The 2011 CERN – Latin-American School of High-Energy Physics

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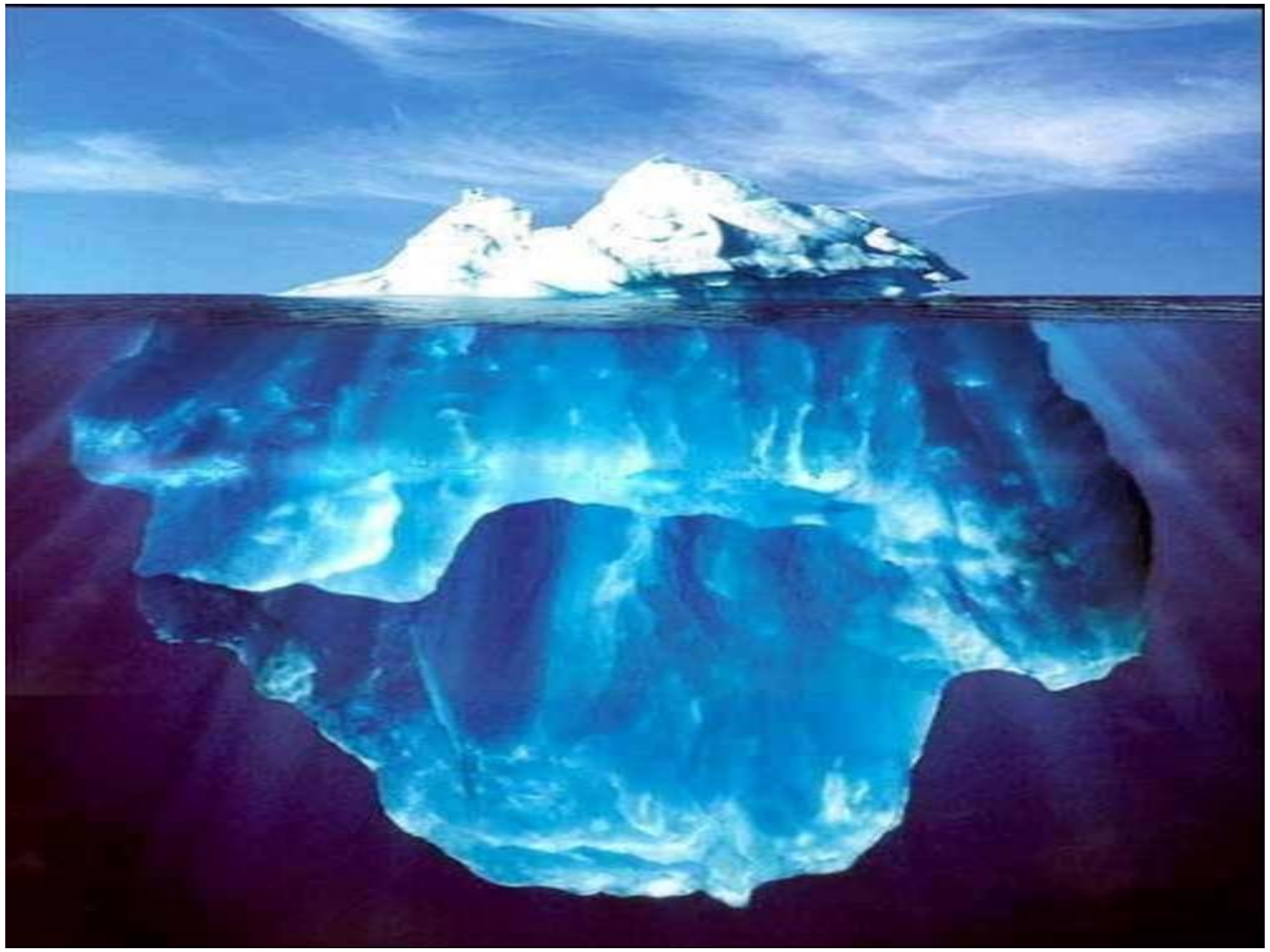
QFT and the EW Standard Model



Just the tip of the iceberg...

Outline

- Why Quantum Field Theory?
- Quantisation
- Kinematical symmetries
- Global symmetries
- Local symmetries
- Discrete symmetries
- Broken symmetries
- Scale symmetries, renormalisation
- Standard Model symmetries
- Amusing examples throughout time permitting

All this in four lectures...

Do we really need it?

The Schrödinger equation, plus many body physics constructions are very successful in atomic, molecular and solid state physics. The theory of bands, electrical conductivity, atomic bonding, orbitals... are adequately explained in this scheme

$$i\frac{\partial}{\partial t}\Psi(\mathbf{r_1},\mathbf{r_2},...,\mathbf{r_N},t) = \left(\sum_i \frac{(\mathbf{p}_i - e_i\mathbf{A}_i)}{2m_i} + e_i\Phi_i + V(\mathbf{r}_i)\right)\Psi(\mathbf{r}_j,t)$$

$$P(\mathbf{r_1},\mathbf{r_2},...,\mathbf{r_N},t) = |\Psi(\mathbf{r_1},\mathbf{r_2},...,\mathbf{r_N},t)|^2, \qquad \int\prod_{i=1}^{N} d^3\mathbf{r}_i\, P(\mathbf{r_j},t) = 1 \ \ \forall\ t$$

A note on conventions

$$\hbar = c = 1, \qquad \eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1) \qquad \mathbf{F} = \frac{1}{4\pi}\frac{qq'}{r^3}\mathbf{r} \qquad \alpha = \frac{e^2}{4\pi\hbar c} \qquad e \approx .303$$

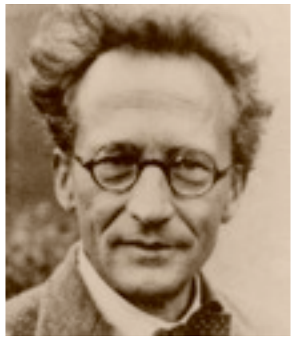
Einstein and Heisenberg complicate our lives

Useful basic formulae. A reminder. Just this once, we reintroduce h and c

$$
\begin{aligned}
p^2 &= \left(\frac{E}{c}\right)^2 - \mathbf{p}^2 = m^2c^2 \\
E &= \pm\sqrt{\mathbf{p}^2c^2 + m^2c^4} \approx \pm(mc^2 + \frac{\mathbf{p}^2}{2m} + \ldots) \\
\Delta x\,\Delta p &\geq \frac{\hbar}{2} \\
\lambda &= \frac{h}{mc} \qquad \text{Compton wavelength} \\
E &= \frac{mc^2}{\sqrt{1-\mathbf{v}^2/c^2}} \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\mathbf{v}^2/c^2}} \\
\Delta p &\geq mc \qquad \Delta E \geq mc^2 \\
(\Delta x)_{\rm min} &\geq \frac{1}{2}\left(\frac{\hbar}{mc}\right)
\end{aligned}
$$

When the uncertainly in momentum is bigger than m c, the uncertainty in energy is larger than m c^2, hence there is enough energy to produce another particle of the same type. In Relativity mass and energy are interchangeable. Hence we cannot localise a particle below its Compton wavelength. If we do, we will not find a single particle, but rather a fairly complicated quantum state with no well-defined number of particles. Particle production by physical processes should be a central part of the theory.

