac{(1-x)^2}{x}$, $|x| < 1$, $s \to t$, $x \to y$, $\{x, 1 \pm x, y, 1 \pm y, x + y, 1 + xy\}$

Various approaches for solving system of DEs with high precision numerically (II)

Methods

(i) Numerical MIs, boundary conditions in unphysical region, SD, contours deformation; Mandal, Zhao, 1812.03060, https://doi.org/10.1007/JHEP03(2019)190;
(ii) Numerical MIs, numerical DEs; Czakon, Niggetiedt, https://doi.org/10.1007/JHEP05(2020)149;
(iii) Boundaries by expansion by regions; Beneke, Smirnov et al, https://doi.org/10.1140/epjc/s10052-012-2139-2;
(iv) Semi-analytically: one-dimensional and generalized series expansions;
Moriello, 1907.13234, https://doi.org/10.1007/JHEP01(2020)150
Hidding ('DiffExp'), 2006.05510 , https://doi.org/10.1016/j.cpc.2021.108125
Becchetti et al, 2112.07578, https://inspirehep.net/literature/1989984

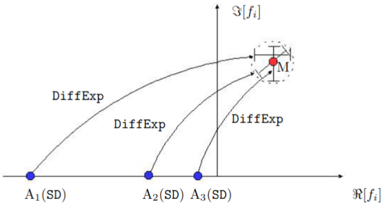
(v) ...

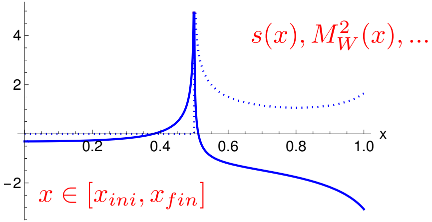
Some applications:

1907.13156, 1911.06308, https://doi.org/10.1007/JHEP06(2020)093, 2-loop Higgs+jet
2112.07578, https://inspirehep.net/literature/1989984, 2-loop mixed QCD-electroweak corrections to $gg \to Hg$,
2001.03008, http://dx.doi.org/10.1007/JHEP05(2020)149 3-loop form factor, production of an off-shell Higgs boson in gluon fusion in QCD,
2106.05296, https://doi.org/10.1007/JHEP09(2021)152, 4-loop MSbar quark mass relations,

I highlight two recent approaches which give many digits solutions

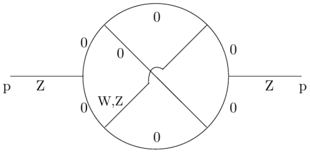
1. I. Dubovyk, A. Freitas, JG, K. Grzanka, M. Hidding, J. Usovitsch, 'Evaluation of multi-loop multi-scale Feynman integrals for precision physics', https://arxiv.org/abs/2201.02576.
   Automation with (see talk by Johann Usovitsch, Caesar project):
   - KIRA (IBPs) - finite number of MIs;
   - Reduze by A. von Manteuffel, E. Panzer, R. M. Schabinger - choice of MIs basis for boundary conditions without ϵ singularities;
   - pySecDec - calculation of boundary conditions in Euclidean points;
   - DiffExp - transport of solutions to the Minkowskian points.



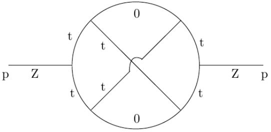


MIs with high accuracy*, results (see Johann's talk for details)

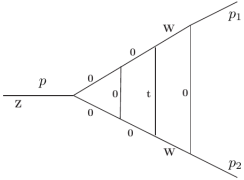
With present SD and MB tools we can not get results for 3-loop EWPOs at the e^+e^- Z-resonance peak.



