Two-to-three scattering amplitudes in QCD

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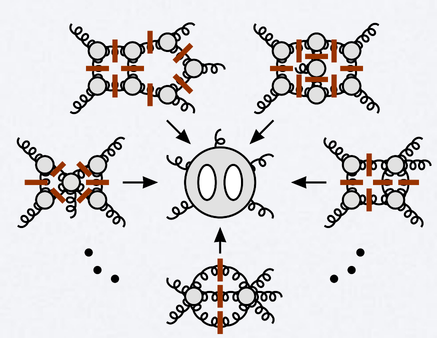
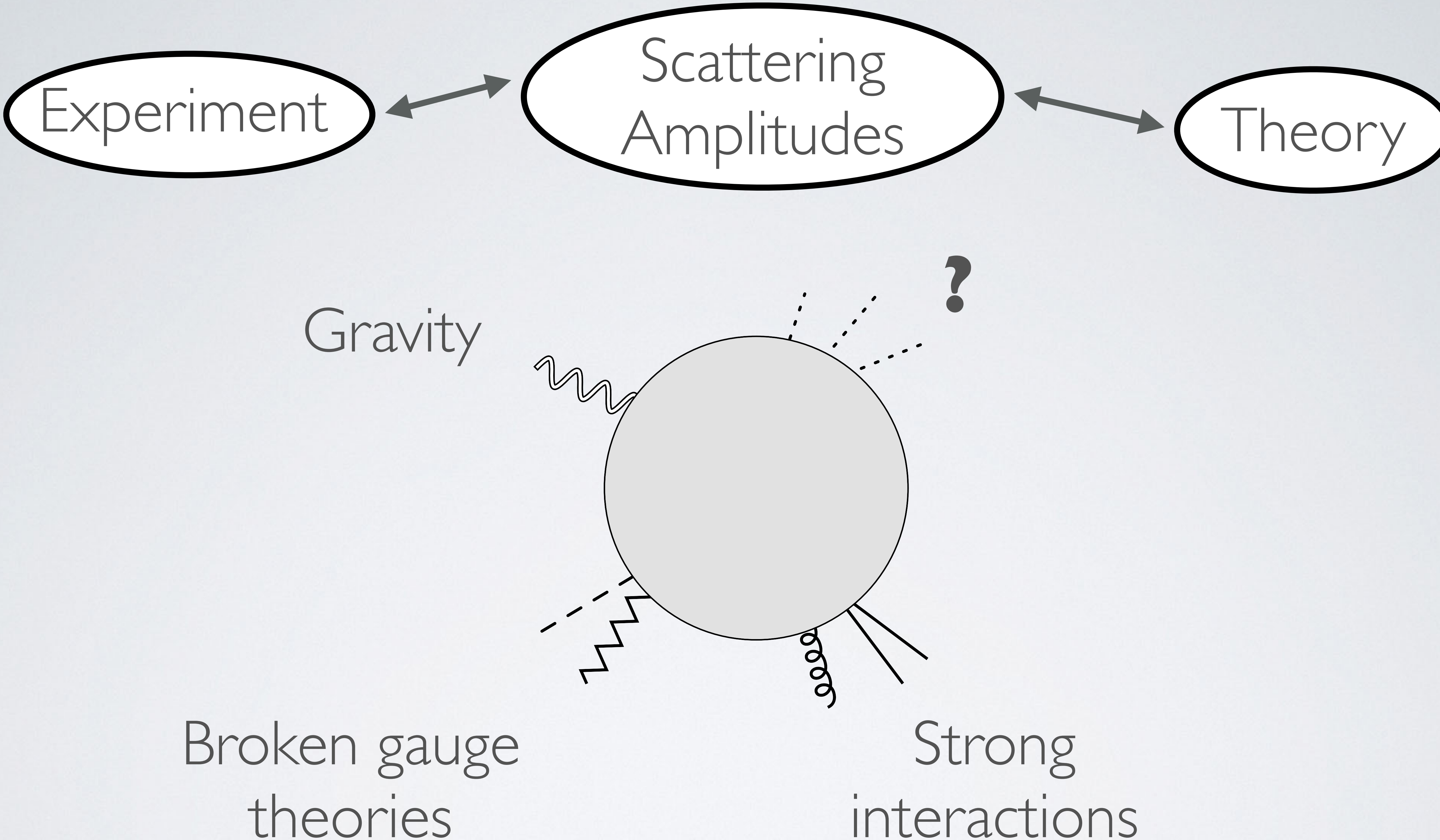
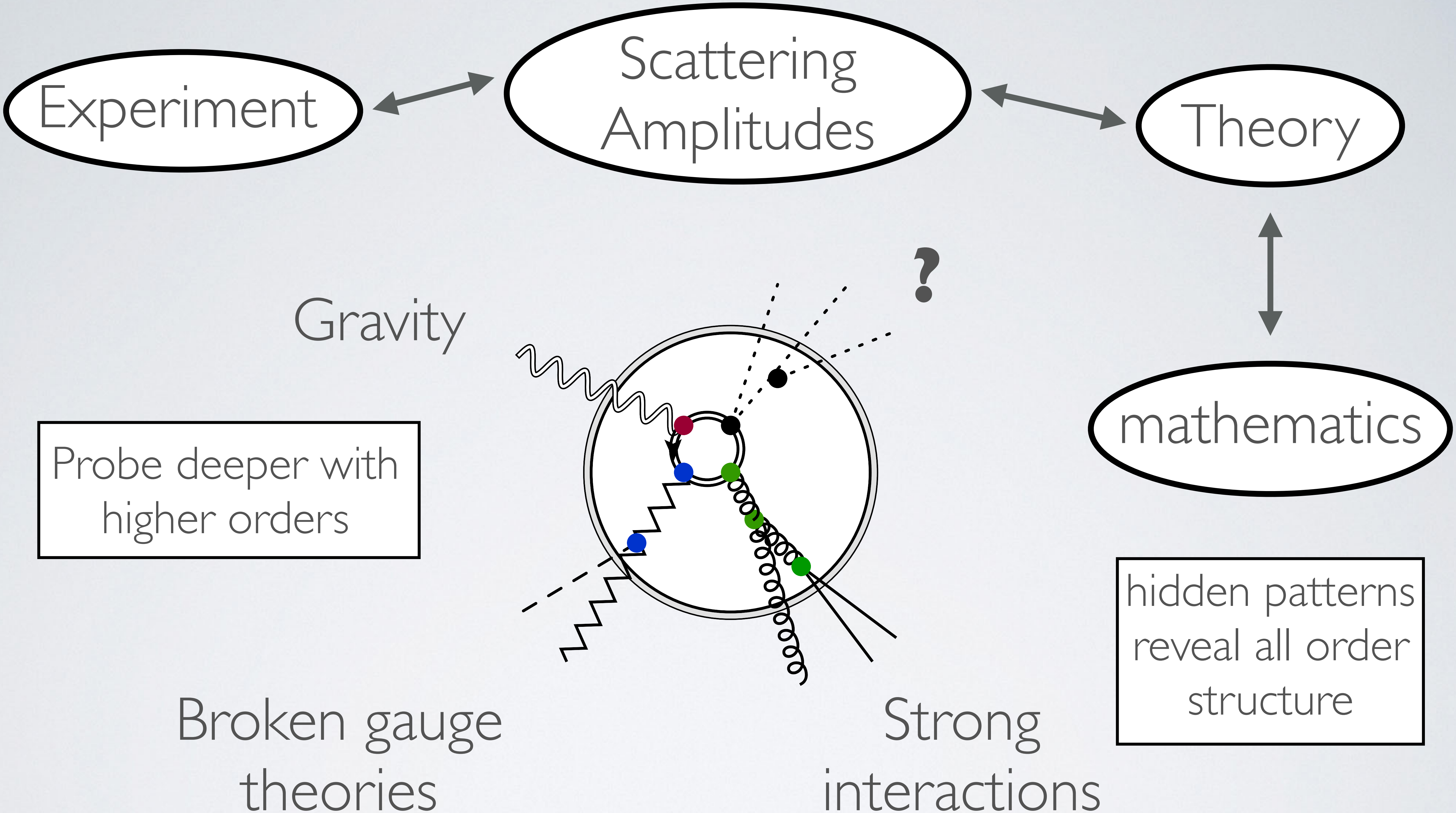
based on work with Brønnum-Hansen, Chaubey, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Marzucca, Moodie, Peraro, Kryś, Zoia

7th September 2021

Colloquium for Demokritos, Athens




Experiment
Scattering Amplitudes
Theory
Gravity
?
Broken gauge theories
Strong interactions


Experiment
Scattering Amplitudes
Theory
Gravity
?
mathematics
Probe deeper with higher orders
hidden patterns reveal all order structure
Broken gauge theories
Strong interactions

$$\mathcal{A}_{h,c,\ldots}(i \to f) : \mathbb{M} \mapsto \mathbb{C}$$

Quantum numbers:
spin, colour charge etc.

Kinematics: momenta in (e.g.)
Minkowski space, masses etc.

$$\sigma = \frac{1}{(\text{flux})} \int d\Phi_f \sum_Q \langle |\mathcal{A}_Q(i \to f)|^2 \rangle$$

Cross section:
more generally,
differential
observables*

Phase-space integral:
over final state kinematics

Squared amplitude:
summed over states and
averaged initial

* apologies for over simplified picture! Most QFTs filled with **IR divergences** - must also sum over unresolved radiation (i→f +X). Observables must be well defined with respect to this radiation or 'infrared safe'.

