Noether's Theorem

Quantum mechanical realisation of Symmetries (Wigner's theorem). In a QM theory physical symmetries are maps among states that preserve probability amplitudes (their modulus). They can be unitary or anti-unitary

$$|\alpha\rangle \longrightarrow |\alpha'\rangle, \qquad |\beta\rangle \longrightarrow |\beta'\rangle$$

$$|\langle\alpha|\beta\rangle| = |\langle\alpha'|\beta'\rangle|.$$

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle$$ unitary

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle^*$$ anti-unitary T-reversal, CPT

For continuous symmetries we have Noether's celebrated theorem: If under infinitesimal transformations, AND WITHOUT USING THE EQUATIONS OF MOTION you can show that:

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$$

$$\delta_\varepsilon \mathscr{L} = \partial_\mu K^\mu$$

then there is a conserved current in the theory

In formulas:

$$\begin{aligned}\delta_\varepsilon \mathscr{L} &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\partial_\mu \delta_\varepsilon \phi + \frac{\partial \mathscr{L}}{\partial \phi}\delta_\varepsilon \phi \\ &= \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi\right) + \left[\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right)\right]\delta_\varepsilon \phi \\ &= \partial_\mu K^\mu .\end{aligned}$$

$$\partial_\mu J^\mu = 0$$

$$\boxed{J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi - K^\mu}$$

With a conserved charge that generates the symmetry:

$$Q \equiv \int d^3x J^0(t,\mathbf{x}) \qquad \frac{dQ}{dt} = \int d^3x\, \partial_0 J^0(t,\mathbf{x}) = -\int d^3x\, \partial_i J^i(t,\mathbf{x}) = 0,$$

$$\delta\phi = i[\phi, Q].$$

Space-time translations -- Energy-Momentum
Lorentz transformation-- Angular momentum and CM motion
Phase rotation -- abelian and non-abelian charges

Useful examples

Massive Dirac fermions:

$$\mathscr{L} = i\overline{\psi}_j \partial\!\!\!/ \psi_j - m\overline{\psi}_j \psi_j \qquad \psi_i \longrightarrow U_{ij}\psi_j \qquad U \in U(N) \;\; N \text{the number of fermions}$$

$$U = \exp(i\alpha^a T^a), \qquad (T^a)^\dagger = T^a$$

$$j^{\mu\, a} = \overline{\psi}_i T^a_{ij} \gamma^\mu \psi_j \qquad \partial_\mu j^\mu = 0 \qquad Q^a = \int d^3x \, \psi_i^\dagger T^a_{ij} \psi_j$$

$$[Q^a, H] = 0. \qquad \mathscr{U}(\alpha) = e^{i\alpha^a Q^a}.$$

When U is the identity, we have fermion number, or charge

In the m=0 we have more symmetry: CHIRAL SYMMETRY, rotate L,R fermions independently

$$\mathcal{L} = i\bar{\psi}_{jL} \partial\!\!\!/ \psi_{Lj} + i\bar{\psi}_{jR} \partial\!\!\!/ \psi_{Rj}$$

$$\psi_{L,R} \to U_{L,R} \psi_{L,R} \qquad U(N)_L \times U(N)_R$$

Wigner-Weyl mode


Imagine we have a symmetry that is a symmetry of the ground state

$$[Q^a, H] = 0. \qquad \mathscr{U}(\alpha)|0\rangle = |0\rangle \qquad Q^a|0\rangle \;=\; 0$$

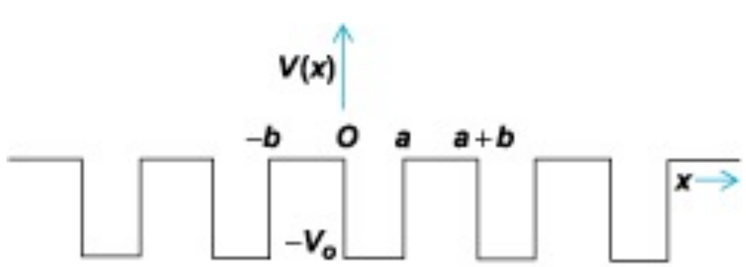
Then the states of the theory fall into multiplets of the symmetry group

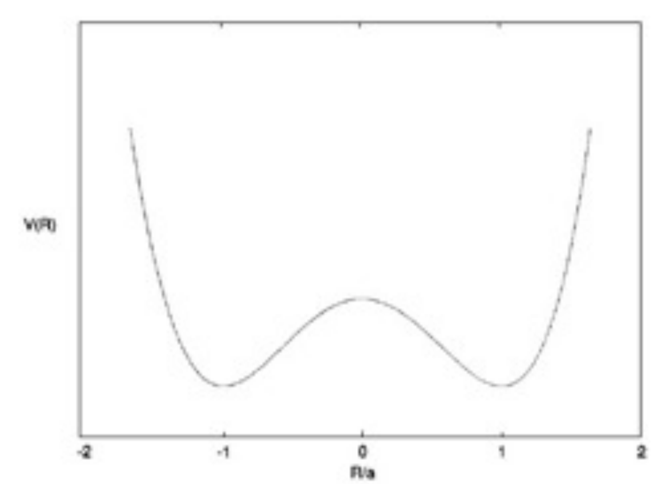
$$\mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1} = U_{ij}(\alpha)\phi_j$$

$$|i\rangle = \phi_i|0\rangle$$

$$\mathscr{U}(\alpha)|i\rangle = \mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1}\mathscr{U}(\alpha)|0\rangle = U_{ij}(\alpha)\phi_j|0\rangle = U_{ij}(\alpha)|j\rangle$$