Klein paradoxes...



Another way to see the same problem is to consider a particle in a potential barrier in the simplest relativistic generalisation of the Schroedinger equation, the Klein-Gordon equation

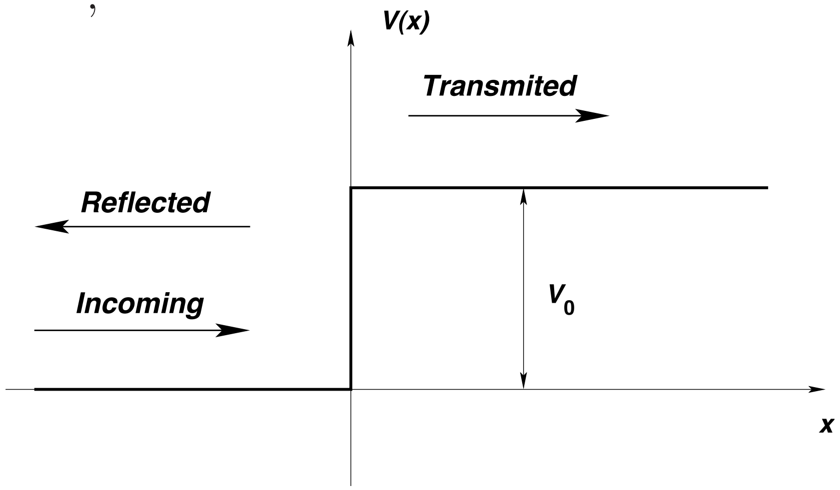
$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(t,\mathbf{x}) = 0$$

$$p_1 = \sqrt{E^2 - m^2}, \qquad p_2 = \sqrt{(E-V_0)^2 - m^2}$$

$$T = \frac{2p_1}{p_1 + p_2}, \qquad R = \frac{p_1 - p_2}{p_1 + p_2}$$

$$\psi_I(t,x) = e^{-iEt + ip_1 x} + Re^{-iEt - ip_1 x},$$
$$\psi_{II}(t,x) = Te^{-iEt + p_2 x},$$

$$\psi_I(t,0) = \psi_{II}(t,0)$$
$$\partial_x \psi_I(t,0) = \partial_x \psi_{II}(t,0)$$

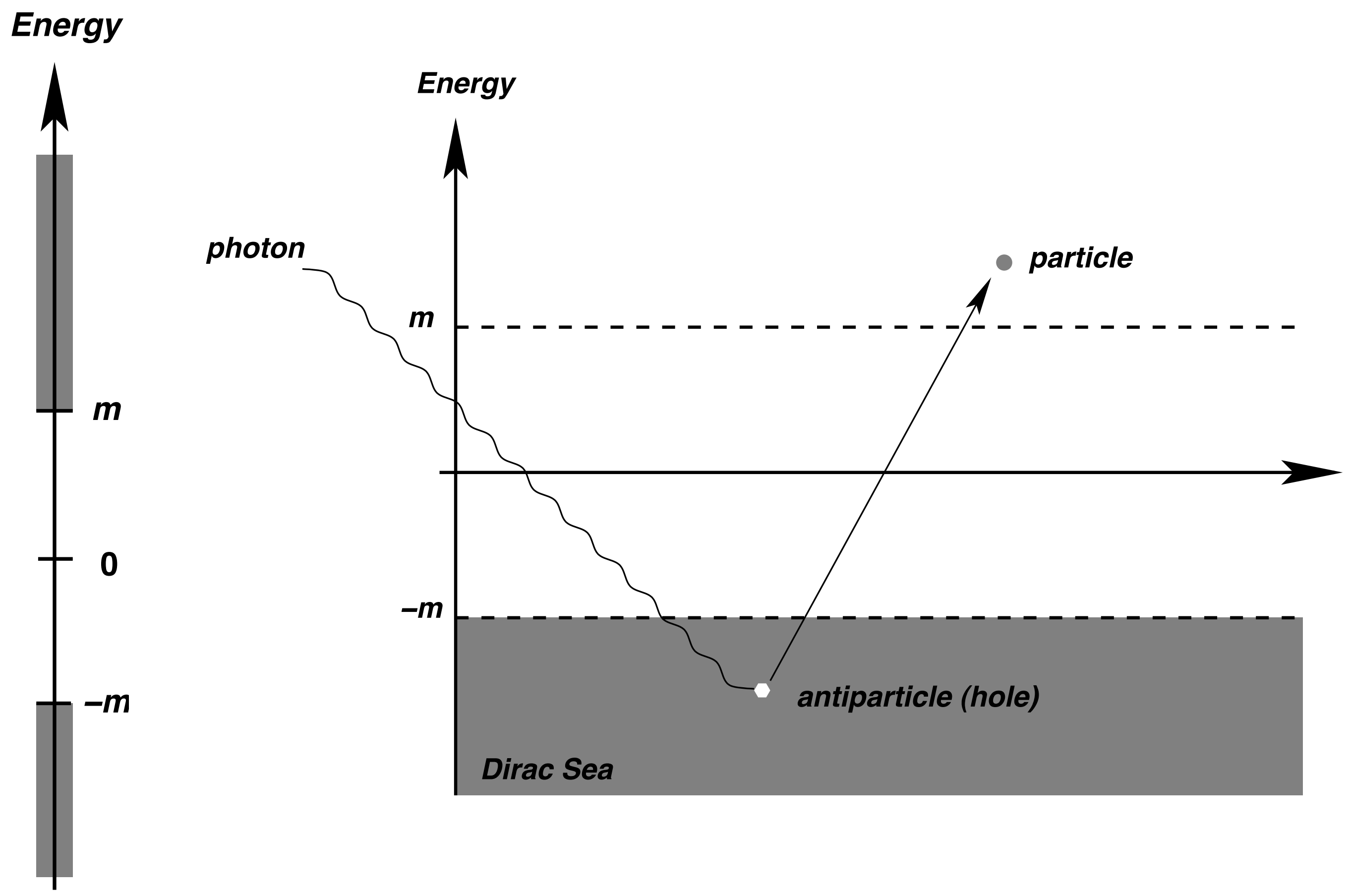


Three cases to consider

$$1)\ E - m > V_0 \qquad 2)\ E - m < V_0 \qquad 3)\ V_0 > 2m \qquad V_0 - 2m < E - m < V_0$$

In the third case we have the strange situation that we have transmitted wave with negative kinetic energy $E - m - V_0$

In the equation that bears his name, Dirac also found the problem with negative energy states. In his case however he found a rather ingenious way to solve the problem. Since he was describing electrons, he decided to simply fill all the negative energy states, this way Pauli's principle would guarantee stability. His equation also predicted the existence of anti-particles, although at the beginning he was reluctant to accept it. With the Dirac sea we have a simple way to understand anti-electrons = positrons (more later)



An energetic photon can make a hole. The absence of a negative energy state with negative charge manifests itself as a particle of positive energy and positive charge:

the positron

- We still have a multi-particle theory after all
- This does not work for bosons…
- We should give up the wave equation approach

Beating a dead horse...