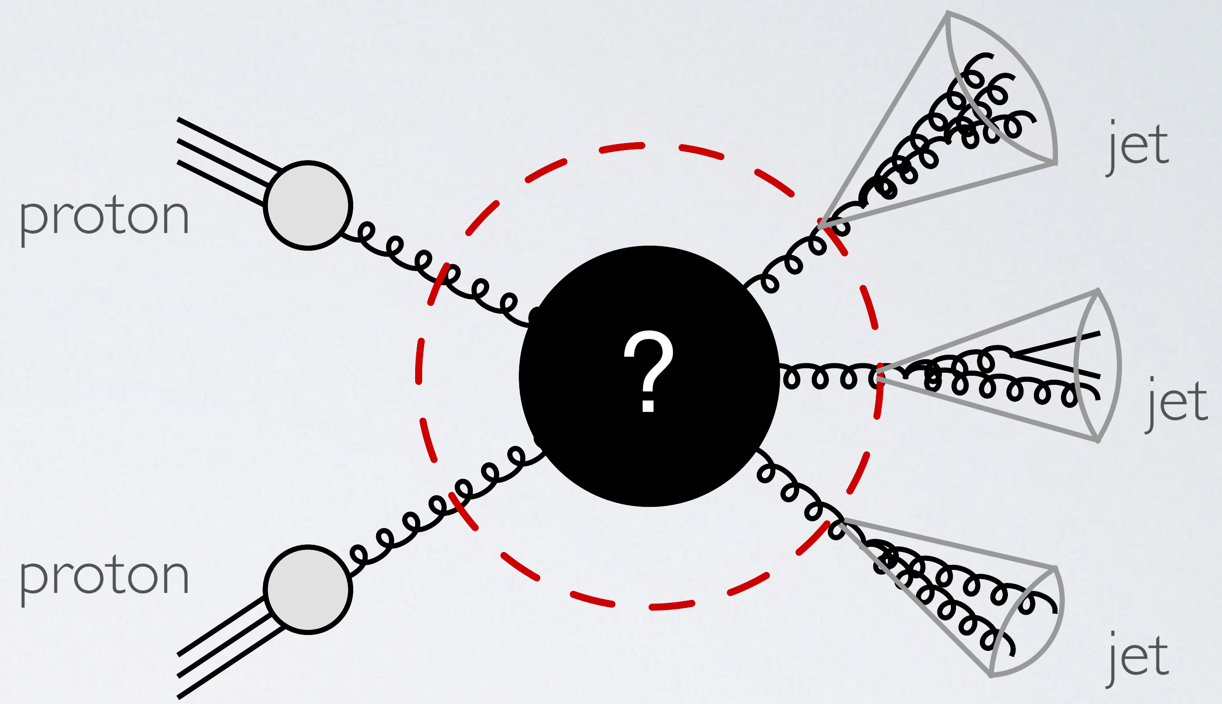
UV divergences also appear and must be renormalised into couplings or treated with an EFT expansion

precision frontier: 2 → 3 scattering

(2 → 3)/(2 → 2) ratio quantities become accessible

systematic errors cancel

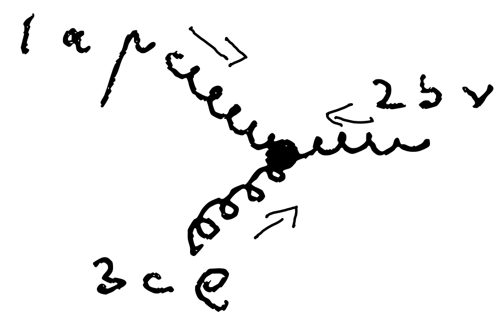
high precision observables



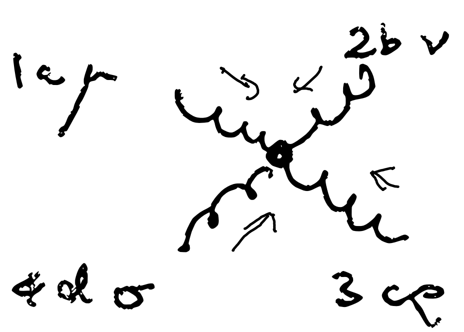
process	precision observables
$pp \to 3j$	jet multiplicity ratios, α_s at high energies, 3-jet mass
$pp \to \gamma\gamma + j$	background to Higgs p_T, signal/background interference effects
$pp \to H + 2j$	Higgs p_T, Higgs coupling through vector boson fusion (VBF)
$pp \to V + 2j$	Vector boson p_T, W^+/W^- ratios and multiplicity scaling
$pp \to VV + j$	backgrounds to p_T spectra for new physics decaying via vector boson

$\mathcal{L} \to \mathcal{A}$

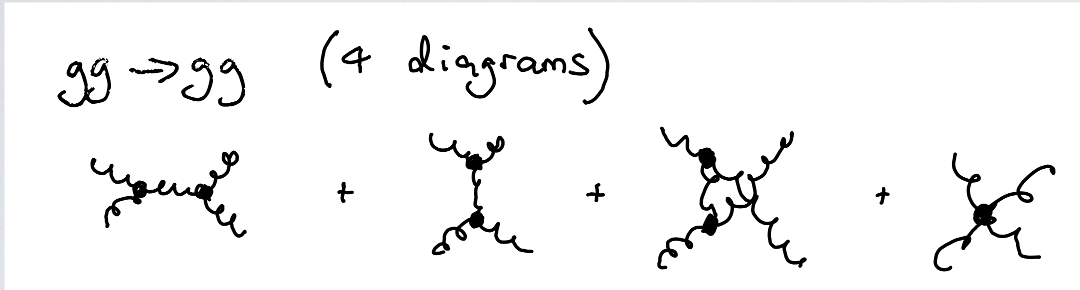
$$\mathcal{L}_{YM} = -\frac{1}{4}\left(F^{\mu\nu a}\, F_{\mu\nu a}\right)$$



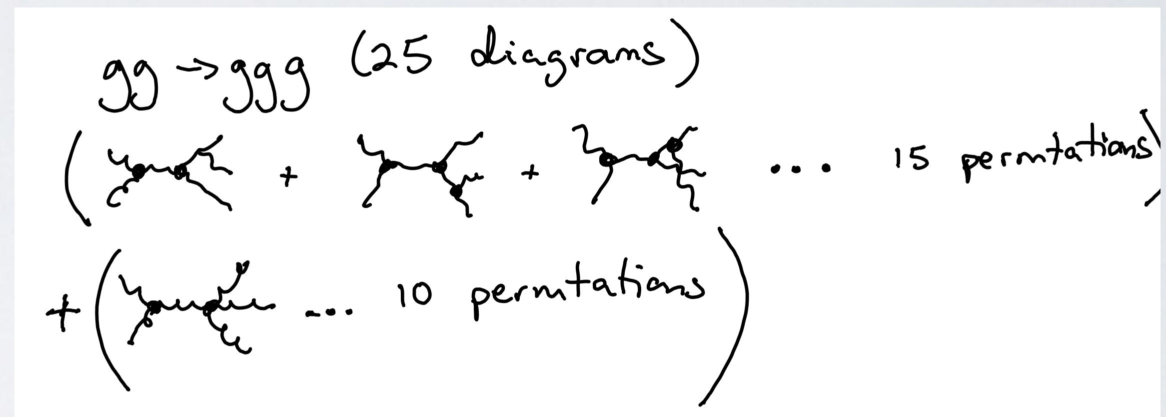
$$g_s f^{abc}\left(g^{\mu\nu}(p_1-p_2)^{\rho} + g^{\nu\rho}(p_2-p_3)^{\mu} + g^{\rho\nu}(p_3-p_1)^{\mu}\right)$$

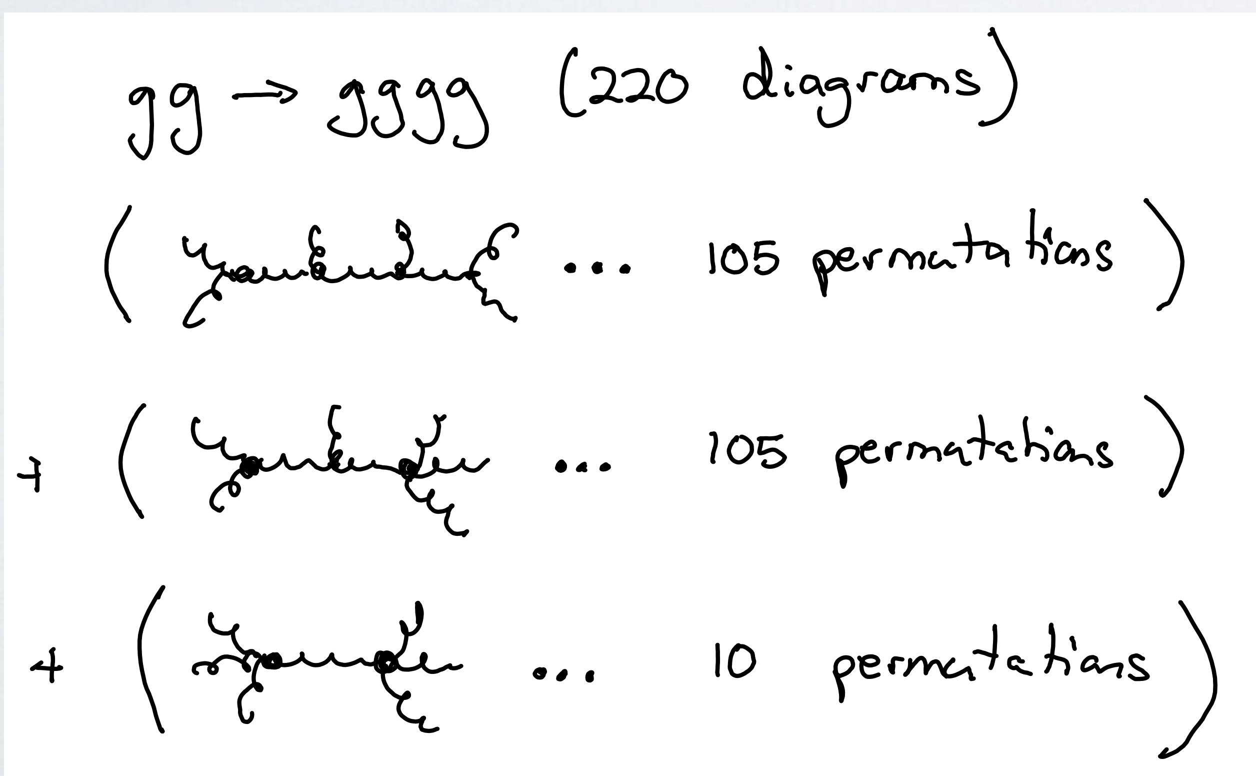


$$i g_s^2\left\{f^{abx} f^{xcd}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{acx} f^{xbd}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{adx} f^{xcb}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma}\right)\right\}$$