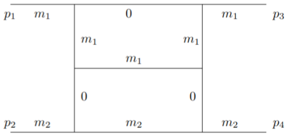
IhNp1



taNPl1



vtwPl



box2l

$$
\begin{aligned}
I_{\text{box2l}}[2,1,1,1,1,1,1,0,0,s,t,m_1^2,m_2^2] &= +0.000328707579/\epsilon^2 \\
&- (0.0014129475 - 0.0020653306\, i)/\epsilon \\
&- (0.005702737 - 0.000485980\, i) + \mathcal{O}(\epsilon), \\
&\quad 55 \text{ MIs}, s = 2, t = 5, m_1^2 = 4, m_2^2 = 16.
\end{aligned}
$$

2. A set of Jan 27 2022 papers by Zhi-Feng Liu, Yan-Qin Ma and Xiao Liu: https://inspirehep.net/literature/2020677, https://inspirehep.net/literature/2020676, https://inspirehep.net/literature/2020880 and 1711.09572 https://inspirehep.net/literature/1639025.

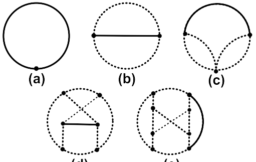
$$\widetilde{I}_{\vec{\nu}}(\eta) = \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{D/2}} \right) \frac{\widetilde{\mathcal{D}}_{K+1}^{-\nu_{K+1}} \cdots \widetilde{\mathcal{D}}_N^{-\nu_N}}{\widetilde{\mathcal{D}}_1^{\nu_1} \cdots \widetilde{\mathcal{D}}_K^{\nu_K}}.$$

$$\widetilde{\mathcal{D}}_1 = \ell_1^2 - m^2 + \mathrm{i}\eta$$

$$I_{\vec{\nu}} = \lim_{\eta \to 0^+} \widetilde{I}_{\vec{\nu}}(\eta)$$

$$\mathrm{i}\frac{\partial}{\partial \eta} \vec{\widetilde{J}}(\eta) = A(\eta)\widetilde{J}(\eta)$$

Key point: boundary conditions at $\eta \to \infty$ are single mass scale bubble integrals, solved iteratively.



$$I_{\vec{\nu}} = \int \left(\prod_{i=1}^{L} \frac{D\ell_i}{\pi^{D/2}}\right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}}, \quad \mathcal{D}_1 = \ell_1^2 - m^2 + 0^+$$

$$\widehat{I}_{\vec{\nu}'}(\ell_1^2) = \int \left(\prod_{i=2}^{L} \frac{D\ell_i}{\pi^{D/2}}\right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_2^{\nu_2} \cdots \mathcal{D}_K^{\nu_K}}, \quad I_{\vec{\nu}} = \{\Gamma[...]\}\widehat{I}_{\vec{\nu}'}(-m^2)$$

L-loop ↑ $(L\text{-}1)$-loop ↑

$$\begin{aligned} I[(e)] = & - 2.073855510286740\epsilon^{-2} - 7.812755312590133\epsilon^{-1} \\ & - 17.25882864945875 + 717.6808845492140\epsilon \\ & + 8190.876448160049\epsilon^2 + 78840.29598046500\epsilon^3 \\ & + 566649.1116484678\epsilon^4 + 3901713.802716081\epsilon^5 \\ & + 23702384.71086095\epsilon^6 + 14214293.68205112\epsilon^7, \end{aligned}$$

10 orders in ϵ, 16-digit precision.

Such an exact boundary point can be transported by `DiffExp` *to any physical point* → Johann's talk.

Expected precision in 2040

Conclusion of the 2018 Workshop

J. Gluza

"We anticipate that, at the beginning of the FCC-ee campaign of precision measurements, the theory will be precise enough not to limit their physics interpretation. This statement is however conditional to sufficiently strong support by the physics community and the funding agencies, including strong training programmes".

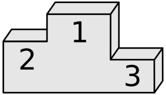
- **Numerical evaluation with three-loops calculations:**

arXiv:1901.02648

	$\delta\Gamma_Z$ [MeV]	δR_l $[10^{-4}]$	δR_b $[10^{-5}]$	$\delta \sin^{2,l}_{eff}\theta$ $[10^{-6}]$
Present EWPO theoretical uncertainties				
EXP-2018	2.3	250	66	160
TH-2018	0.4	60	10	45
EWPO theoretical uncertainties when FCC-ee will start				
EXP-FCC-ee	~~0.1~~ 0.025	10	$2 \div 6$	~~6~~ 3
TH-FCC-ee	0.07	7	3	7

- 500 person-years needed over 20 years – Recognized as strategic priority.