The spectrum of the theory is classified in terms of multiplets of the symmetry group. This is the case of the Hydrogen atom. The Hamiltonian is rotational invariant, the ground state is an s-wave state, hence all excited states fall into degenerate representations of the rotation group: 1s, 2s, 2p, 2s, 3p, 3d,… In QM (finite number of d.o.f.) this is always the case (tunnelling, band theory in solids)



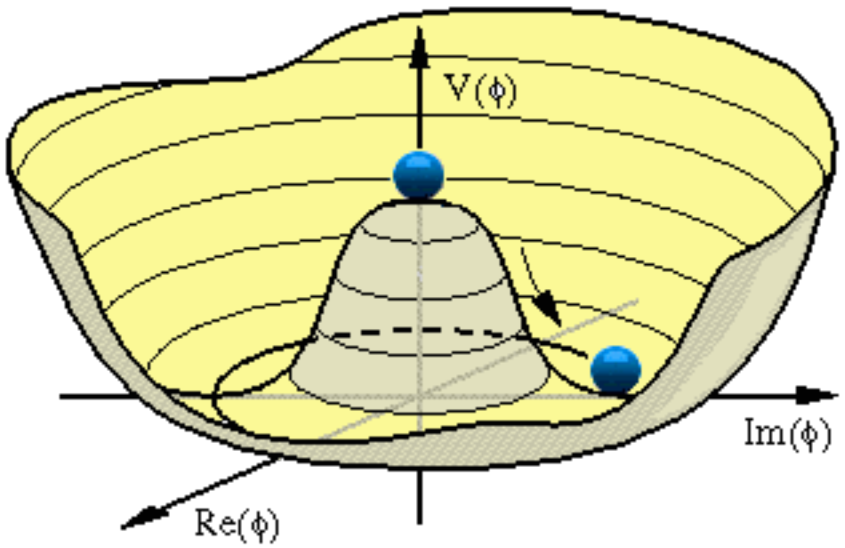


Nambu-Goldstone mode

Sometimes also called hidden symmetry. The symmetry is spontaneously broken by the vacuum

$$[Q^a, H] = 0. \qquad Q^a|0\rangle \neq 0.$$

Consider a collection of N scalar fields with a global symmetry group G



$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi^i \partial^\mu \varphi^i - V(\varphi^i) \qquad \delta\varphi^i = \varepsilon^a (T^a)^i_j \varphi^j$$

$$H[\pi^i, \varphi^i] = \int d^3x \left[\frac{1}{2}\pi^i \pi^i + \frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i) \right]$$

$$\mathscr{V}[\varphi^i] = \int d^3x \left[\frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i) \right]$$

The minima satisfy

$$\langle\varphi^i\rangle \qquad V(\langle\varphi^i\rangle) = 0, \qquad \nabla\varphi = \mathbf{0} \qquad \left.\frac{\partial V}{\partial \varphi^i}\right|_{\varphi^i = \langle\varphi^i\rangle} = 0$$

$$T^a = \{H^\alpha, K^A\} \qquad (H^\alpha)^i_j \langle\varphi^j\rangle = 0 \qquad (K^A)^i_j \langle\varphi^j\rangle \neq 0.$$

unbroken broken

Nambu-Goldstone mode

The masses are given by the second derivatives of the potential (assuming canonical normalisation)

$$M^2_{ij} \equiv \left.\frac{\partial^2 V}{\partial\varphi^i \partial\varphi^j}\right|_{\varphi=\langle\varphi\rangle}$$

Invariance

$$\delta V(\varphi) = \varepsilon^a \frac{\partial V}{\partial \varphi^i} (T^a)^i_j \varphi^j = 0 \qquad \frac{\partial^2 V}{\partial\varphi^i \partial\varphi^k}(T^a)^i_j\varphi^j + \frac{\partial V}{\partial\varphi^i}(T^a)^i_k = 0$$

$$M^2_{ik}(T^a)^i_j\langle\varphi^j\rangle = 0. \qquad M^2_{ik}(K^A)^i_j\langle\varphi^j\rangle = 0$$

For every broken generator there is a massless scalar field

The argument works at the full quantum level

The fields acquiring a VEV need not be elementary

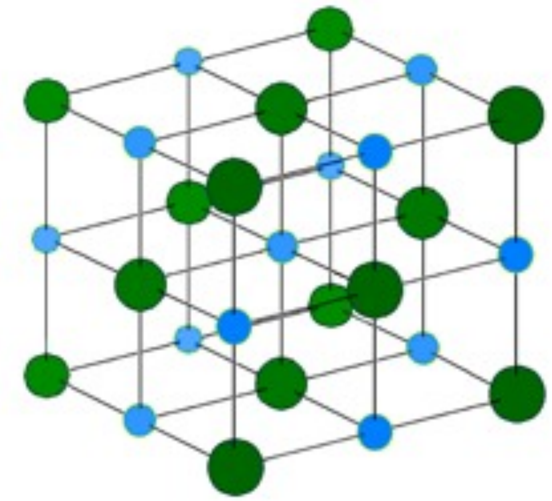
Simplest example:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$$