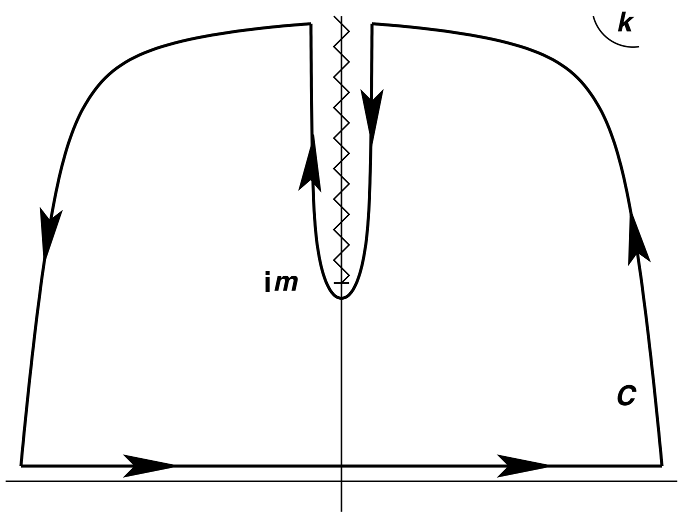
If we still insist against all odds, and decide to violate locality, but to eliminate once and for all the negative energy states by choosing our free Hamiltonian as follows:

$$H = \sqrt{-\nabla^2 + m^2}$$

$$\psi(0,\mathbf{x}) = \delta(\mathbf{x})$$

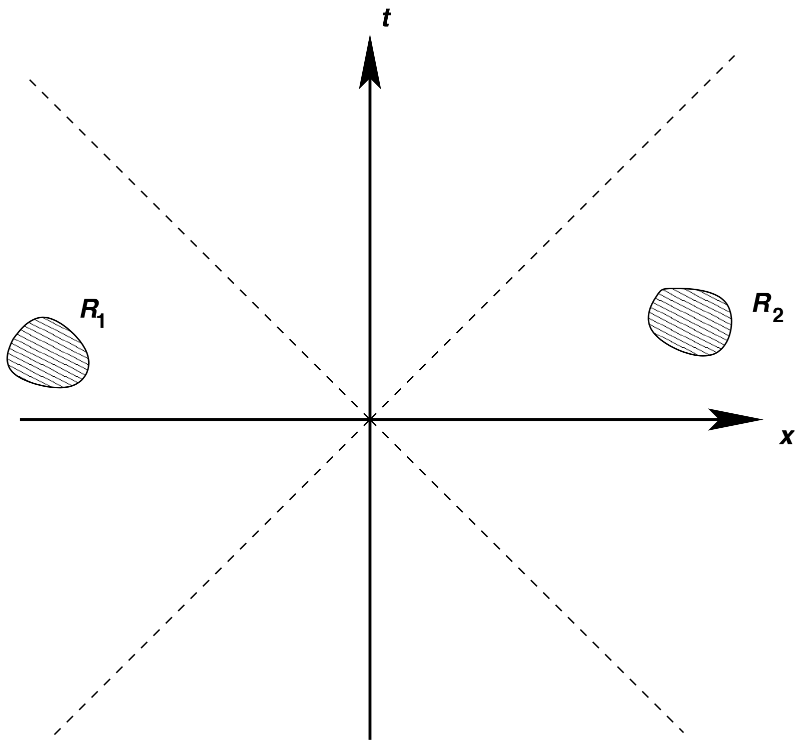
$$\psi(t,\mathbf{x}) = e^{-it\sqrt{-\nabla^2+m^2}}\,\delta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x} - it\sqrt{k^2+m^2}}.$$

$$\psi(t,\mathbf{x}) = \frac{1}{2\pi^2|\mathbf{x}|}\int_{-\infty}^{\infty} k\,dk\, e^{ik|\mathbf{x}|}\, e^{-it\sqrt{k^2+m^2}}.$$



Oops!! we have violated causality! For any t>0 and any |x|, this wave function does not vanish!...

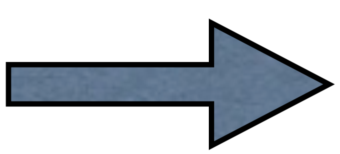
Relativistic causality



Microscopic causality, Locality in Special Relativity imposes important constraints into what are observables. The light-cone decrees the causal structure of space-time. Physical measurements should be compatible with it

$$[\mathcal{O}(x), \mathcal{O}(y)] = 0, \qquad \text{if } (x-y)^2 < 0.$$

- The world is Quantum
- Particle Wave Duality
- Special Relativity
- Microscopic Causality



LQFT

From classical to quantum fields

In scattering experiments we observe asymptotic free particles characterised by their energy-momentum charge and other quantum numbers. Ignore for the moment everything but energy-momentum. In the NR-case we describe the one-particle states by kets carrying a unitary rep. of the rotation group.

$$|\mathbf{p}\rangle \in \mathscr{H}_1, \qquad \langle \mathbf{p}|\mathbf{p}'\rangle = \delta(\mathbf{p}-\mathbf{p}') \qquad \int d^3p\, |\mathbf{p}\rangle\langle\mathbf{p}| = \mathbf{1}. \qquad \mathscr{U}(R)|\mathbf{p}\rangle = |R\mathbf{p}\rangle \qquad \widehat{P}^i = \int d^3p\, |\mathbf{p}\rangle p^i \langle \mathbf{p}|$$

To deal with multi-particle states it is convenient to introduce creation and annihilation operators, this leads to the Fock space of states, built out of the vacuum by acting with creation operators:

$$|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle, \qquad a(\mathbf{p})|0\rangle = 0 \qquad \langle 0|0\rangle = 1$$

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta(\mathbf{p}-\mathbf{p}'), \qquad [a(\mathbf{p}), a(\mathbf{p}')] = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0,$$

We need relativistic invariance, hence we need to find ways to count states in an invariant way. This is necessary also when we deal with decay rates and cross sections. We need to count final states in a way consistent with Lorentz invariance. We can easily construct such an invariant phase space volume:

$$\int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\,\theta(p^0)\,f(p)$$ to integrate over p0, we use a nice identity:

$$\delta[g(x)] = \sum_{x_i = \text{zeros of } g} \frac{1}{|g'(x_i)|}\delta(x-x_i) \qquad \delta(p^2-m^2) = \frac{1}{2p^0}\delta\left(p^0 - \sqrt{\mathbf{p}^2+m^2}\right) + \frac{1}{2p^0}\delta\left(p^0 + \sqrt{\mathbf{p}^2+m^2}\right)$$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}$$ with $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2+m^2}$ and $(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}')$ are invariant