- Concerning (very) accurate calculation of MIs, the progress in recent two years is gigantic (in terms of quality and quantity);
- At the moment the most robust method for precision calculations seems to be DEs;

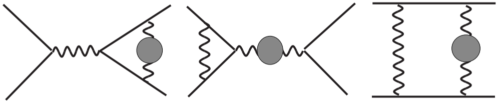


- Other discussed methods will become efficient too $\longrightarrow$ cross checks needed for physical results;
- *Developed strategies for multi-loop evaluations along with (improving all the time) tools are already at the level to meet severe experimental demands of future colliders, in particular FCC-ee experiment, and will enhance HL-LHC precision physics studies.*
- Still, many things to be done (other talks) $\rightarrow$ The real challenge for the young generation!

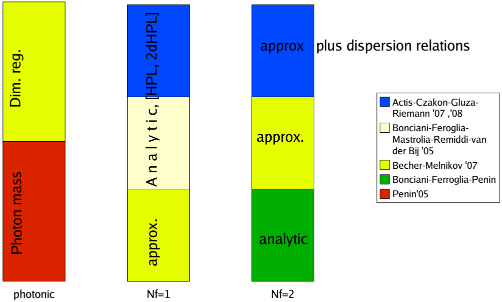
Backup slides

`Thank you for listening!`

Everywhere, cross-checks are necessary. 2-loop massive Bhabha case.



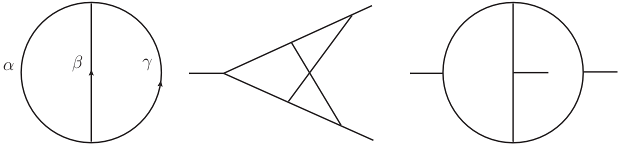
(a) (b) (c)



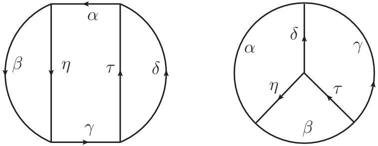
AMBREv3.m:

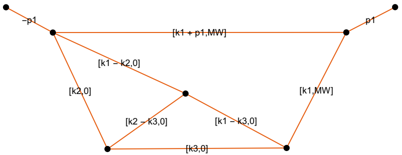
- topology based factorization - chain diagrams, Kinoshita '74

2-loop:



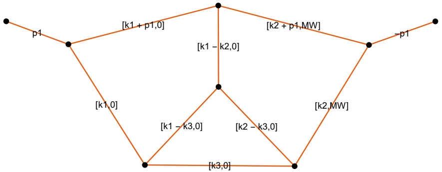
3-loop:





1-dim

$-18.779406962 - 6.390785027i$



4-dim

$-22.5213 + 4.74442i \pm (0.001 + 0.001i)$

$$I = -\frac{1}{(-s)^{1+3\epsilon}} \int_{-i\infty}^{+i\infty} \prod_{i=1}^{4} dz_i \left(-\frac{M_W^2}{s}\right)^{z_3} \frac{\Gamma(-\epsilon - z_1)\Gamma(-z_1)\Gamma(1+2\epsilon+z_1)}{\Gamma(1-2\epsilon)\Gamma(1-3\epsilon-z_1)}$$
$$\times \frac{\Gamma(-2\epsilon - z_{12})\Gamma(1-\epsilon+z_2)\Gamma(1+z_{12})\Gamma(1+\epsilon+z_{12})\Gamma(1+3\epsilon+z_3)\Gamma(1-\epsilon-z_4)}{\Gamma(1-z_2)\Gamma(2+\epsilon+z_{12})}$$
$$\times \frac{\Gamma(-\epsilon-z_2)\Gamma(-z_2)\Gamma(1+z_3-z_4)\Gamma(-z_4)\Gamma(-z_3+z_4)\Gamma(-3\epsilon-z_3+z_4)}{\Gamma(1-4\epsilon-z_3)\Gamma(2+2\epsilon+z_3-z_4)}.$$