$$\phi \to \phi + c$$

Its own NG-boson

Phonons are NG bosons



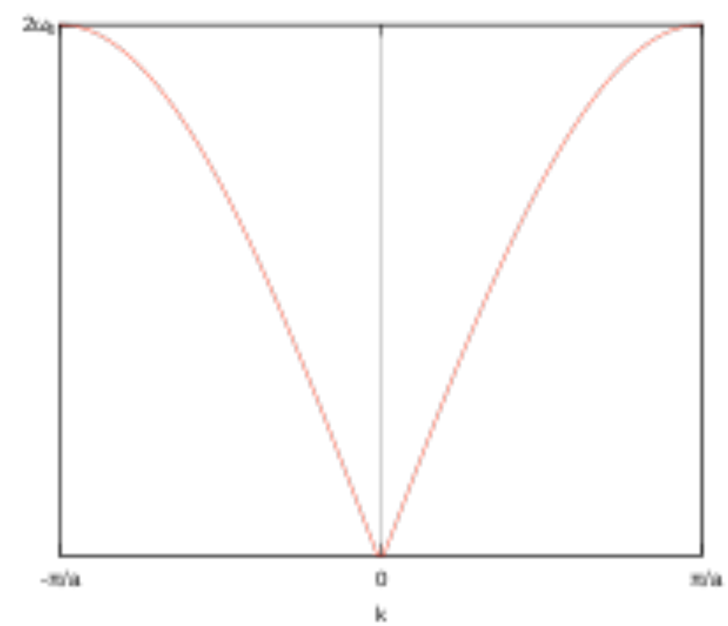
A liquid is translationally invariant

The crystal after solidification has discrete translational symmetry

The low energy excitation of the lattice contain acoustic phonons

Their dispersion relation is as for NG bosons

They propagate at the speed of sound



$$\omega(k) \;=\; 2\omega|\sin(ka/2)|$$

Order parameters

❖ The notion of symmetry breaking is intimately connected with the theory of phase transitions in CMP

❖ It is quite frequent that in going from one phase to another the symmetry of the ground state (vacuum) changes

❖ In real physical systems this is what we see with magnetic domains in magnetic material below the Curie point

❖ In going from one phase to the other, some parameters change in a noticeable way. These are the order parameters.

❖ In liquid-solid transition it is the density

❖ In magnetic materials it is the magnetisation

❖ In the Ginsburg-Landau theory of superconductivity, the Cooper pairs acquire a VEV. This breaks U(1) inside the superconductor and thus explains among other things the Meissner effect. The Cooper pairs are pairs of electrons bound by the lattice vibrations. In ordinary superconductors their size is several hundred Angstroms.

❖ The order parameters need not be elementary fields...