non-abelian structure quickly leads to an explosion of terms

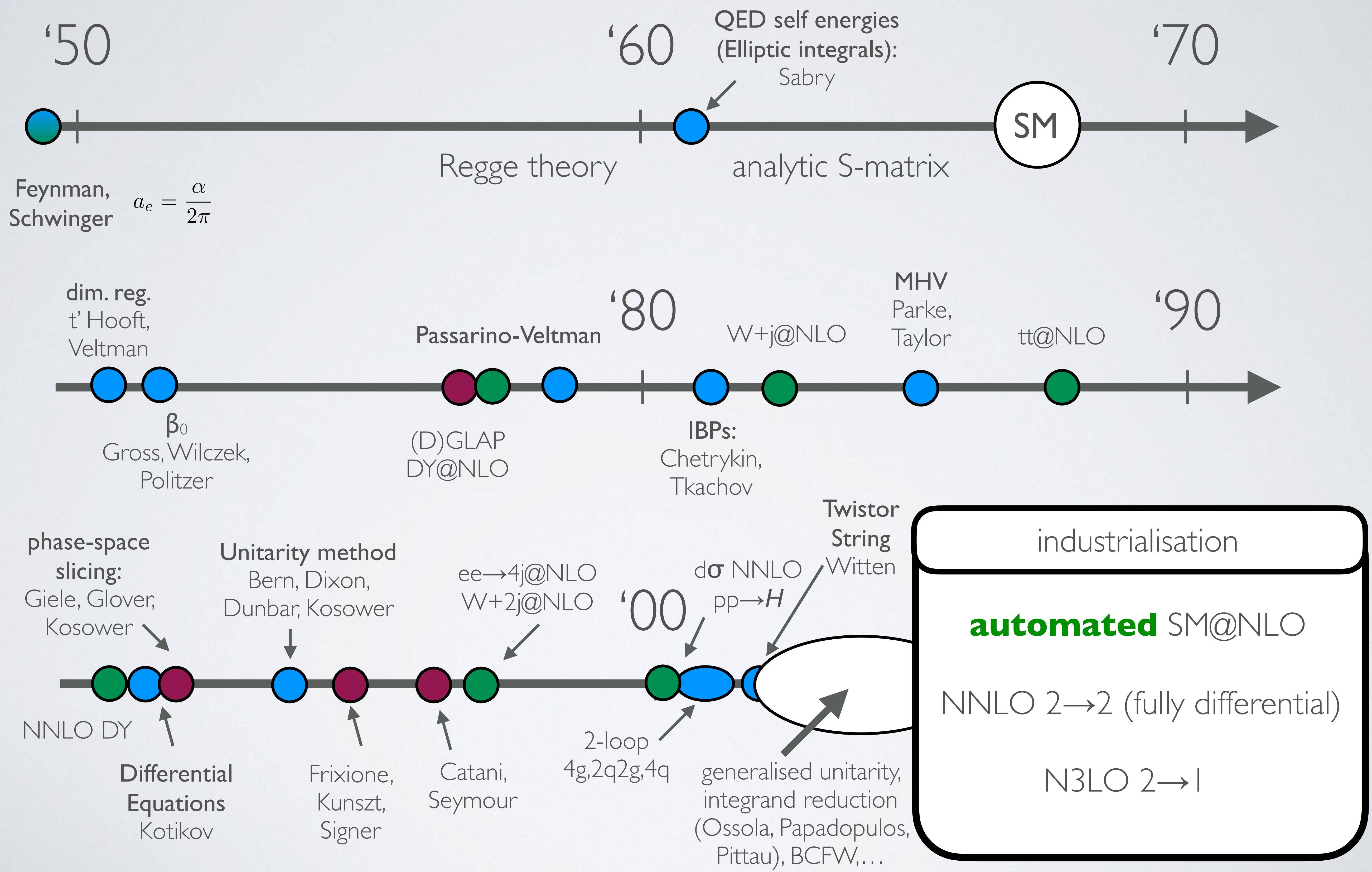




at loop level we find new - an much harder problems - associated with how to evaluate loop integrals…

a selected history

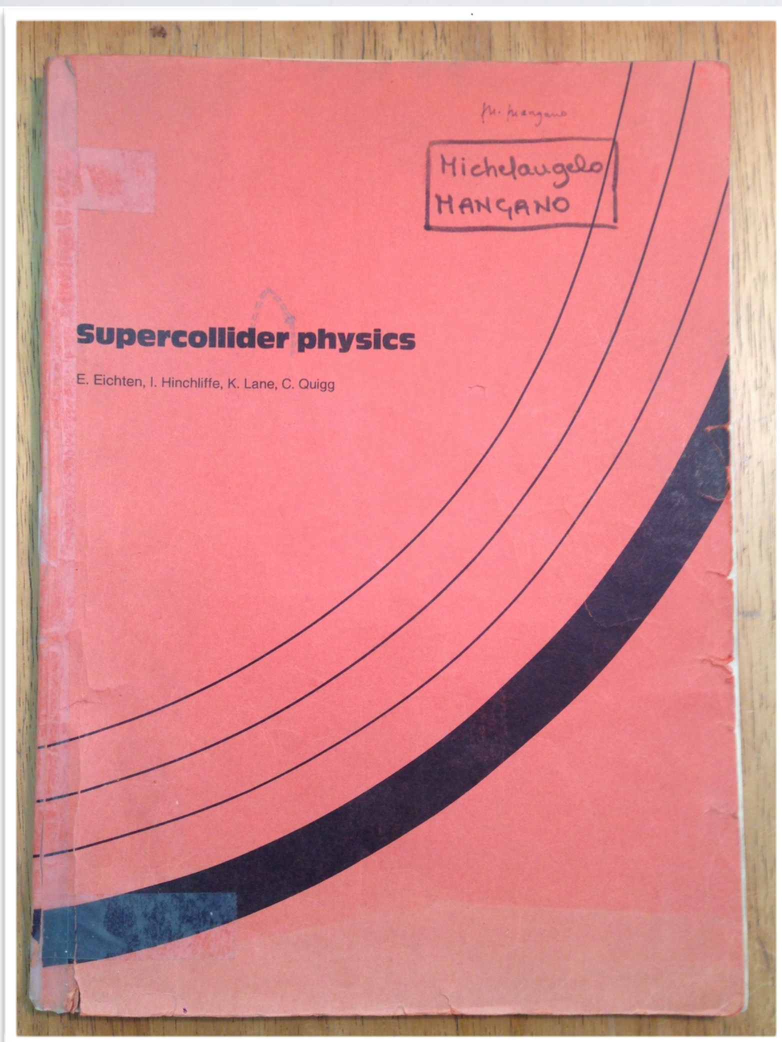
increasingly precise predictions in gauge theory



experiment drives theory

[taken from talk by Michelangelo Mangano at MHV@30 (2016)]

Reviews of Modern Physics, Vol. 56, No. 4, October 1984



For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two$\to$four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Then, in 1986….