Now proceed by imitation of the NR case, with the non-trivial result that we have a unitary representation of the Lorentz group

$$|p\rangle = (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}\,|\mathbf{p}\rangle \qquad \langle p|p'\rangle = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}') \qquad \hat{P}^\mu = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|p\rangle p^\mu \langle p| \qquad \mathscr{U}(\Lambda)|p\rangle = |\Lambda^\mu_{\ \nu}p^\nu\rangle \equiv |\Lambda p\rangle$$

$$\langle 0|0\rangle = 1$$

$$\alpha(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}a(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}'),$$

$$\alpha^\dagger(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}a^\dagger(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha(\mathbf{p}')] = [\alpha^\dagger(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = 0.$$

$$|f\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}f(\mathbf{p})\alpha^\dagger(\mathbf{p})|0\rangle$$

Let us construct some observable in this theory. It will be an operator depending on space time, and satisfying some simple conditions:

- Hermiticity $\quad \phi(x)^\dagger = \phi(x).$
- Microcausality $\quad [\phi(x),\phi(y)] = 0, \qquad (x-y)^2 < 0.$
- Translational invariance $\quad e^{i\hat{P}\cdot a}\phi(x)e^{-i\hat{P}\cdot a} = \phi(x-a)$
- Lorentz invariance $\quad \mathscr{U}(\Lambda)^\dagger\phi(x)\mathscr{U}(\Lambda) = \phi(\Lambda^{-1}x).$
- Linearity

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[f(\mathbf{p},x)\alpha(\mathbf{p}) + g(\mathbf{p},x)\alpha^\dagger(\mathbf{p})\right].$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[e^{-iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}\alpha(\mathbf{p}) + e^{iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}}\alpha^\dagger(\mathbf{p})\right]$$

+ve energy

-ve energy

We have obtained from first principles the quantisation of the Klein-Gordon field. There are more straightforward ways, but the procedure shows how to implement the basis principles of the theory, Lorentz invariance, locality and positivity of the spectrum

Some important properties

$$[\phi(t,\mathbf{x}), \partial_t \phi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}).$$

$$[\phi(x), \phi(x')] = i\Delta(x-x')$$

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = 0$$

$$i\Delta(x-y) = -\mathrm{Im} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-iE_{\mathbf{p}}(t-t')+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}$$

$$= \int \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2-m^2)\varepsilon(p^0) e^{-ip\cdot(x-x')}$$

$$\Delta(x-y) = 0 \qquad \text{for } (x-y)^2 < 0$$

The construction is free of paradoxes. It satisfies the KG equation because the +ve and -ve energy plane waves satisfy it. Of course with a free field we do not go very far…

We should design more powerful techniques leading to similar properties by for more general theories where interactions can take place.

There are two general approaches: the canonical-formalism, and the Feynman path integral. We will briefly introduce the first, just as a reminder.

Remember: PHYSICS is where the ACTION is!

Proceed by analogy with ordinary QM

$$S[x,\dot{x}] = \int dt\, L(x,\dot{x})$$

$$L = \sum_i \frac{1}{2} m_i \dot{\mathbf{x}}_i^2 - V(\mathbf{x})$$

$$S[\phi(x)] \equiv \int d^4x \mathscr{L}(\phi, \partial_\mu\phi) = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right)$$

$$\mathbf{x}_a,\ \dot{\mathbf{x}}_a \longleftrightarrow \phi(\mathbf{x},0),\ \dot{\phi}(\mathbf{x},0)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\partial_\mu\left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\partial\mathscr{L}}{\partial\phi} = 0 \quad \Longrightarrow \quad (\partial_\mu\partial^\mu + m^2)\phi = 0$$

canonical momenta

$$p = \frac{\partial L}{\partial \dot{x}}$$

$$\pi(x) \equiv \frac{\partial\mathscr{L}}{\partial(\partial_0\phi)} = \frac{\partial\phi}{\partial t}$$

$$H = \sum_i p_i\, \dot{x}^i - L$$

$$H \equiv \int d^3x \left(\pi\frac{\partial\phi}{\partial t} - \mathscr{L}\right) = \frac{1}{2}\int d^3x \left[\pi^2 + (\nabla\phi)^2 + m^2\right].$$

$$[q^i, p_j] = i\hbar$$

$$[\phi(t,\mathbf{x}), \partial_t\phi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}).$$

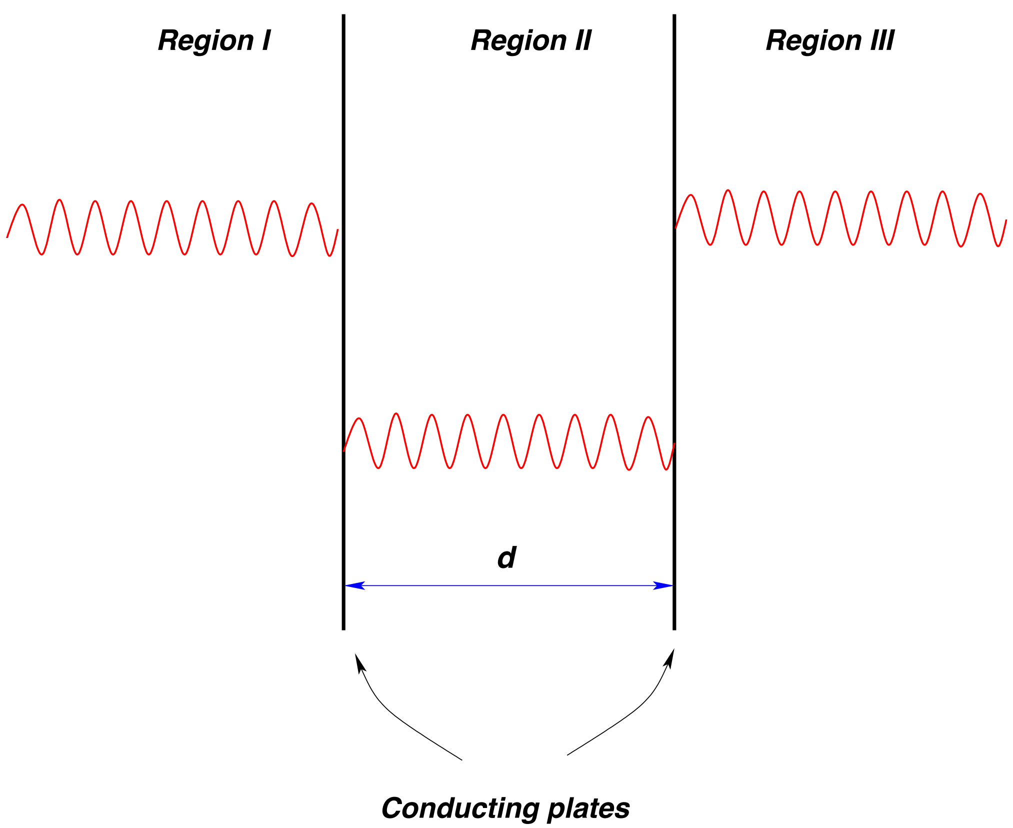
Expanding in solutions to the KG equations and performing the canonical quantisation, we recover the algebra of creation and annihilation operator we had before, but we get a surprise