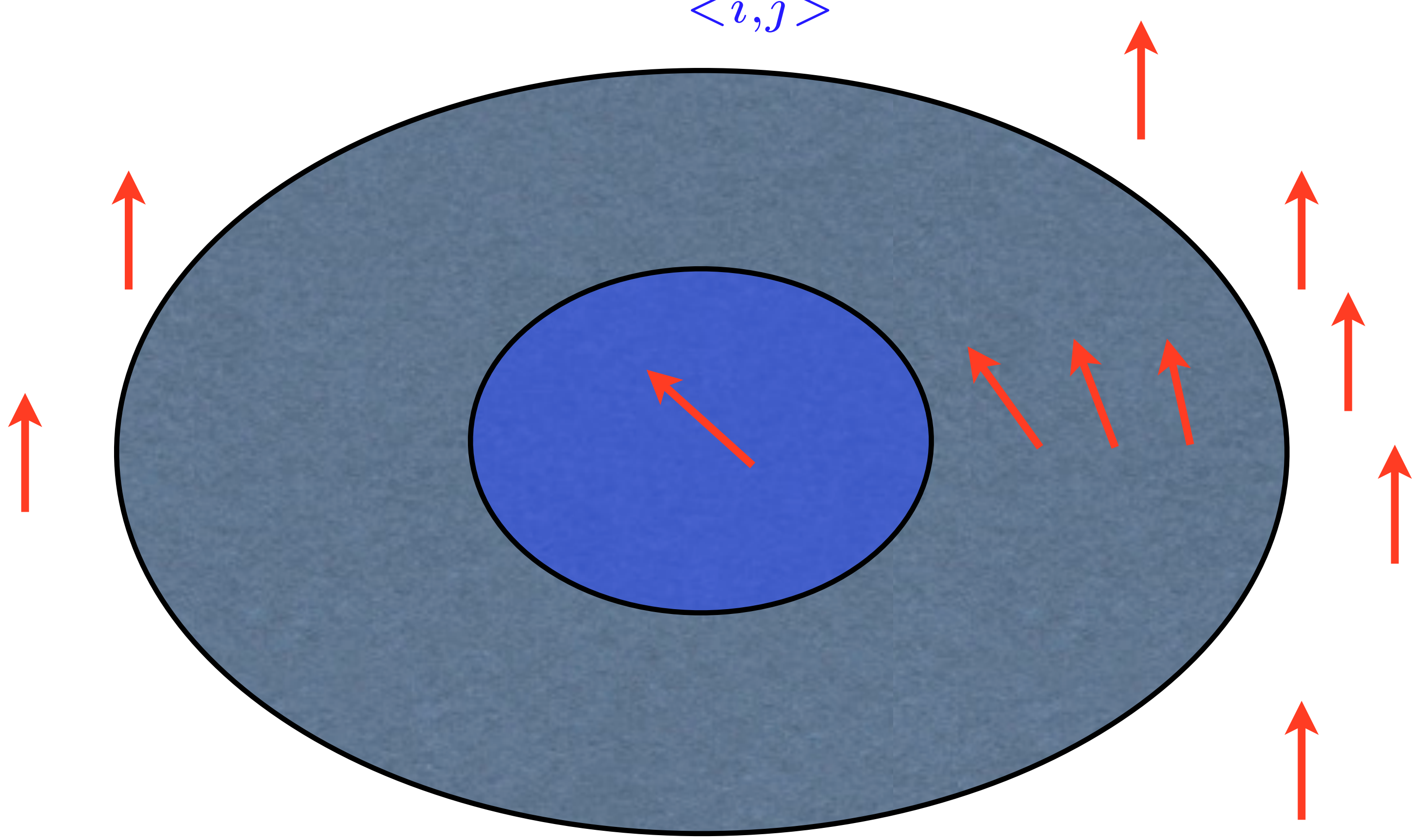
Misconceptions, vacuum degeneracy

By abuse of language we often hear, or say that in theories with SSB there is vacuum degeneracy. This is fact is not the case, at least in LQFT. In understanding this we will also understand why there are massless states in theories with SSB. N is the volume in the example. The Heisenberg model of magnetism. H is rotational invariant above the critical temperature, and magnetised below it

$|0\rangle$ $\qquad$ $|\theta\rangle$

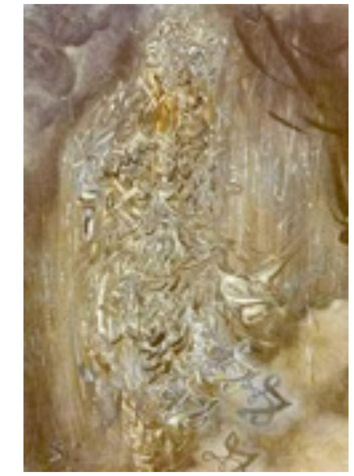
$$H = -J \sum_{<i,j>} \mathbf{s}_i \cdot \mathbf{s}_j$$

$$\begin{aligned} \langle 0|\theta\rangle &= (\cos(\theta/2))^N \\ &\to 0 \qquad N \to \infty \end{aligned}$$



By making the transitions very slowly we can manage to make this configuration to have as small an energy as we wish. Hence we have a continuum spectrum above zero. This is the sign of a massless particle, the NG-boson

Pions



In HEP they provide the only observed NG bosons

The order parameter is not an elementary field

To find other NG bosons in the SM we have to go to the Higgs sector, and there they are "eaten" to provide masses for the W and Z vector bosons

In QCD there are no fundamental scalars. Consider just two flavors u,d. We have chiral symmetry

$$\begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \longrightarrow M_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \qquad G = \underbrace{SU(2)_L \times SU(2)_R}_{SU(2)_V} \times U(1)_B \times U(1)_A$$

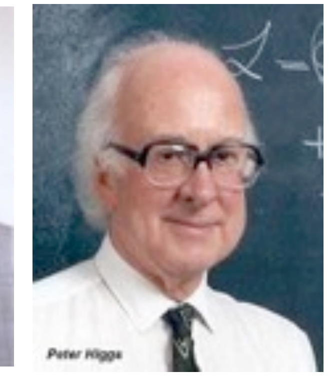
$$q^f_\alpha \qquad f = u, d, \;\; \alpha = 1, 2, 3$$

$$\langle \bar{q}^f \cdot q^{f'} \rangle = \Lambda^3_{\chi SB}\, \delta^{ff'} \qquad \bar{q}^f \cdot q^{f'} \simeq \Lambda^3_{\chi SB}\, e^{i\pi^a \sigma^a / f_\pi}$$

These are the pions.
This is an IR property of QCD, not accessible to Pert.Th.
Low-E pion theorems, chiral Lagrangians….

The BEH mechanism





Notice we say the mechanism, not necessary the particle! In gauge theories one cannot just add a mass for the gauge bosons. This badly destroys the gauge symmetry and the theory is inconsistent.

BEH showed that in gauge theories with SSB the NG bosons are "eaten" by the gauge bosons to become massive but preserving the basic properties of the gauge symmetry. Ex. Abelian Higgs model

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - \frac{\lambda}{4}\left(\varphi^\dagger\varphi - \mu^2\right)^2 \qquad \varphi \longrightarrow e^{i\alpha(x)}\varphi, \qquad A_\mu \longrightarrow A_\mu + \partial_\mu\alpha(x)$$

$$\langle\varphi\rangle = \mu e^{i\vartheta_0} \longrightarrow \mu e^{i\vartheta_0 + i\alpha(x)} \qquad \varphi(x) = \left[\mu + \frac{1}{\sqrt{2}}\sigma(x)\right]e^{i\vartheta(x)}$$

Take the unitary gauge

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\mu^2 A_\mu A^\mu + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}\lambda\mu^2\sigma^2$$
$$-\lambda\mu\sigma^3 - \frac{\lambda}{4}\sigma^4 + e^2\mu A_\mu A^\mu\sigma + e^2 A_\mu A^\mu\sigma^2.$$

$$m_\gamma^2 = 2e^2\mu^2$$

The simplest example is the GL and BCS theory of superconductivity, in this case the "Higgs" particle is composite, it is an object of charge made of two bound electrons that get a "VEV" (Cooper pairs) that get a VEV in the superconducting state. The photon is massive in this state. This explains among other things the Meissner effect.