[Parke, Taylor PRL 56 (1986) 2459]

$$A^{(0)}(1^+, 2^+, 3^+, \ldots, n^+) = 0$$

$$A^{(0)}(1^-, 2^+, 3^+, \ldots, n^+) = 0$$

$$A^{(0)}(1^-, 2^-, 3^+, \ldots, n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1)n \rangle \langle n1 \rangle}$$

Colour ordering: manage gauge group factors ⇒ reduce number of independent terms

Spinor-helicity formalism: [Berends, Kleiss, de Causmaeker, Gastmans, Troost, Wu, Gunion, Kunszt, Giele, Kujif, Xu, Zhang, Chang]

$$\langle ab \rangle = \sqrt{|(p_a + p_b)^2|} \exp(i\theta_{ab})$$

↙

$$(p_a + p_b)^2 = s_{ab}$$

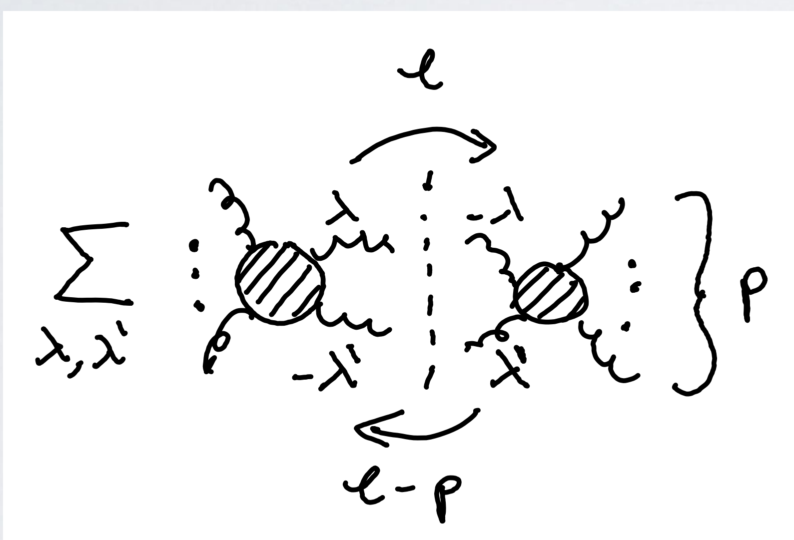
$$\langle ab \rangle^* = \pm [ab]$$

on-shell simplicity

S-matrix elements are simpler than the
Feynman diagram representation suggests

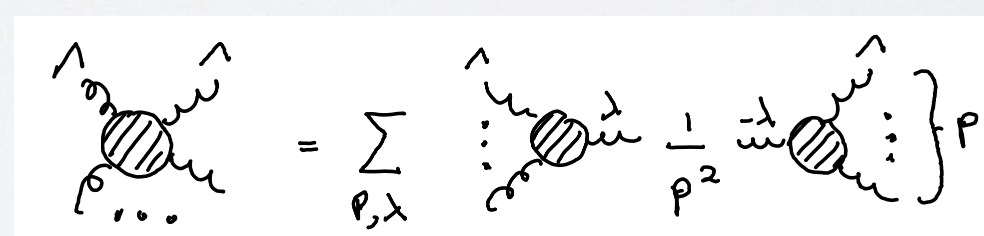
unitarity cuts

[Bern, Dixon, Dunbar, Kosower (1994)]



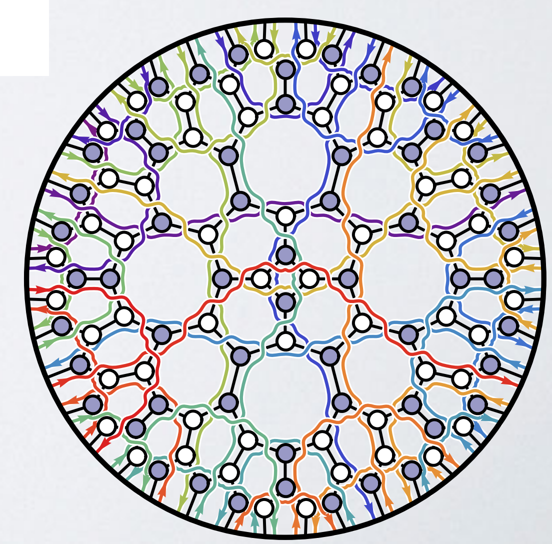
BCFW recursion, on-shell diagrams

[Britto, Cachazo, Feng, Witten (2005)][Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka (2012)]



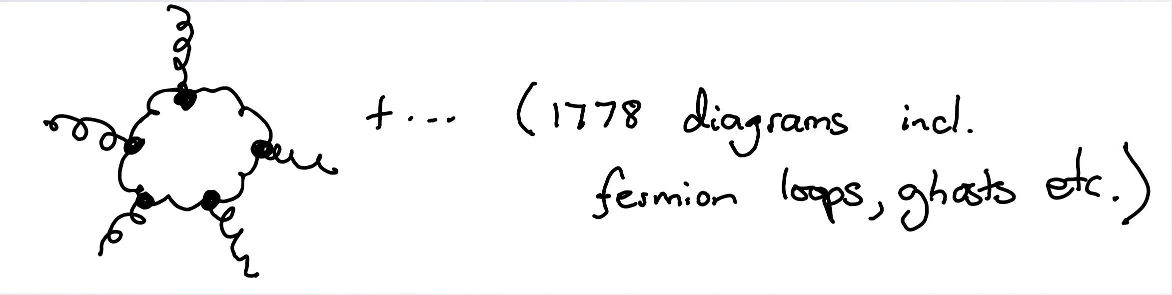
geometric formulation of gauge theory?

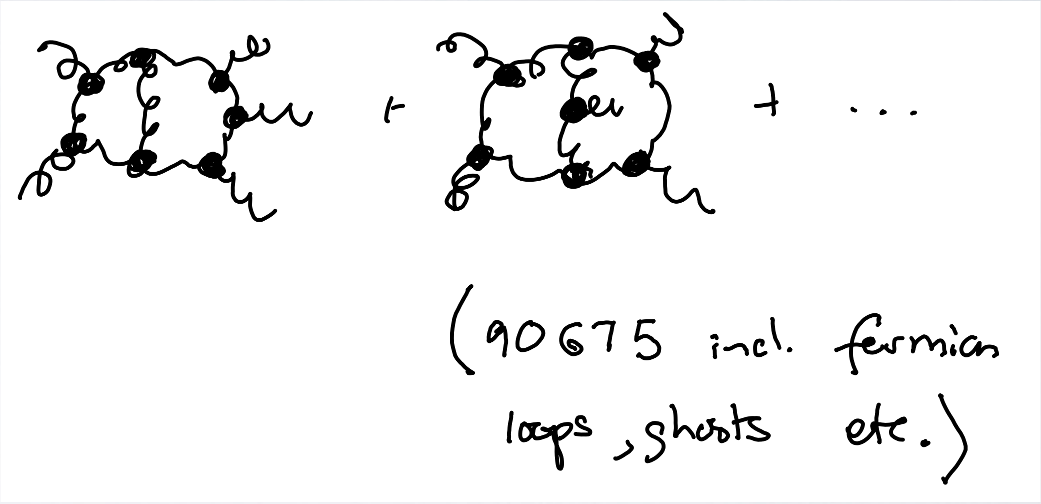
'amplitudehedron' [Arkani-Hamed, Trnka (2013]



where are the bottlenecks?

the number of diagrams is not a good measure of complexity





- easy to generate (e.g. QGRAF)
- efficient processing with computer algebra (e.g. FORM)
- often very large intermediate expressions
- cancellations between diagrams

accurate matrix elements
(perturbative)

non-perturbative

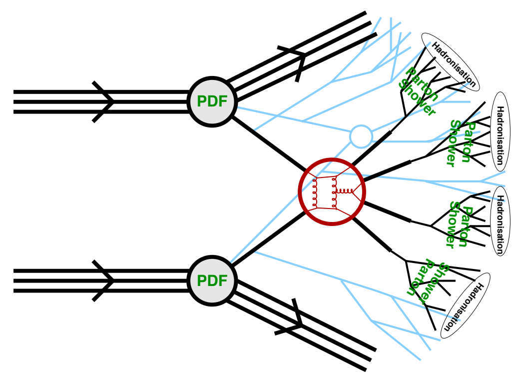
PDFs, hadronisation…

MC event generator

phase-space sampling

parton shower / resummation

underlying event



how can we make reliable theoretical error estimates?

perturbative error

LO > 50%

NLO 20-30%

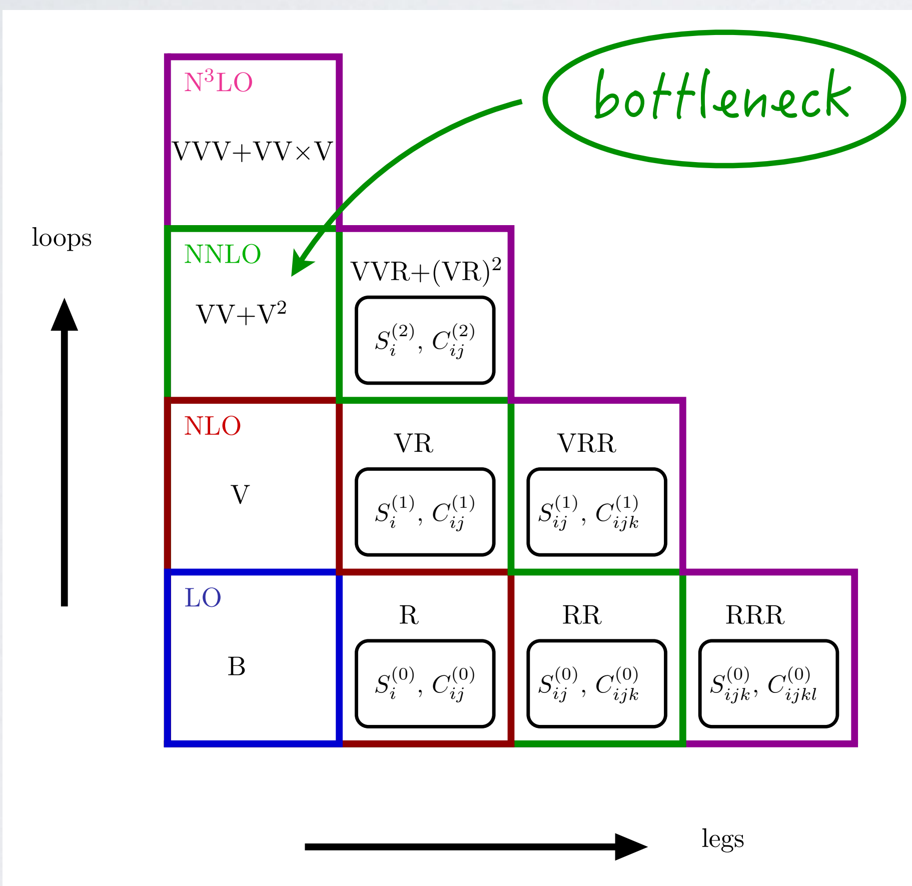
NNLO 5-10%

infra-red subtraction/
regularisation

precision QCD predictions

$$d\sigma = d\sigma^{\mathrm{LO}} + \alpha_s d\sigma^{\mathrm{NLO}} + \alpha_s^2 d\sigma^{\mathrm{NNLO}}$$