Casimir effect

Writing the products of creation and ann. operators in NORMAL ORDERING i,e, annihilation operators to the right, we get rid of the sum of the zero point energy of the infinite number of oscillators in the field. In infinite space we subtract it, or simply normal order. When we do not have translational invariance, something interesting happens

$$\begin{aligned}\widehat{H} &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left[\widehat{\alpha}^\dagger(\mathbf{p})\widehat{\alpha}(\mathbf{p}) + (2\pi)^3 E_{\mathbf{p}}\,\delta(\mathbf{0})\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}E_{\mathbf{p}}\widehat{\alpha}^\dagger(\mathbf{p})\widehat{\alpha}(\mathbf{p}) + \frac{1}{2}\int d^3p\, E_{\mathbf{p}}\delta(\mathbf{0})\end{aligned}$$

$$E(d)_{\text{reg}} = E(d)_{\text{vac}} - E(\infty)_{\text{vac}}$$



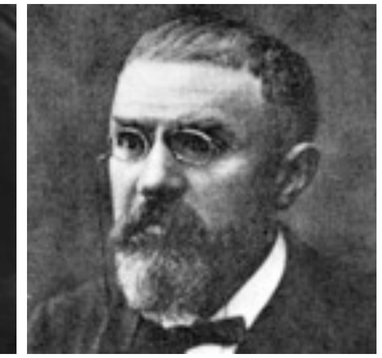
The force per unit area is the derivative of this quantity with respect to d divided by the area of the plates. The result is finite and attractive, the Casimir force! Which has been measured (of course for the electromagnetic field)

$$P_{\text{Casimir}} = -\frac{\pi^2}{240}\frac{1}{d^4}$$

Lorentz and Poincaré Groups

In trying to systematically construct viable QFTs it is useful to understand the representations of the Lorentz (and Poincaré) groups.

The Hilbert space of states has to carry a unitary representation of the Lorentz group, so that quantum amplitudes are consistent with Unitarity and Relativistic Invariance. The fields themselves however, transform under finite dimensional representations. They are much easier to study. Just recall the usual rotation group SU(2). The Lorentz group, also known as SO(3,1) preserves the Minkowski metric


$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu \qquad \mu, \nu = 0, 1, 2, 3$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

$$\det \Lambda = \pm 1 \qquad (\Lambda^0_0)^2 - \sum_{i=1}^{3} (\Lambda^i_0)^2 = 1$$

- $\mathfrak{L}_+^\uparrow$: proper, orthochronous transformations with $\det \Lambda = 1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}_-^\uparrow$: improper, orthochronous transformations with $\det \Lambda = -1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}_-^\downarrow$: improper, non-orthochronous transformations with $\det \Lambda = -1, \Lambda^0{}_0 \leqslant -1$.
- $\mathfrak{L}_+^\downarrow$: proper, non-orthochronous transformations with $\det \Lambda = 1, \Lambda^0{}_0 \leqslant -1$.

$$\mathfrak{L}_+^\uparrow \xrightarrow{\mathscr{P}} \mathfrak{L}_-^\uparrow, \qquad \mathfrak{L}_+^\uparrow \xrightarrow{\mathscr{T}} \mathfrak{L}_-^\downarrow, \qquad \mathfrak{L}_+^\uparrow \xrightarrow{\mathscr{P}\mathscr{T}} \mathfrak{L}_+^\downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Lorentz and Poincaré Groups

$$R(\mathbf{e},\varphi) = e^{-i\varphi\,\mathbf{e}\cdot\mathbf{J}}$$

$$B(\mathbf{u},\lambda) = e^{-i\lambda\,\mathbf{u}\cdot\mathbf{M}}$$

Rotations and boosts generate Lorentz transformation, hence six parameter and six generators of infinitesimal transformations. The algebra is easy to obtain and "diagonalise"

$$\begin{aligned}[]
[J_i, J_j] &= i\varepsilon_{ijk} J_k,\\
[J_i, M_k] &= i\varepsilon_{ijk} M_k,\\
[M_i, M_j] &= -i\varepsilon_{ijk} J_k
\end{aligned}$$

$$J_k^{\pm} = \frac{1}{2}(J_k \pm iM_k)$$

$$\begin{aligned}[]
[J_i^{\pm}, J_j^{\pm}] &= i\varepsilon_{ijk} J_k^{\pm},\\
[J_i^{+}, J_j^{-}] &= 0.
\end{aligned}$$

The representations of each SU(2) are labelled by a single integer or half integer "angular" momentum s=0, 1/2, 1, 3/2, … Under parity

$$(\mathbf{s}_+, \mathbf{s}_-)$$

Representation	Type of field
$(\mathbf{0},\mathbf{0})$	Scalar
$(\frac{1}{2},\mathbf{0})$	Right-handed spinor
$(\mathbf{0},\frac{1}{2})$	Left-handed spinor
$(\frac{1}{2},\frac{1}{2})$	Vector
$(\mathbf{1},\mathbf{0})$	Selfdual antisymmetric 2-tensor
$(\mathbf{0},\mathbf{1})$	Anti-selfdual antisymmetric 2-tensor

$$\begin{aligned}
\mathbf{J} &\xrightarrow{P} \mathbf{J}\\
\mathbf{M} &\rightarrow -\mathbf{M}\\
\mathbf{J}^{\pm} &\rightarrow \mathbf{J}^{\mp}\\
(\mathbf{s_1}, \mathbf{s_2}) &\rightarrow (\mathbf{s_2}, \mathbf{s_1})
\end{aligned}$$

$$\mathbf{J} = \mathbf{J}^+ + \mathbf{J}^-$$

$$(\mathbf{s}_+, \mathbf{s}_-) = \sum_{\mathbf{j}=|\mathbf{s}_+-\mathbf{s}_-|}^{\mathbf{s}_++\mathbf{s}_-} \mathbf{j}$$

Luis Alvarez-Gaume Natal Lectures March 23-26 2011

Weyl spinors

The simplest representations have fundamental physical importance, they are called Weyl spinors. Clearly they are representations of the connected component of SO(3,1), but not of parity, since parity interchanges the representations

$$J_i^+ = \frac{1}{2}\sigma_i, \qquad J_i^- = 0 \qquad \text{for} \qquad (\tfrac{1}{2}, \mathbf{0}),$$

$$J_i^+ = 0, \qquad J_i^- = \frac{1}{2}\sigma_i \qquad \text{for} \qquad (\mathbf{0}, \tfrac{1}{2}).$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$u_\pm \longrightarrow e^{-\frac{i}{2}(\theta\mathbf{n} \mp i\beta)\cdot\sigma} u_\pm \qquad\qquad u_\pm \longrightarrow e^{i\theta} u_\pm$$