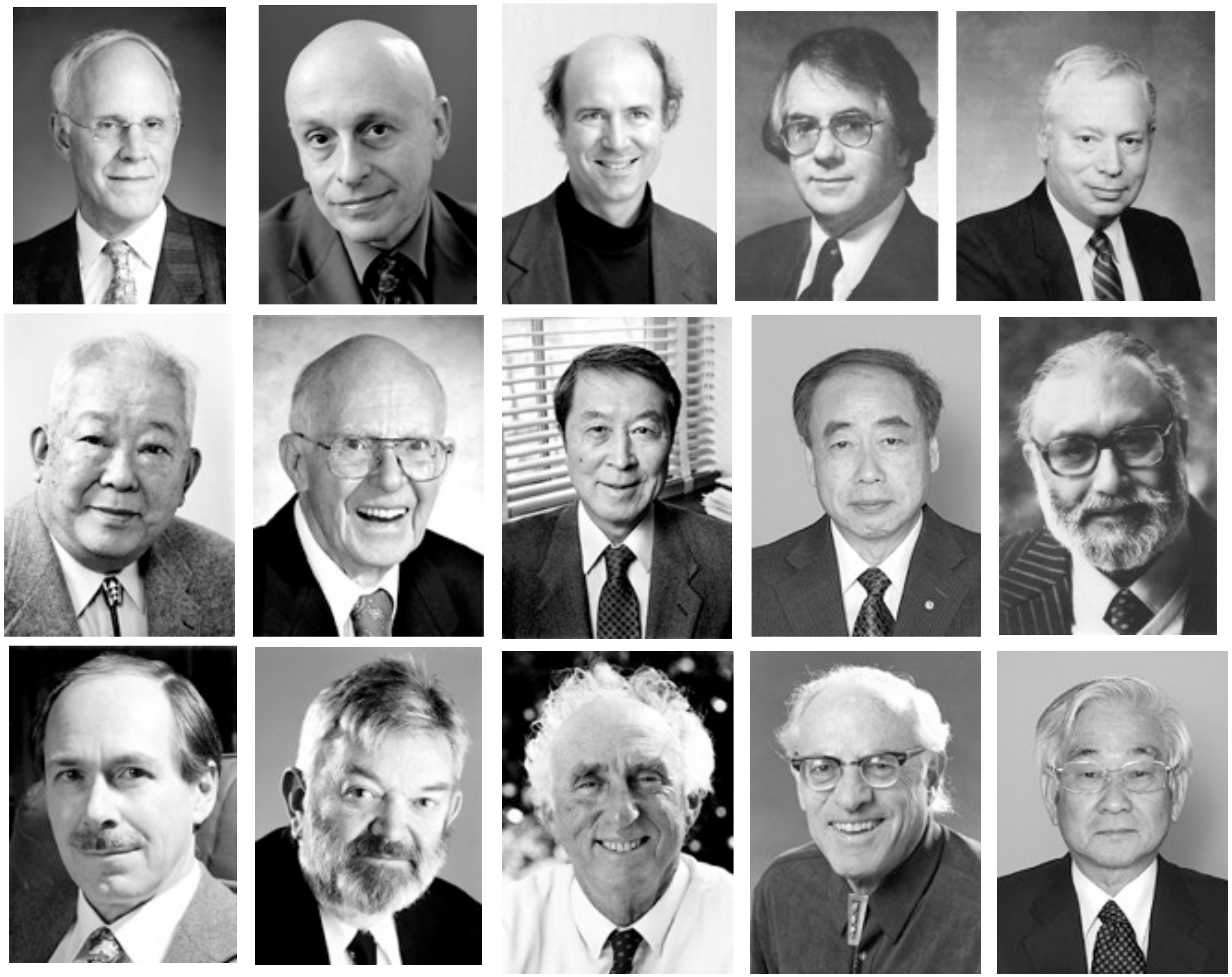
The Standard Model

The physics that led to the SM is the combined results of many people over more than a hundred years, some of the them were awarded the Nobel Prize in Physics. They appear in the pages that follow. We can think of the beginning of the SM Odyssey with the discovery of the electron by Thomson in 1897.

Try putting names to the faces as well as the associated contributions.