~10-30 % (NLO) **~1-10 %** (NNLO)



complicated integrals

complicated phase-space

(d)σ N3LO (2⟶1)

[Anasastiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger]
[Berhing, Melnikov, Rietkerk, Tancredi, Wever]
[Dulat, Mistlberger, Pelloni] [Duhr, Dulat, Mistlberger]

towards dσ N3LO 2⟶2

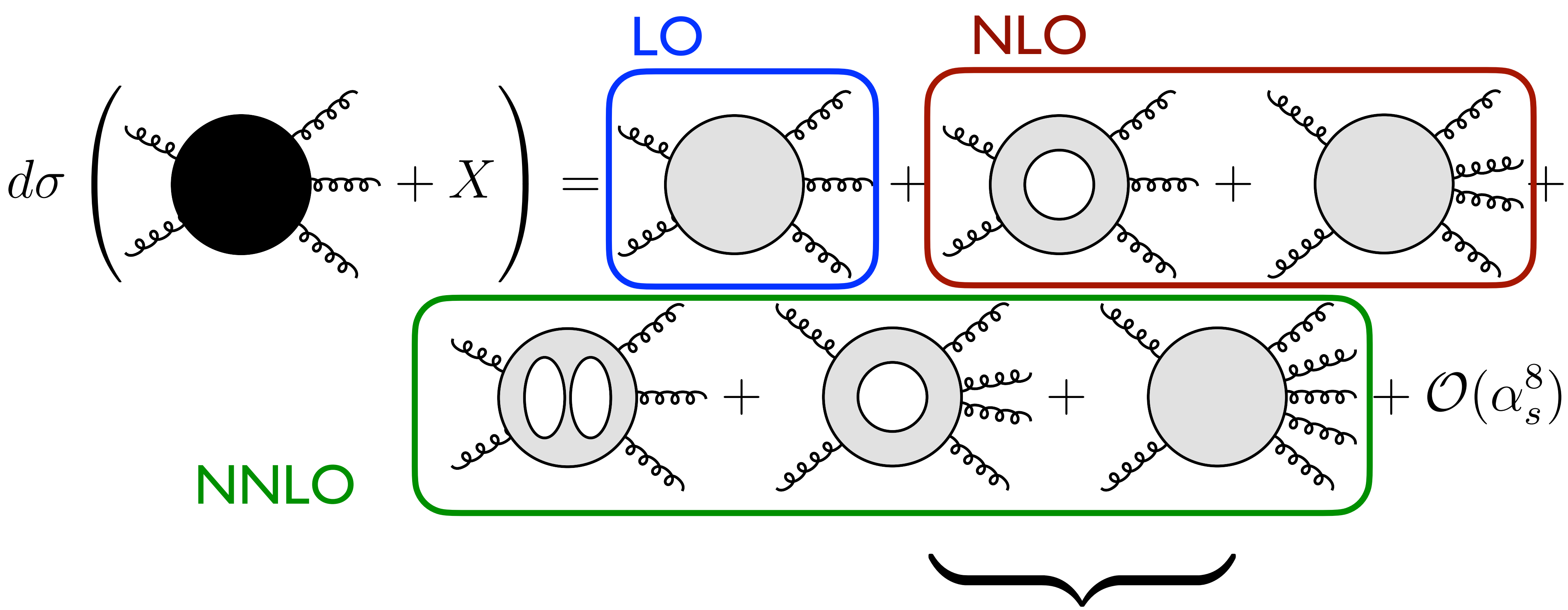
3-loop 4-point amplitudes
[Ahmed, Henn, Mistlberger]
[Jin, Luo] [Caola, von Manteuffel, Tancredi]

dσ NNLO (fully differential 2⟶3)

qq⟶3γ [Chawdhry, Czakon, Mitov, Poncelet]
[Kallweit, Sotnikov, Wiesemann]
qq⟶γγj [Chawdhry, Czakon, Mitov, Poncelet]
pp⟶3j [Czakon, Mitov, Poncelet]

don't forget! (N)NLO EW, mass effects, resummation, showers…

precision frontier: 2 → 3 scattering at two loops



infrared subtraction problem is highly non-trivial, plenty of new ideas needed here too of course!

bare amplitudes

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) \mathcal{F}_i(\epsilon, \{p\})$$

rational functions

integrals/special functions

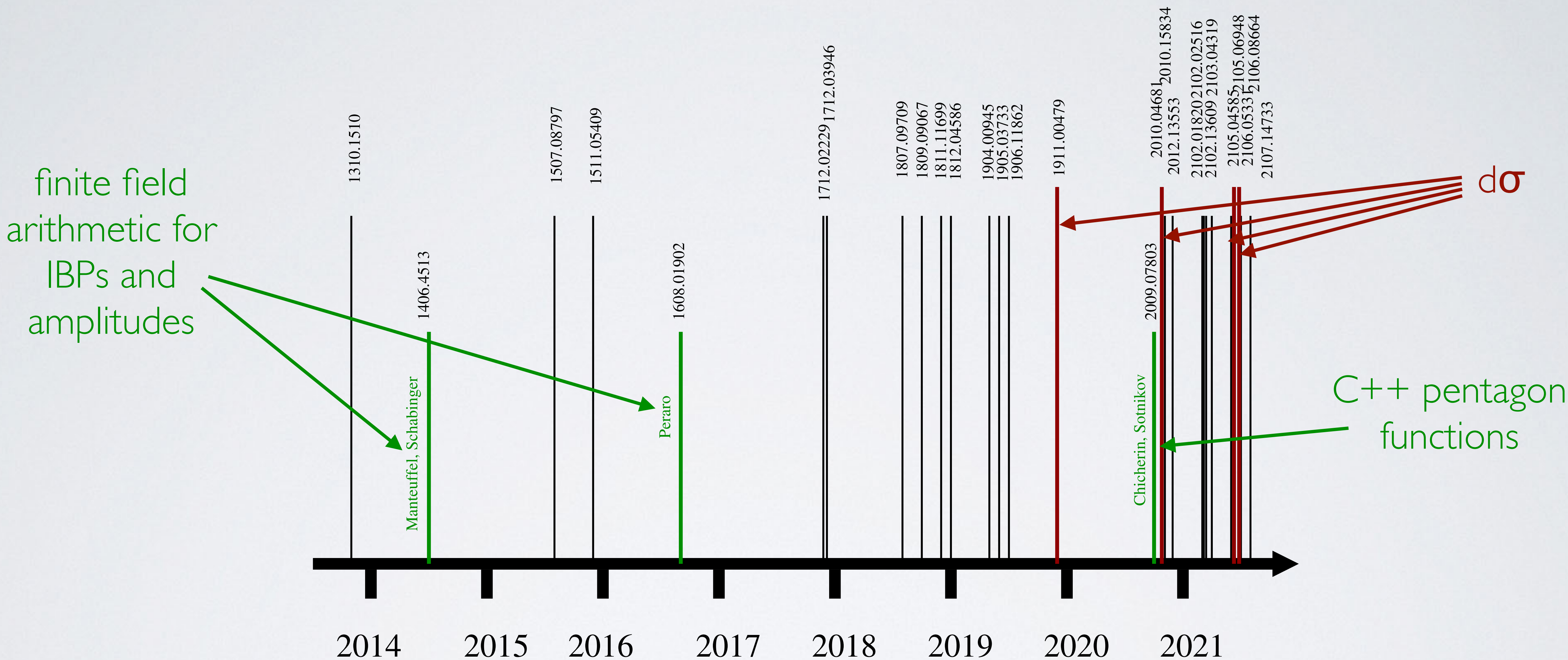
finite remainders

$$F^{(L)} = A^{(L),4-2\epsilon} - \sum_{k=1}^{L} I^{(k),4-2\epsilon} A^{(L-k),4-2\epsilon}$$

universal IR/UV poles

[Catani (1998)][Becher, Neubert (2009)]
[Magnea, Gardi (2009)]

precision frontier: 2 → 3 scattering at two-loops in QCD



Abreu, Agarwal, SB, Brønnum-Hansen, Buccioni, Chawdhry, Chicherin, Czakon, Dixon, Dormans, Febres Cordero, Gehrmann, Hartanto, Heinrich, Henn, Herrmann, Ita, Kraus, Kryś, Lo Presti, Mitev, Mitov, Mogull, Ochirov, O'Connell, Page, Papadopoulos, Pascual, Peraro, Poncelet, Ruf, Sotnikov, Tancredi, Tommasini, von Manteuffel, Wasser, Wever, Zeng, Zhang, Zoia, ...