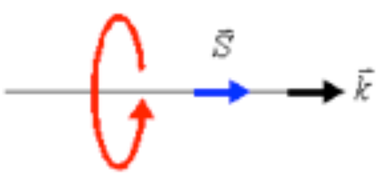
Consider for simplicity this global symmetry: fermion number

$$\sigma_\pm^\mu = (\mathbf{1}, \pm\sigma_i) \qquad u_+^\dagger \sigma_+^\mu u_+ \qquad u_-^\dagger \sigma_-^\mu u_-$$

$$\mathscr{L}_{\text{Weyl}}^\pm = i u_\pm^\dagger (\partial_t \pm \sigma\cdot\nabla) u_\pm = i u_\pm^\dagger \sigma_\pm^\mu \partial_\mu u_\pm$$

$$(\partial_0 \pm \sigma\cdot\nabla) u_\pm = 0 \qquad u_\pm(x) = u_\pm(k) e^{-ik\cdot x} \qquad k^2 = k_0^2 - \mathbf{k}^2 = 0 \qquad (|\mathbf{k}| \mp \mathbf{k}\cdot\sigma) u_\pm = 0$$

$$u_+ : \qquad \frac{\sigma\cdot\mathbf{k}}{|\mathbf{k}|} = 1,$$



positive helicity, right handed antineutrinos

$$u_- : \qquad \frac{\sigma\cdot\mathbf{k}}{|\mathbf{k}|} = -1$$

negative helicity, left handed, neutrinos

Charge conjugation and Majorana masses

We know that under parity, the L,R Weyl spinors are exchanged. Another way to exchange them is via complex conjugation, later to be related to charge conjugation

$$\begin{aligned} M_L &= e^{-\frac{i}{2}\theta\cdot\sigma-\frac{1}{2}\beta\cdot\sigma} & \det M_L &= 1 & \det M &= \epsilon_{ab} M_{a1} M_{b2} \\ M_R &= e^{-\frac{i}{2}\theta\cdot\sigma+\frac{1}{2}\beta\cdot\sigma} & \det M_R &= 1 & \det M \epsilon_{ab} &= \epsilon_{cd} M_{ca} M_{db} \end{aligned} \qquad \epsilon = i\,\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Using $\sigma^* = -\sigma_2\,\sigma\,\sigma_2$

$$\begin{aligned} \psi_L^c &= \sigma_2\,\psi_L^* \quad \text{transforms like} \quad \psi_R \\ \psi_R^c &= \sigma_2\,\psi_R^* \quad \text{transforms like} \quad \psi_L \end{aligned}$$

- We can express any theory fully in terms of L or R fermions.
- Charge conjugation and parity exchange L and R
- A parity invariant theory requires L,R spinors at the same time
- We can construct a mass for pure L spinors if we ignore fermion number
- Fermions are anticommuting

$$\mathscr{L}^{\pm}_{\text{Weyl}} = i u_\pm^\dagger \sigma_\pm^\mu \partial_\mu u_\pm + \frac{m}{2}\left(\varepsilon_{ab} u_\pm^a u_\pm^b + \text{h.c.}\right)$$

$$\epsilon_{ab} u^a\, u^b = u^1\, u^2 - u^2\, u^1$$

Most general Majorana mass, Takagi factorisation

$$\frac{1}{2}\left(M_{IJ}\,\epsilon_{ab}\,u^{a,I}\,u^{b,J} + \text{h.c.}\right),$$

$$I,J = 1,\dots N_F, \quad M_{IJ} = M_{JI} \text{ complex}$$

$$M = U \begin{pmatrix} m_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & m_{N_F} \end{pmatrix} U^T$$

m_i are positive square roots of $MM^\dagger$

This is the most general fermion mass matrix!!! It includes CKM, in fact it is more general

Weyl + parity: Dirac

$(\frac{1}{2},\mathbf{0})\oplus(\mathbf{0},\frac{1}{2})$

$$P: u_\pm \longrightarrow u_\mp \qquad \psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \qquad \left.\begin{aligned} i\sigma_+^\mu \partial_\mu u_+ &= m u_- \\ i\sigma_-^\mu \partial_\mu u_- &= m u_+ \end{aligned}\right\} \Longrightarrow \quad i\begin{pmatrix} \sigma_+^\mu & 0 \\ 0 & \sigma_-^\mu \end{pmatrix}\partial_\mu \psi = m\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\psi$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_-^\mu \\ \sigma_+^\mu & 0 \end{pmatrix} \qquad \overline{\psi} \equiv \psi^\dagger \gamma^0 = \psi^\dagger \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \mathscr{L}_{\mathrm{Dirac}} = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$$

DIRACOLOGY

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

$$P_+\psi = \begin{pmatrix} u_+ \\ 0 \end{pmatrix} \qquad P_-\psi = \begin{pmatrix} 0 \\ u_- \end{pmatrix}$$

$$\begin{aligned} \mathrm{Tr}\,\gamma^\mu\gamma^\nu &= 4\,\eta^{\mu\nu} \\ \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta &= 4\,\eta^{\mu\nu}\eta^{\alpha\beta} - 4\,\eta^{\mu\alpha}\eta^{\beta\nu} + 4\,\eta^{\mu\beta}\eta^{\alpha\nu} \\ \mathrm{Tr}\,\gamma_5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu &= 4\,i\,\epsilon^{\alpha\beta\mu\nu} \end{aligned}$$

We look for +ve and -ve energy solutions as usual

$$u(k,s)e^{-ik\cdot x}$$

$$(\not{k} - m)u(k,s) = 0$$

$$v(k,s)e^{ik\cdot x}$$

$$(\not{k} + m)v(k,s) = 0$$

$$k^2 = m^2$$

$$\begin{aligned} \overline{u}(\mathbf{k},s)u(\mathbf{k},s) &= 2m, \\ \overline{u}(\mathbf{k},s)\gamma^\mu u(\mathbf{k},s) &= 2k^\mu, \\ \sum_{s=\pm\frac{1}{2}} u_\alpha(\mathbf{k},s)\overline{u}_\beta(\mathbf{k},s) &= (\not{k} + m)_{\alpha\beta} \end{aligned}$$

$$\begin{aligned} \overline{v}(\mathbf{k},s)v(\mathbf{k},s) &= -2m, \\ \overline{v}(\mathbf{k},s)\gamma^\mu v(\mathbf{k},s) &= 2k^\mu, \\ \sum_{s=\pm\frac{1}{2}} v_\alpha(\mathbf{k},s)\overline{v}_\beta(\mathbf{k},s) &= (\not{k} - m)_{\alpha\beta} \end{aligned}$$