This is an amusing exercise in HEP history

Who is who in the Standard Model



Who is who in the Standard Model

Who is who in the Standard Model

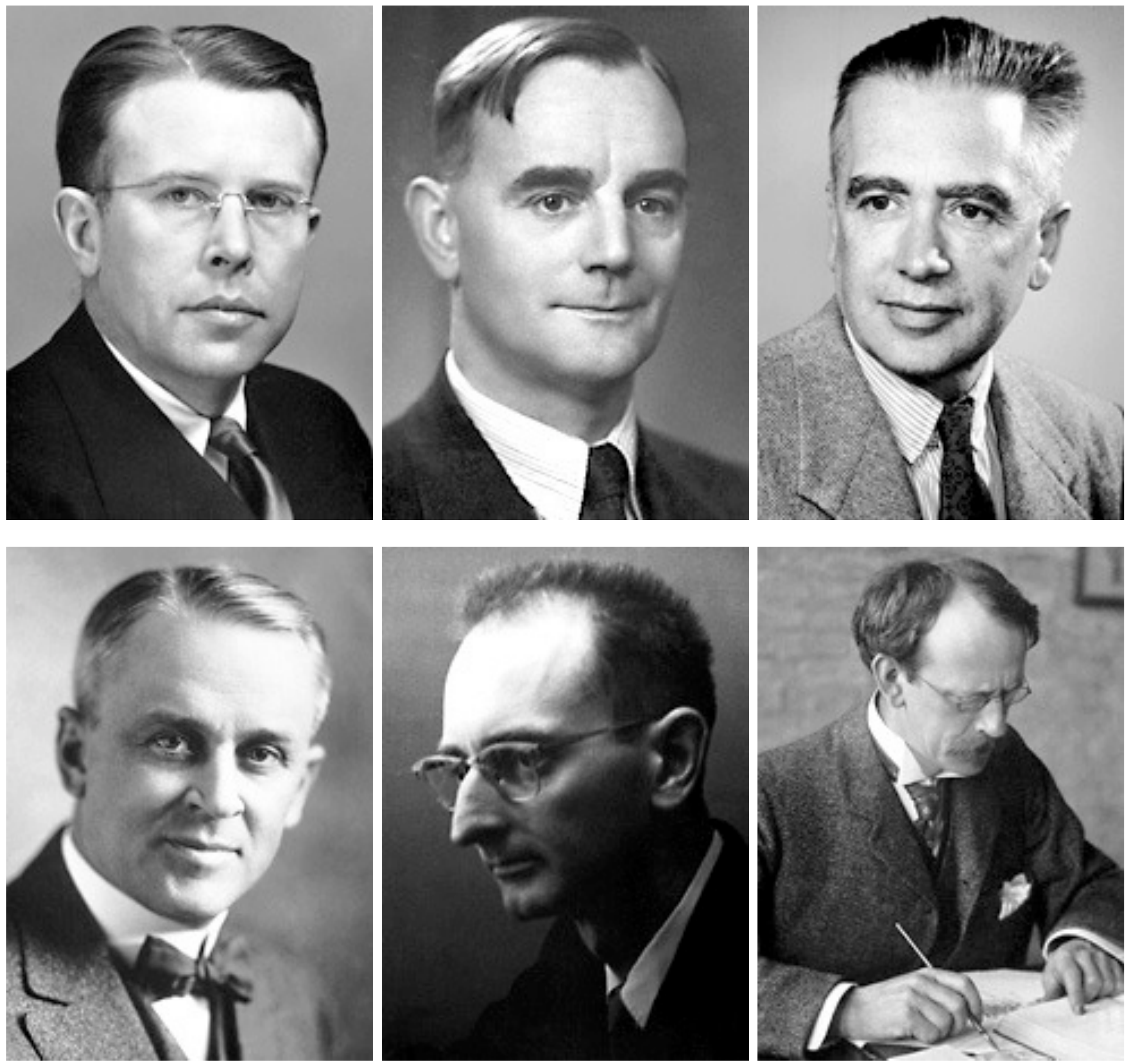









Who is who in the Standard Model



How many did you recognise?

Gauge couplings: colour

There are three gauge groups in the theory, the colour group SU(3) and the electroweak group SU(2)xU(1) of weak isospin and hypercharge. Y and T3 mix to generate electric charge and the photon

$$SU(3)_c \times SU(2) \times U(1)_Y \to SU(3) \times U(1)_Q$$

QCD by itself is a perfect theory in many ways

$$\mathscr{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_{f=1}^{6} \overline{Q}^f \left(i\not{D} - m_f \right) Q^f. \qquad Q^f_i \longrightarrow U(g)_{ij} Q^f_j, \qquad \text{with} \quad g \in \mathrm{SU(3)}$$

Isospin as an approximate symmetry:

$$\mathscr{L} = (\overline{u}, \overline{d}) \begin{pmatrix} i\not{D} - \frac{m_u + m_d}{2} & 0 \\ 0 & i\not{D} - \frac{m_u + m_d}{2} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} - \frac{m_u - m_d}{2} (\overline{u}, \overline{d}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

Once the electroweak sector is included the story of the masses is far more complicated (see later)

The EW group has four generators

$$\mathbf{W}_\mu = W^+_\mu T^- + W^-_\mu T^+ + W^3_\mu T^3, \qquad \mathbf{B}_\mu = B_\mu Y.$$

$$Q = T^3 + Y.$$

$$[Q, T^\pm] = \pm T^\pm, \qquad [Q, T^3] = [Q, Y] = 0.$$

$$A_\mu = B_\mu \cos\theta_w + W^3_\mu \sin\theta_w,$$

$$Z_\mu = -B_\mu \sin\theta_w + W^3_\mu \cos\theta_w$$

$$\begin{aligned} D_\mu &= \partial_\mu - ig\mathbf{W}_\mu - ig'\mathbf{B}_\mu \\ &= \partial_\mu - igW^+_\mu T^-_R - igW^-_\mu T^+_R - igW^3_\mu T^3_R - ig' B_\mu Y_R, \end{aligned}$$