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) \mathcal{F}_i(\epsilon, \{p\})$$

integrals/special functions

integration-by-parts identities: [Chetyrkin, Tkachov (1981)]

$$\int d^D k \frac{\partial}{\partial k^\mu} \frac{v^\mu}{\prod_i (k-q_i)^2} = 0$$

Algorithmic solution: large linear algebra problem [Laporta (2000)]

differential equations: Kotikov (1991), Bern, Dixon, Kosower (1993), Remiddi (1997), Gehrmann, Remiddi (2000), Henn (2013), Papadopoulos (2014)

'canonical form'

$$\frac{d}{ds}\vec{MI}(s,\epsilon) = \epsilon A(s).\vec{MI}(s,\epsilon)$$

↙ iterated integrals

↘ poly-logarithms?

finite field arithmetic

not a new idea - used in many computer algebra systems

solving IBP systems: e.g. FINRED [von Manteuffel],
KIRA+FIREFLY [Maierhoefer, Usovitsch, Uwer, Klappert, Lange]

framework for amplitude computations: FINITEFLOW [Peraro (2019)]

```
(* take some, reasonably large, prime number *)
FFPrimeNo[1]
(* all quantities evaluated modulo a prime number *)
Mod[-3,FFPrimeNo[1]]
Mod[87+FFPrimeNo[1],FFPrimeNo[1]]
Solve[b*3==87,Modulus→FFPrimeNo[1]][[1]]
(* already implemented in Mathematica *)
Mod[87/3+FFPrimeNo[1],FFPrimeNo[1]]
```

```
9 223 372 036 854 775 643
9 223 372 036 854 775 640
87
{b → 29}
29
```

NB: multiplicative inverse

extremely efficient solutions to linear algebra systems

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) \mathcal{F}_i(\epsilon, \{p\})$$

rational functions

multiple numerical (mod prime) evaluations can used to reconstruct complete analytic information

Newton (polynomial) and Thiele (rational) interpolation

```
(* implement the Newton interpolation algorithm *)
NewtonReconstruct[z_, zvalues_List, fvalues_List, primeno_]:=Module[{res,maxdegree,aa,eqs,sol},
maxdegree = Length[zvalues]-1;
res = Sum[aa[r]*Product[(z-zvalues[[i+1]]),{i,0,r-1}],{r,0,maxdegree}];
eqs = Equal@@@Transpose[{res /. ({Rule[z,#]}&/@zvalues),fvalues}];
sol = Solve[eqs,Table[aa[i],{i,0,Length[fvalues]-1}],{Modulus->primeno}];
Return[res/. sol[[1]]];
]
```

```
fff[z_]:=15/2*z+119/6*z^2;
values = {19,44,78};
FFRatMod[fff/@values,FFPrimeNo[0]]
test = NewtonReconstruct[z,values,%,FFPrimeNo[0]]
Collect[%,z,FFRatRec[#,FFPrimeNo[0]]&]
```

{6 148 914 691 236 524 491, 6 148 914 691 236 555 916, 121 251}

6 148 914 691 236 524 491 + 1257 (-19 + z) + 1 537 228 672 809 129 317 (-44 + z) (-19 + z)

$$\frac{15\,z}{2} + \frac{119\,z^2}{6}$$

Rational external kinematics: e.g. Momentum Twistors (Hodges)

Trivial parallelisation of sample points

putting everything together:
a summary of latest results

new results!

massless 5-particle scattering

a basis of pentagon functions identified!

Gehrmann, Henn, Lo Presti (2018)

Chicherin, Henn, Mitev (2018)

Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia (2020)

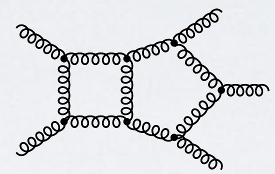
Abreu, Dixon, Herrmann, Page, Zeng (2020)

efficient numerical evaluations for all master integrals

Sotnikov, Chicherin (2020)

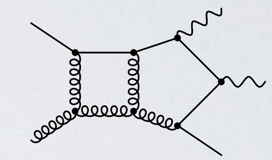
fast numerical codes for evaluation in physical region

pp→3j



Abreu, Febres-Cordero, Ita Page, Sotnikov [2102.13609]

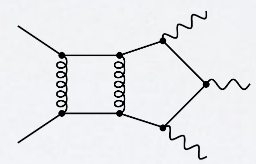
pp→γγ +j



Agarwal, Buccioni, von Manteuffel, Tancredi [2102.01820]

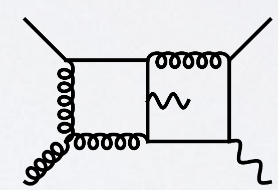
Chawdhry, Czkaon, Mitov Poncelet [2103.04319]

pp→γγγ

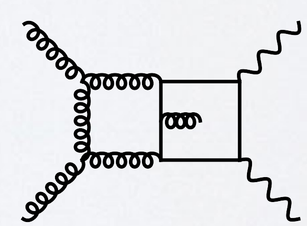


Chawdhry, Czkaon, Mitov Poncelet [2103.04319]

Abreu, Page, Pascual, Sotnikov [2010.15834]



Agarwal, Buccioni, von Manteuffel, Tancredi [2105.04585]



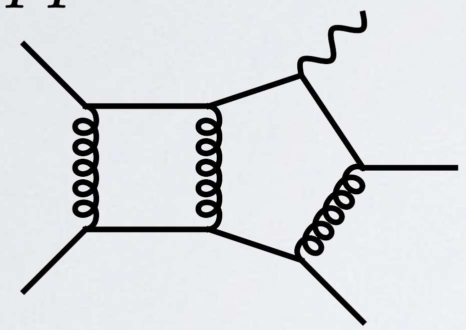
SB, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Marzucca, Moodie, Peraro, Krys, Zoia [2106.08664]

new results!

5-particle scattering with an off-shell leg

analytic finite remainders.
numerical evaluation with generalised series expansions

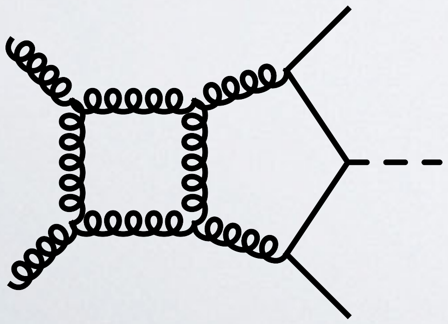
$pp \to W b\bar{b}$



leading colour, on-shell W, massless b

SB, Hartanto, Zoia
[2102.02516]

$pp \to H b\bar{b}$



leading colour, massless b

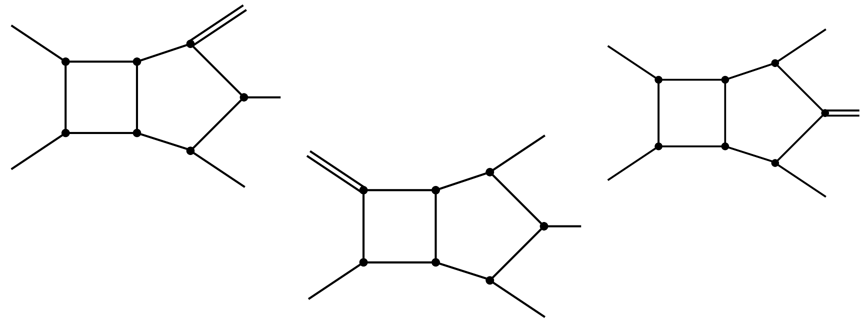
SB, Hartanto, Kryś, Zoia
[2107.14733]

all planar integrals known!

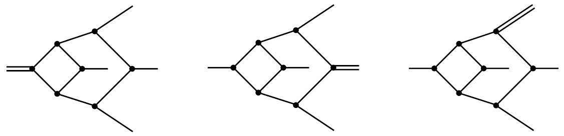
[Papadopoulos Tommasini, Wever (2019)]
[Padadopoulos, Wever (2019)]

[Abreu, Ita, Moriello, Page, Tschernow, Zeng (2020)]

[Canko, Padadopoulos, Syrrakos (2020)]
[Syrrakos (2020)]



non-planar hexa-box



[Abreu, Ita, Page, Tschernow 2107.14180]

new results!

analytic scattering amplitudes with massive propagators

well studied process - fully analytic form still challenging

numerical solutions very successful

[Baernreuther, Czakon, Chen, Fiedler, Poncelet (2008-2018)]

analytic solutions for qq→tt and all non-elliptic sectors of gg→tt known

[Bonciani, Ferroglia, Gehrmann, Studerus, von Manteuffel, Di Vita, Laporta, Mastrolia, Primo, Schubert, Becchetti, Casconi, Lavacca (2009-2019)]

$$gg \to t\bar{t}$$

leading colour helicity amplitudes with top-quark loops

SB, Chaubey, Hartanto, Marzucca [2102.13450]

$$e^+e^- \to \mu^+\mu^-$$

complete analytic form

Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Ronca, Schubert, Torres Bobadilla, Tramontano [2106.13179]

and now…a few technical details

SB, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia [2106.08664]

SB, Hartanto, Zoia [2102.02516]

SB, Hartanto, Kryś, Zoia [2107.14733]

SB, Chaubey, Hartanto, Marzucca [2102.02516]

computational framework

QGRAF + FORM/MATHEMATICA + rational phase-space (Momentum Twistors) → colour ordered helicity amplitudes

$$M^{(2)}(\{p\},\epsilon) = \sum_i c_i(\{p\},\epsilon)\mathcal{F}_i(\{p\},\epsilon)$$

↓ IBPs

$$M^{(2)}(\{p\},\epsilon) = \sum_i d_i(\{p\},\epsilon)\mathrm{MI}_i(\{p\},\epsilon)$$