Quantisation

We repeat the bosonic arguments, except for the fact that we have now anti-commutation relations between electron and positron creation-annihilation operators

$$\widehat{\psi}_\alpha(t,\vec{x}) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left[u_\alpha(\vec{k},s)\,\widehat{b}(\vec{k},s)e^{-i\omega_{\vec{k}}t+i\vec{k}\cdot\vec{x}} + v_\alpha(\vec{k},s)\,\widehat{d}^{\,\dagger}(\vec{k},s)e^{i\omega_{\vec{k}}t-i\vec{k}\cdot\vec{x}}\right].$$

$$\{\widehat{\psi}_\alpha(t,\mathbf{x}),\widehat{\psi}^\dagger_\beta(t,\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}$$

$$\begin{aligned}\{b(\mathbf{k},s),b^\dagger(\mathbf{k}',s')\} &= (2\pi)^3(2\omega_{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'},\\ \{b(\mathbf{k},s),b(\mathbf{k}',s')\} &= \{b^\dagger(\mathbf{k},s),b^\dagger(\mathbf{k}',s')\}=0.\\ \{d(\mathbf{k},s),d^\dagger(\mathbf{k}',s')\} &= (2\pi)^3(2\omega_{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'},\\ \{d(\mathbf{k},s),d(\mathbf{k}',s')\} &= \{d^\dagger(\mathbf{k},s),d^\dagger(\mathbf{k}',s')\}=0.\end{aligned}$$

$$\hat{H} = \frac{1}{2}\sum_{s=\pm\frac{1}{2}}\int\frac{d^3k}{(2\pi)^3}\left[b^\dagger(\mathbf{k},s)b(\mathbf{k},s)-d(\mathbf{k},s)d^\dagger(\mathbf{k},s)\right]$$

$$\widehat{H} = \sum_{s=\pm\frac{1}{2}}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left[\omega_{\vec{k}}b^\dagger(\vec{k},s)b(\vec{k},s)+\omega_{\vec{k}}d^\dagger(\vec{k},s)d(\vec{k},s)\right] - 2\int d^3k\,\omega_{\vec{k}}\delta(\vec{0}).$$

We have a conserved charge and current

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \qquad \partial_\mu j^\mu = 0 \qquad Q = e\int d^3x\, j^0$$

The two-point function or Feynman propagator is:

$$S_{\alpha\beta}(x_1,x_2) = \langle 0|T\left[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\right]|0\rangle$$

$$T\left[\psi_\alpha(x)\overline{\psi}_\beta(y)\right] = \theta(x^0-y^0)\psi_\alpha(x)\overline{\psi}_\beta(y) - \theta(y^0-x^0)\overline{\psi}_\beta(y)\psi_\alpha(x).$$

Introducing gauge fields

The canonical gauge field is the electromagnetic field. The first one that was understood as a gauge field. For some time this symmetry sounded like a luxury. In fact the classical theory can be formulated exclusively in terms of the E,B field that are manifestly gauge invariant. This is not so in the quantum theory, where we need to use the vector and scalar potentials. There are new, non-local observables. They are responsible for the Bohm-Aharonov effect and the quantisation of electric charge (if there is a single monopole in the Universe, (Dirac)).

What we have learned is that all fundamental interactions known to us are mediated by suitable generalisations of the EM field. They are gauge theories. In fact it seems as though Nature abhors global symmetries. It appears that all the known global symmetries are just low-energy accidents. All symmetries in the UV should be local.

We do not know why this should be so. String Theory is the only theory where this fact finds an explanation. Unfortunately there is no evidence for it at this moment...

E&M in Quantum Mechanics

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla \times \mathbf{B} = \frac{\partial}{\partial t}\mathbf{E}$$

Classical EM

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

$$\partial_\mu F^{\mu\nu} = j^\mu \qquad j^\mu = (\rho, \mathbf{j})$$
$$\varepsilon^{\mu\nu\sigma\eta}\partial_\nu F_{\sigma\eta} = 0, \qquad A^\mu = (\varphi, \mathbf{A})$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Classical EM in relativistic form

Coupling to QM requires the gauge potentials and a non-trivial transformation of the wave function, this gives subtle consequences to gauge symmetry

$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + e\varphi\right]\Psi$$

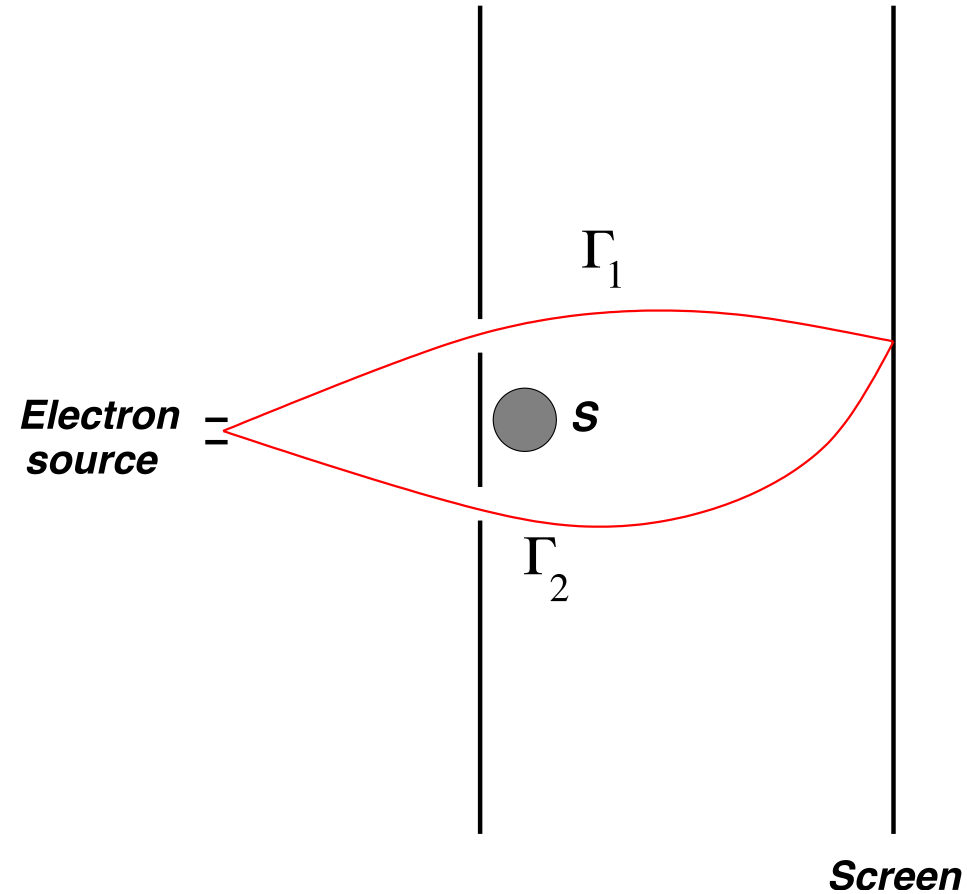
$$\Psi(t,\mathbf{x}) \longrightarrow e^{-ie\,\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x})$$

$$\varphi(t,\mathbf{x}) \to \varphi(t,\mathbf{x}) + \frac{\partial}{\partial t}\varepsilon(t,\mathbf{x}), \qquad \mathbf{A}(t,\mathbf{x}) \to \mathbf{A}(t,\mathbf{x}) + \nabla\varepsilon(t,\mathbf{x}).$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

Non-local observables

$$\Psi = e^{ie\int_{\Gamma_1}\mathbf{A}\cdot d\mathbf{x}}\Psi_1^{(0)} + e^{ie\int_{\Gamma_2}\mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)}$$
$$= e^{ie\int_{\Gamma_1}\mathbf{A}\cdot d\mathbf{x}}\left[\Psi_1^{(0)} + e^{ie\oint_{\Gamma}\mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)}\right]$$



$$U = \exp\left[ie\oint_{\Gamma}\mathbf{A}\cdot d\mathbf{x}\right]$$

This is the Aharonov-Bohm effect. The phase factor, and its non-abelian generalisation are known as "Wilson loops" or holonomies of the gauge field. Note that classically there would be no effect. The Lorentz force equation only involves E,B hence the electrons would not see the solenoid at all!!