$$e = g\sin\theta_w = g'\cos\theta_w.$$

$$\begin{aligned} D_\mu = {} & \partial_\mu - igW^+_\mu T^-_R - igW^-_\mu T^+_R - iA_\mu (g\sin\theta_w T^3_R + g'\cos\theta_w Y_R) \\ & - iZ_\mu (gT^3_R \cos\theta_w - g' Y_R \sin\theta_w). \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\text{gauge}} = {} & -\frac{1}{2} W^+_{\mu\nu} W^{-\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{ig}{2} \cos\theta_w W^+_\mu W^-_\nu Z^{\mu\nu} \\ & + \frac{ie}{2} W^+_\mu W^-_\nu F^{\mu\nu} - \frac{g^2}{2} \left[(W^+_\mu W^{+\mu})(W^-_\mu W^{-\mu}) - (W^+_\mu W^{-\mu})^2 \right] \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{\text{matter}} = {} & \sum_{i=1}^{3} \Big(i\overline{\mathbf{L}}^j \not{D} \mathbf{L}^j + i\overline{\ell}^j_R \not{D} \ell^j_R \\ & + i\overline{\mathbf{Q}}^j \not{D} \mathbf{Q}^j + i\overline{U}^j_R \not{D} U^j_R + i\overline{D}^j_R \not{D} D^j_R \Big) \end{aligned}$$

$$W^\pm_{\mu\nu} = \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu \mp ie \left(W^\pm_\mu A_\nu - W^\pm_\nu A_\mu \right) \mp ig\cos\theta_w \left(W^\pm_\mu Z_\nu - W^\pm_\nu Z_\mu \right)$$

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$\frac{g}{2\cos\theta_w} Z_\mu \overline{\nu}_\ell \gamma^\mu \nu_\ell, \qquad \frac{g}{\cos\theta_w} \left(-\frac{1}{2} + \sin^2\theta_w \right) Z_\mu \overline{\ell}_L \gamma^\mu \ell_L.$$

Fermion quantum numbers

The fundamental fermions come in three families with the same quantum numbers with respect to the gauge group

Leptons					
i (generation)	1	2	3	T^3	Y
$\mathbf{L}^i$	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$-\frac{1}{2}$
ℓ^i_R	e^-_R	μ^-_R	τ^-_R	0	-1

Quarks					
i (generation)	1	2	3	T^3	Y
$\mathbf{Q}^i$	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{pmatrix} c \\ s \end{pmatrix}_L$	$\begin{pmatrix} t \\ b \end{pmatrix}_L$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$	$\frac{1}{6}$
U^i_R	u_R	c_R	t_R	0	$\frac{2}{3}$
D^i_R	d_R	s_R	b_R	0	$-\frac{1}{3}$

In principle one could add sterile neutrinos, right handed neutrinos who are singlets under the gauge group. They would generate Dirac masses for the known neutrinos

Higgs couplings

The Higgs couplings responsible for the masses of the leptons and the current algebra masses of the quarks are:

$$\mathbf{H} = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \qquad \tilde{\mathbf{H}} \equiv i\sigma^2 \mathbf{H}^* = \begin{pmatrix} H^{0*} \\ H^{+*} \end{pmatrix} \qquad \mathscr{L}_{\text{Higgs}} = (D_\mu \mathbf{H})^\dagger D^\mu \mathbf{H} - V(\mathbf{H}, \mathbf{H}^\dagger), \qquad V(\mathbf{H}, \mathbf{H}^\dagger) = \frac{\lambda}{4}\left(\mathbf{H}^\dagger \mathbf{H} - \mu^2\right)^2$$

$$Y(\mathbf{H}) = \tfrac{1}{2}$$

$$\mathscr{L}^{(\ell)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3}\left(C^{(\ell)}_{ij}\,\overline{\mathbf{L}}^i\,\mathbf{H}\,\ell^j_R + C^{(\ell)*}_{ji}\,\overline{\ell}^i_R \mathbf{H}^\dagger\,\mathbf{L}^j\right)$$

$$\mathscr{L}^{(q)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3}\left(C^{(q)}_{ij}\,\overline{\mathbf{Q}}^i\,\mathbf{H}\,D^j_R + C^{(q)*}_{ji}\,\overline{D}^i_R\,\mathbf{H}^\dagger\,\mathbf{Q}^j\right) - \sum_{i,j=1}^{3}\left(\tilde{C}^{(q)}_{ij}\,\overline{\mathbf{Q}}^i\,\tilde{\mathbf{H}}\,U^j_R + \tilde{C}^{(q)*}_{ji}\,\overline{U}^i_R\,\tilde{\mathbf{H}}^\dagger\,\mathbf{Q}^j\right).$$

The most general Lagrangian compatible with the gauge symmetry and up to dimension 4, so that the theory is renormalisable. One H gets its VEV the masses are generated from the Yukawa couplings. Use unitary gauge. The gauge fields get masses from the kinetic term

$$\mathbf{H}(x) = e^{i\mathbf{a}(x)\cdot\frac{\sigma}{2}}\begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}}h(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}}h(x) \end{pmatrix}$$

CKM matrix

$$\mathscr{L}^{(\ell)}_{\text{mass}} = -(\overline{e}_L, \overline{\mu}_L, \overline{\tau}_L)\, M^{(\ell)} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} + \text{h.c.} \qquad \mathscr{L}^{(q)}_{\text{mass}} = -(\overline{d}_L, \overline{s}_L, \overline{b}_L)\, M^{(q)} \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} - (\overline{u}_L, \overline{c}_L, \overline{t}_L)\, \widetilde{M}^{(q)} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} + \text{h.c.}$$

$$M^{(\ell,q)}_{ij} = \mu C^{(\ell,q)}_{ij}, \qquad \widetilde{M}^{(q)}_{ij} = \mu \widetilde{C}^{(q)}_{ij}.$$

$$V_L^{(\ell)\dagger} M^{(\ell)} V_R^{(\ell)} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \qquad V_L^{(q)\dagger} M^{(q)} V_R^{(q)} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \qquad \widetilde{V}_L^{(q)\dagger} \widetilde{M}^{(q)} \widetilde{V}_R^{(q)} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