↓ IR/UV sub + expansion to function basis

$$F^{(2)}(\{p\}) = \sum_i e_i(\{p\})\mathrm{mon}_i(f_j^{(w)})$$

↓ linear relations, univariate apart, polynomial reconstruction

complete reduction setup implemented in FINITEFLOW

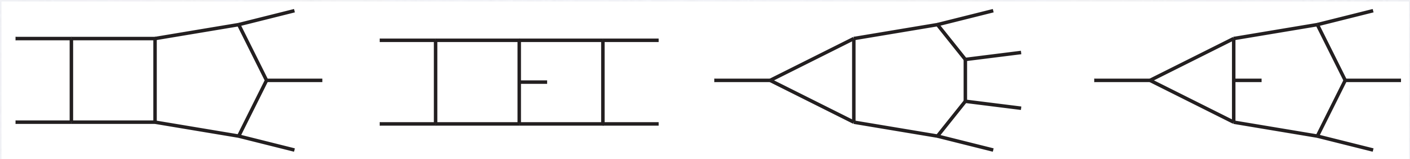
IBPs generated with help from LITERED/ FINITEFLOW

$gg \to \gamma\gamma g$

SB, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia [2106.08664]

$$\mathcal{A}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) = g_s g_e^2 \left(Q_u^2 N_u + Q_d^2 N_d\right) f^{a_1 a_2 a_3} \sum_{\ell=1}^{\infty} \left(n_\epsilon \frac{\alpha_s}{4\pi}\right)^\ell A^{(\ell)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma)$$

$$\begin{aligned} A^{(1)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) &= A_1^{(1)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma)\,, \\ A^{(2)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) &= N_c A_1^{(2)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) \\ &\quad + \frac{1}{N_c} A_2^{(2)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) + n_f A_3^{(2)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) \end{aligned}$$



# permutations:	84	18	120	21

IBPs with LITERED/FINITEFLOW. syzygy relations for planar sectors

analytic reconstruction

polynomial degrees in momentum twistor parametrisation

finite remainder	original	stage 1	stage 2	stage 3*	stage 4*
$F^{(2)}_{1;1}(1^-,2^-,3^+,4^+,5^+)$	69/60	28/20	24/0	19/10	11/5
$F^{(2)}_{1;0}(1^-,2^-,3^+,4^+,5^+)$	78/69	44/35	43/0	21/10	16/9
$F^{(2)}_{1;1}(1^-,2^+,3^+,4^-,5^+)$	59/55	30/27	29/0	18/15	17/4
$F^{(2)}_{1;0}(1^-,2^+,3^+,4^-,5^+)$	89/86	38/36	38/0	20/16	17/3
$F^{(2)}_{1;1}(1^+,2^+,3^+,4^-,5^-)$	40/42	25/27	25/0	15/18	15/0
$F^{(2)}_{1;0}(1^+,2^+,3^+,4^-,5^-)$	66/66	32/33	32/0	13/13	12/3

linear relations

univariate apart

factor matching

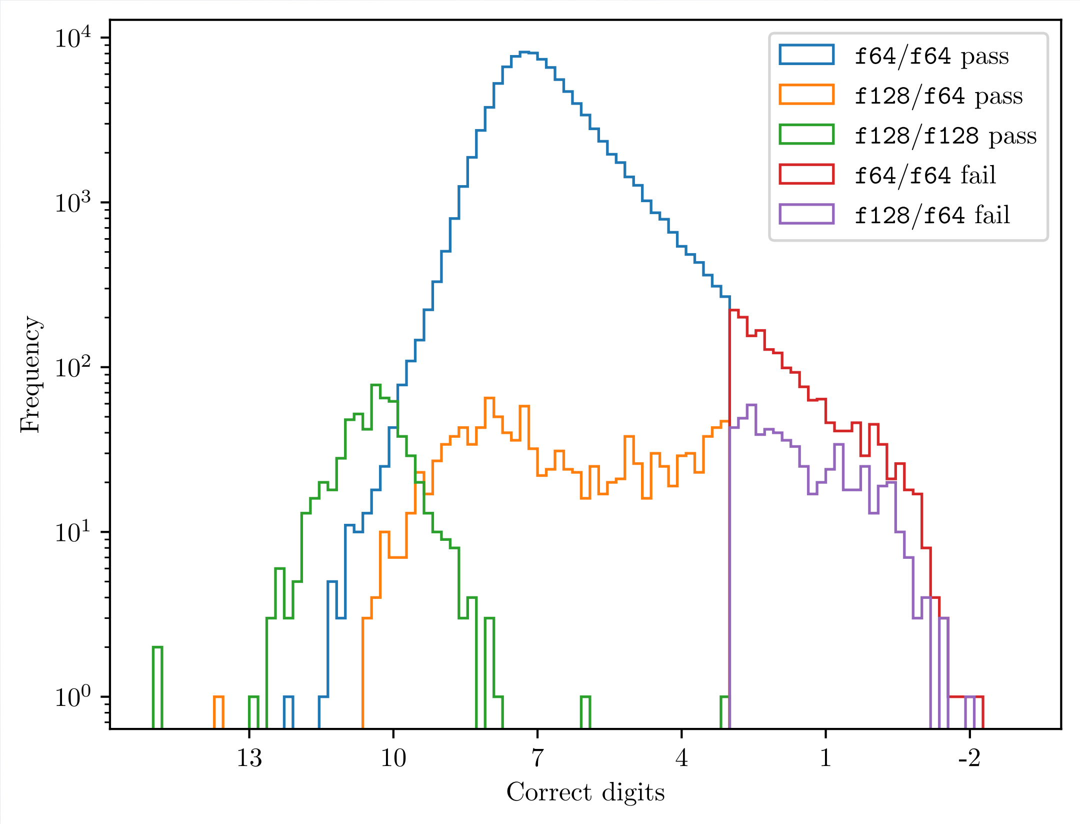
univariate apart
+factor matching

numerical performance

complete colour and
helicity summed hard functions

C++ code available at
https://bitbucket.org/njet/njet

100,000 points over
physical phase-space



average evaluation time including
stability tests and higher precision
corrections ~ **26s per point**

Aside: framework also used to
compute leading colour $pp \to 3j$

full agreement with 2102.13609v2

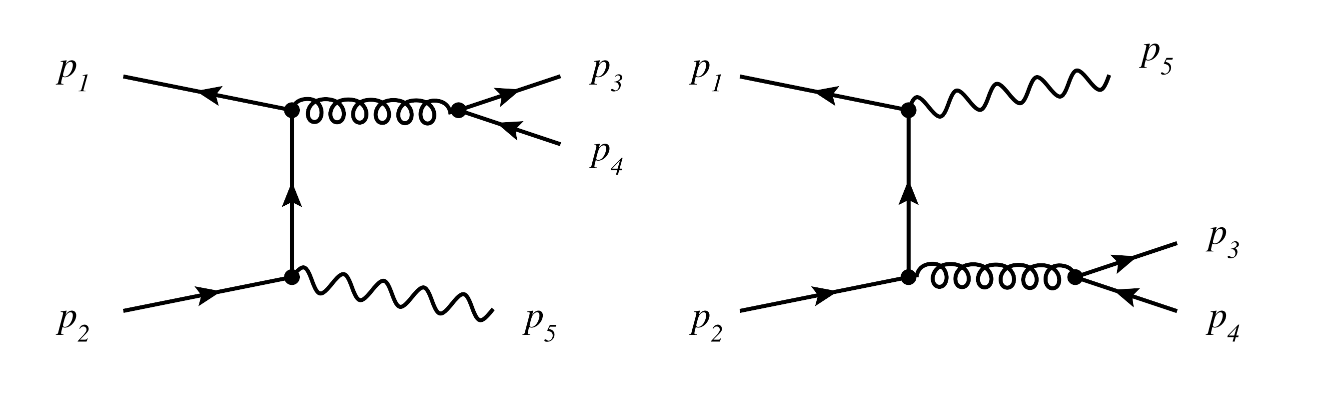
will be included in next NJET version

$pp \to Wb\bar{b}$

SB, Hartanto, Zoia [2102.02516]

two leading order diagrams

background to associated HW production



$$\bar{d}(p_1) + u(p_2) \to b(p_3) + \bar{b}(p_4) + W^+(p_5)$$

6 scalar invariants

$$s_{12} = (p_1 + p_2)^2\,, \quad s_{23} = (p_2 - p_3)^2\,, \quad s_{34} = (p_3 + p_4)^2\,,$$
$$s_{45} = (p_4 + p_5)^2\,, \quad s_{15} = (p_1 - p_5)^2\,, \quad s_5 = p_5^2\,. \qquad (4)$$

- leading colour approximation
- on-shell W boson

$$i \sum_{\lambda} \varepsilon_W^{\mu *}(p_5, \lambda) \varepsilon_W^{\nu}(p_5, \lambda) = -g^{\mu\nu} + \frac{p_5^\mu p_5^\nu}{m_W^2}.$$

special functions

58 letters including 3 square roots

thanks to work of [Abreu et al (2020)]:

↓

also in this work: numerical evaluation with generalised series expansions [Moriello (2019)]

$$d\overrightarrow{\mathrm{MI}} = \epsilon \sum_{i=1}^{58} a_i d\log w_i \, \overrightarrow{\mathrm{MI}}$$

this form allows easy expansion to Chen's iterated integrals

[Chen (1977)]

obey shuffle relations ⇒

minimal basis of independent function

[Canko, Papadopoulos, Syrakkos (2020)]