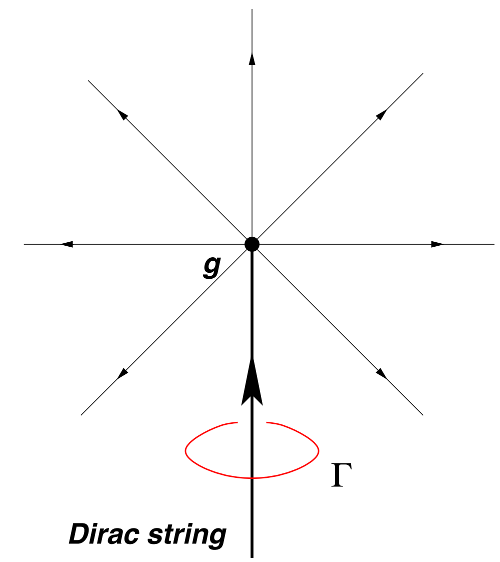
$$m\frac{du^{\mu}}{d\tau} = eF^{\mu\nu}u_{\nu}$$

Magnetic monopoles: Dirac and charge quantisation

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla \times \mathbf{B} = \frac{\partial}{\partial t}\mathbf{E}$$

$$\mathbf{E} - i\mathbf{B} \longrightarrow e^{i\theta}(\mathbf{E} - i\mathbf{B})$$

For angle = 90 E and B get exchanged



The symmetry extend to matter if we have magnetic sources:

$$\rho - i\rho_m \longrightarrow e^{i\theta}(\rho - i\rho_m), \qquad \mathbf{j} - i\mathbf{j}_m \longrightarrow e^{i\theta}(\mathbf{j} - i\mathbf{j}_m).$$

Consider a magnetic pole:

$$\nabla \cdot \mathbf{B} = g\,\delta(\mathbf{x}). \qquad B_r = \frac{1}{4\pi}\frac{g}{|\mathbf{x}|^2}, \qquad B_\varphi = B_\theta = 0 \qquad A_\varphi = \frac{1}{4\pi}\frac{g}{|\mathbf{x}|}\tan\frac{\theta}{2}, \qquad A_r = A_\theta = 0.$$

The Dirac string can be changed by gauge transformations, in doing QM it has to be unobservable. Then we can do a "A-B" like argument (Dirac did it 20 years earlier). We should not forget the fact that there is a factor of $\hbar c$

$$e^{ieg} = 1 \qquad eg = 2\pi n$$

$$q_1 g_2 - q_2 g_1 = 2\pi n.$$

Electromagnetic Fields and Photons

Ignoring sources, the E&M field is a "free field"

$$\mathscr{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)$$

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

The electric field is the momentum p and the vector potential the "coordinate" q

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

$$\partial_\mu F^{\mu\nu} = 0 \qquad 0 = \partial_\mu \partial^\mu A^\nu - \partial_\nu\left(\partial_\mu A^\mu\right) = \partial_\mu \partial^\mu A^\nu$$

To be able to invert, we need to fix the gauge: $\partial_\mu A^\mu = 0.$

$$\varepsilon_\mu(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}}$$
$$k^\mu \varepsilon_\mu(\mathbf{k},\lambda) = 0$$
$$k^2 = k_\mu k^\mu = (k^0)^2 - \mathbf{k}^2 = 0$$

As usual, we look for plane wave solutions, we still have residual gauge transformation that can be used to fully fix the gauge

$$\epsilon_\mu(\mathbf{k},\lambda) \to \epsilon_\mu(\mathbf{k},\lambda) + k_\mu\,\chi(\mathbf{k}),\;\; k^2 = 0$$

Now, as usual we expand the field in oscillator and apply CCR. After fully fixing the gauge there are only two physical polarisations. Gauge invariance seems more a redundancy rather than a symmetry in the description of the theory

$$\widehat{A}_\mu(t,\mathbf{x}) = \sum_{\lambda=\pm 1}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}\Big[\varepsilon_\mu(\mathbf{k},\lambda)\widehat{a}(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} + \varepsilon_\mu(\mathbf{k},\lambda)^*\,\widehat{a}^\dagger(\mathbf{k},\lambda)e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}\Big].$$

$$[\widehat{a}(\mathbf{k},\lambda),\widehat{a}^\dagger(\mathbf{k}',\lambda')] = (2\pi)^3(2|\mathbf{k}|)\delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}$$

If we keep all four polarisation by partial gauge fixing, then we get negative probabilities (Gupta-Bleuler, BRST)

$$\delta_{\lambda,\lambda'} \to -\eta_{\lambda,\lambda'}$$

Coupling matter

We imitate the coupling in the Schrödinger equation, this is what used to be called minimal coupling. We make derivatives covariant with respect to space-time dependent changes of phases in the wave-function

$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + e\varphi\right]\Psi \qquad D_\mu\left[e^{ie\varepsilon(x)}\psi\right] = e^{ie\varepsilon(x)}D_\mu\psi.$$

$$\Psi(t,\mathbf{x}) \longrightarrow e^{-ie\,\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x}) \qquad D_\mu = \partial_\mu - ieA_\mu.$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu\varepsilon$$

The rigid phase rotation invariance of the Dirac Lagrangian for electrons is transformed into local phase rotations, a physically more satisfactory concept. This defines the coupling of the electron to the E&M field:

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi, \qquad \mathscr{L}_{\text{QED}}^{(\text{int})} = -eA_\mu\overline{\psi}\gamma^\mu\psi.$$

$$\psi \longrightarrow e^{ie\varepsilon(x)}\psi, \qquad A_\mu \longrightarrow A_\mu + \partial_\mu\varepsilon(x).$$

This is QED, the best tested theory in the history of science,
an example is the gyromagnetic ration of the electron,

$$g\frac{e}{8m}[\gamma^\mu, \gamma^\nu]\,F_{\mu\nu}$$

$$g/2 = 1.00115965218085(76)$$
$$\alpha^{-1} = 137.035999070(98)$$

$$\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c}\vec{s}, \qquad \underbrace{g_\mu = 2}_{\text{Dirac}}(1 + a_\mu)$$