In the quark sector going from to mass eigenstates leaves a matrix of phases in the charged currents, the CKM matrix. Not for neutral currents GIM

$$j^\mu_+ = (\overline{u}_L, \overline{c}_L, \overline{t}_L)\gamma^\mu \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = (\overline{u}'_L, \overline{c}'_L, \overline{t}'_L)\gamma^\mu \widetilde{V}_L^{(q)\dagger} V_L^{(q)} \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} \qquad V \equiv \widetilde{V}_L^{(q)\dagger} V_L^{(q)}$$

Discrete symmetries

- In the classical world, we have invariance uncer P,C,T. All we had was E&M and gravity.

- In QFT they are not guaranteed in fact P,C,T, CP are broken symmetries. The only one that survives so far is CPT. It has several important consequences. CP violation is fundamental in the generation of matter. In the SM we need at least three families

- The existence of antiparticles with the same mass and decay rate

- The connection between spin and statistics

- T-reversal and CPT are the only ones implemented by anti-unitary operators

$$\begin{tikzcd} \mathbf{q}_0, \mathbf{p}_0 \arrow[r, "\mathrm{T}"] \arrow[d, "t"] & \mathbf{q}_0, -\mathbf{p}_0 \\ \mathbf{q}(t), \mathbf{p}(t) \arrow[r, "\mathrm{T}"] & \mathbf{q}(t), -\mathbf{p}(t) \arrow[u, "t"'] \end{tikzcd}$$

Anomalous Symmetries

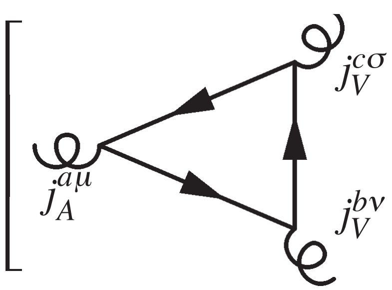
Sometimes symmetries of the classical Lagrangian do not survive quantisation. There are three examples we can cite:

❖Global chiral symmetries, responsible for the electromagnetic decay of the neutral pion

❖Gauged chiral symmetries. This happens when left and right multiplets have different representations of the gauge group. At the one-loop level we find a non-trivial conditions among the quantum numbers necessary to maintain gauge invariance. It suffices to satisfy this condition at the one-loop level

❖Scale invariance. The behaviour of the theory under scale transformation. Rather how physics depends on scales is far more interesting that just dimensional analysis.

$$\langle 0|T\left[j_A^{a\mu}(x)\,j_V^{b\nu}(x')\,j_V^{c\sigma}(0)\right]|0\rangle = \Big[\ \text{(diagram)}\ \Big]_{\text{symmetric}} \propto \pm\ \mathrm{tr}\left[\tau^a_{i,\pm}\{\tau^b_{i,\pm},\tau^c_{i,\pm}\}\right]$$



$$\sum_{i=1}^{N_+}\mathrm{tr}\left[\tau^a_{i,+}\{\tau^b_{i,+},\tau^c_{i,+}\}\right] - \sum_{j=1}^{N_-}\mathrm{tr}\left[\tau^a_{j,-}\{\tau^b_{j,-},\tau^c_{j,-}\}\right] = 0.$$

Anomaly cancellation condition, it has highly non-trivial implications for the family structure

Anomalous Symmetries

quarks: $\begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_{L,\frac{1}{6}}$ $\quad u^\alpha_{R,\frac{2}{3}}$ $\quad d^\alpha_{R,\frac{2}{3}}$

leptons: $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_{L,-\frac{1}{2}}$ $\quad e_{R,-1}$

$(3,2)^L_{\frac{1}{6}} \qquad (1,2)^L_{-\frac{1}{2}}$

$(3,1)^R_{\frac{2}{3}} \qquad (3,1)^R_{-\frac{1}{3}} \qquad (1,1)^R_{-1}$

Anomalies cancel generation by generation. In fact the hypercharge assignments is completely determined if we also impose the traceless-ness of any U(1)

$$\sum_{\text{left}} Y_+^3 - \sum_{\text{right}} Y_-^3 = 3\times 2\times\left(\frac{1}{6}\right)^3 + 2\times\left(-\frac{1}{2}\right)^3 - 3\times\left(\frac{2}{3}\right)^3$$
$$-\,3\times\left(-\frac{1}{3}\right)^3 - (-1)^3 = \left(-\frac{3}{4}\right) + \left(\frac{3}{4}\right) = 0.$$

$\mathrm{SU}(3)^3$	$\mathrm{SU}(2)^3$	$\mathrm{U}(1)^3$
$\mathrm{SU}(3)^2\,\mathrm{SU}(2)$	$\mathrm{SU}(2)\,\mathrm{U}(1)$	
$\mathrm{SU}(3)^2\,\mathrm{U}(1)$	$\mathrm{SU}(2)\,\mathrm{U}(1)^2$	
$\mathrm{SU}(3)\,\mathrm{SU}(2)^2$		
$\mathrm{SU}(3)\,\mathrm{SU}(2)\,\mathrm{U}(1)$		
$\mathrm{SU}(3)\,\mathrm{U}(1)^2$		

Scale invariance, renormalisation