$$I = \frac{3}{s} \int_{-i\infty-\frac{17}{28}}^{+i\infty-\frac{17}{28}} dz_3 \left(-\frac{M_W^2}{s}\right)^{z_3} \frac{\Gamma(-1-z_3)\Gamma(-z_3)(\Gamma(1-z_3)\Gamma(-z_3) - \Gamma(-2z_3))}{\Gamma(1+z_3)\Gamma(-2z_3)}$$
$$\times \Gamma(1+z_3)\psi^{(2)}(1).$$

3-loops, details

All SM massive bosons (and top quark) involved.
$M_Z = 1$, $M_W = \sqrt{0.78}$, $M_H = \sqrt{1.88}$, $M_t = \sqrt{3.6}$,

$$s = M_Z^2 (= M_W^2)$$

,

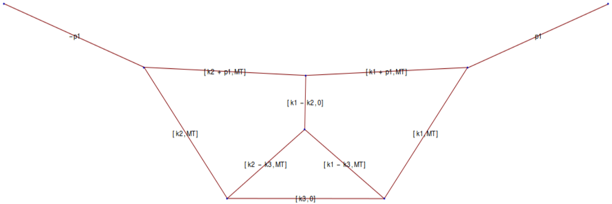
- Self-energies (SE)
  - ta - diagrams with top quark + photon
  - th - diagrams with top quark + W/Z/H
  - lh - diagrams with light quarks + W/Z/H
- △ Mercedes (Merc)
- △ Non-planar (NP)
- △ Planar (PL)

Towards 3-loop results ("Report 1")

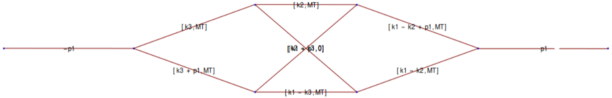
$Z \to b\bar{b}$			
Number of topologies	1 loop	2 loops	3 loops
	1	5	50
Number of diagrams	15	1114	120187
Fermionic loops	0	150	**17580**
Bosonic loops	15	964	102607
QCD / EW	1 / 14	98 / 1016	10405 / 109782

Table: The number of Z decay Feynman diagrams needed to be calculated to meet FCC-ee experimental accuracy. Tadpoles, products of lower loop diagrams and symmetrical diagrams are not included.

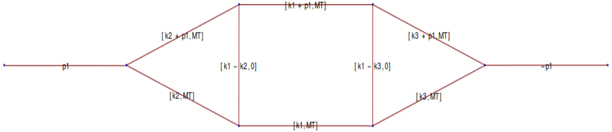
Up to 8 (massive) propagators, tensors up to rank 6.
Vertices are not necessarily more difficult (e.g. MB integrals).



- MERC



- NP



- PL

Examples: pySecDec - status at NNNLO, CPUs: 24, CPU [MHz]: 3000, RAM [GB]: 128, ϵ^0 terms.

Type	Propagators	Masses	Errors (Merc,NP,PL)	Time(Merc,NP,PL)
ta	6	4	10^{-8}, -, -	6s, -, -
	7	5	10^{-9}, 10^{-9}, 10^{-8}	20s, 9s, 9s
	8	6	10^{-9}, 10^{-9}, 10^{-9}	12s, 9s, 5s
th	6	5	10^{-8}, -, -	9s, -, -
	7	6 (H)	10^{-9}, 10^{-9}, 10^{-9}	37s, 19s, 19s
	8	7	10^{-10}, 10^{-10}, 10^{-10}	48s, 16s, 10s
	8	3	10^{-2}, 10^{-3}, 10^{-2}	3h, 4h, 2h
lh	6	1	10^{-5}, 10^{-8}, -	15min, 9s, -
	7	1	10^{-3}, 10^{-3}, 10^{-3}	2h, 2h, 3h
	8	1	NAN, 10^{-1}, 10^{-1}	4h, 6h, 7h