(lengthy) MPL expressions from simplified differential equations

function basis

- use **master integral components as function basis**
  ⇒ $MI_i^{(k)}$ for the $\mathcal{O}(\epsilon^k)$ component of the i^{th} master integral
- high precision evaluation of GPL form (~1000 digits)
  ⇒ **analytic boundaries via PSLQ**
- **determine relations** between integral components by solving linear system
  ⇒ $f_i^{(k)}$ for the function at weight k

derive **new differential equation** for independent integral components

find **analytic cancellation of IR poles**

numerical evaluation

gen. series exp. only with $f_i^{(k)}$ in finite remainder

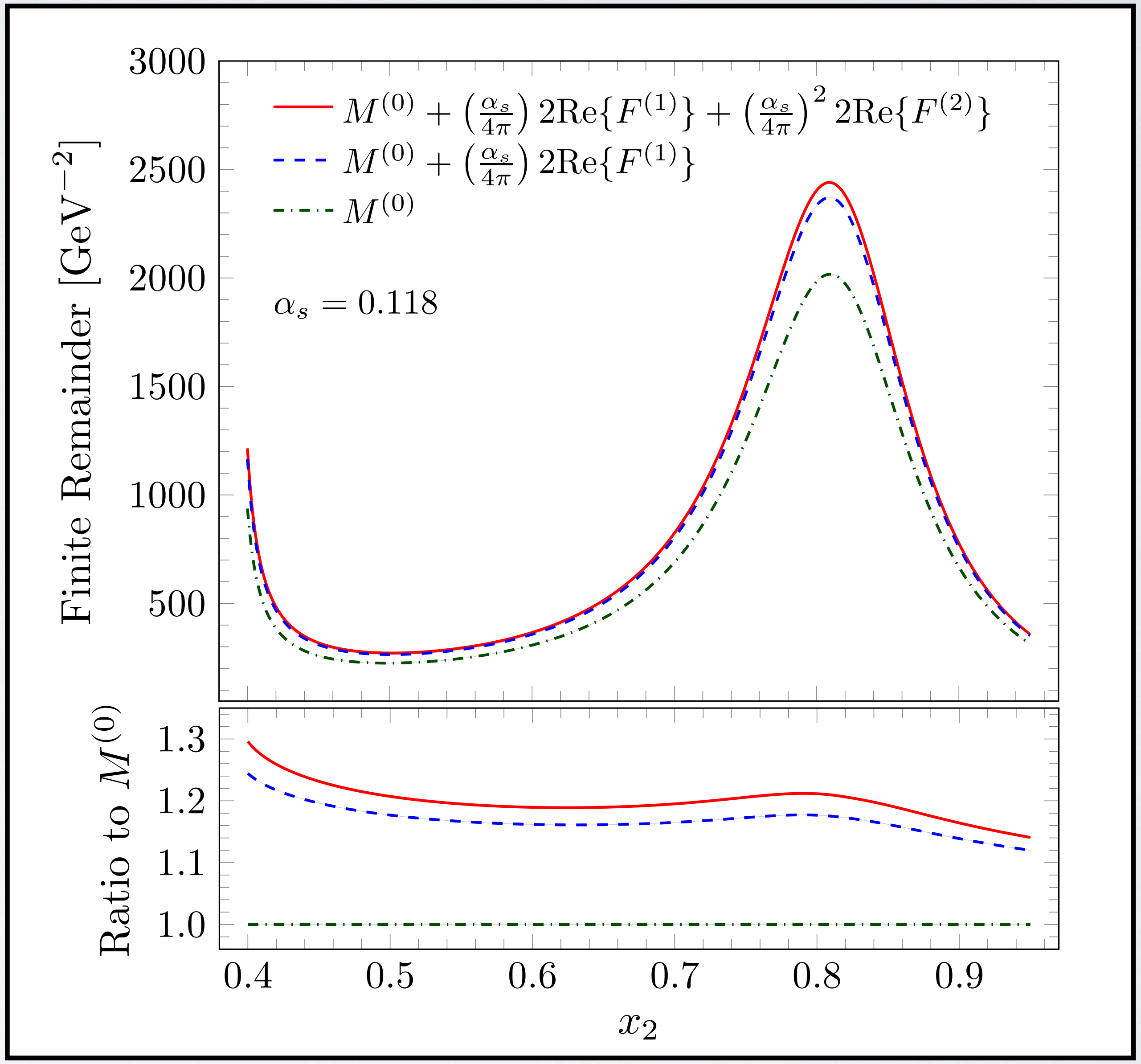
evaluate with DiffExp [Hidding (2020)]

$$p_3 = \frac{x_1\sqrt{s}}{2}\,(1,1,0,0)\;,$$
$$p_4 = \frac{x_2\sqrt{s}}{2}\,(1,\cos\theta,-\sin\phi\sin\theta,-\cos\phi\sin\theta)$$
$$p_5 = \sqrt{s}\,(1,0,0,0) - p_3 - p_4\,,$$

$$\cos\theta = 1 + \frac{2}{x_1 x_2}\left(1 - x_1 - x_2 - \frac{m_W^2}{s}\right)$$

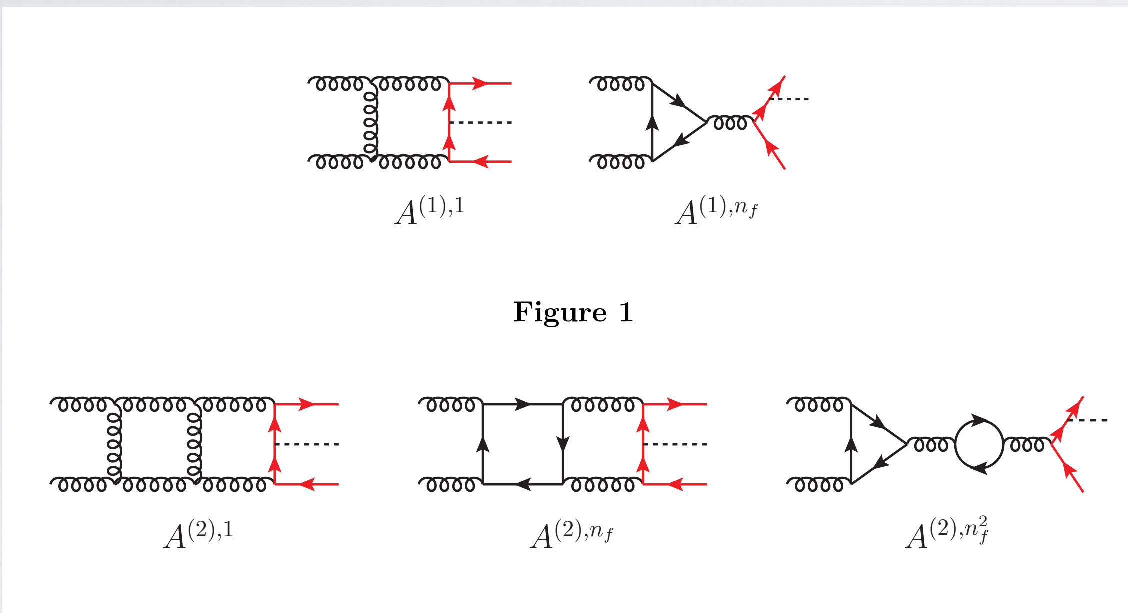
$$s = 1, m_W^2 = 0.1, \phi = 0.1, x_1 = 0.6$$

evaluation time ~ 260s per point



$pp \to H b\bar{b}$

SB, Hartanto, Kryś, Zoia [2107.14733]



$$A^{(0)}(1_{\bar{b}}^+, 2_b^+, 3_g^+, 4_g^+, 5_H) = \frac{s_5}{\langle 23\rangle\langle 34\rangle\langle 41\rangle},$$

$$A^{(0)}(1_{\bar{b}}^+, 2_b^+, 3_g^-, 4_g^-, 5_H) = -\frac{[12]^2}{[23][34][41]},$$

$$A^{(0)}(1_{\bar{b}}^+, 2_b^+, 3_g^+, 4_g^-, 5_H) = \frac{\langle 24\rangle\langle 4|5|1]^2}{s_{234}\langle 23\rangle\langle 34\rangle\langle 2|5|1]} - \frac{s_5[13]^3}{s_{134}[14][34]\langle 2|5|1]},$$

complete set of leading colour two-loop helicity amplitudes (incl n_f terms)

$$0 \to \bar{b}(p_1) + b(p_2) + g(p_3) + g(p_4) + H(p_5)\,,$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{q}(p_3) + q(p_4) + H(p_5)\,,$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{b}(p_3) + b(p_4) + H(p_5)\,,$$

$\bar{b}bggH$	helicity configurations	$r_i(x)$	independent $r_i(x)$	partial fraction in x_5	number of points
$F^{(2),1}$	$++++$	63/57	52/46	20/6	3361
	$+++-$	135/134	119/120	28/22	24901
	$++--$	105/111	105/111	22/12	4797
$F^{(2),n_f}$	$++++$	45/41	45/41	16/6	1381
	$+++-$	94/95	94/95	17/6	1853
	$++--$	89/95	62/69	18/3	2492
$F^{(2),n_f^2}$	$++++$	12/8	9/7	0/0	3
	$+++-$	11/16	11/16	3/0	22
	$++--$	12/20	8/16	8/0	242

$pp \to Hb\bar{b}$

finite remainder basis functions

$$f_i^{(k)} \to h_i^{(k)}$$

7 letters drop out in finite remainders

188 → 23 weight 4 functions

(same basis for $pp \to Wb\bar{b}$)

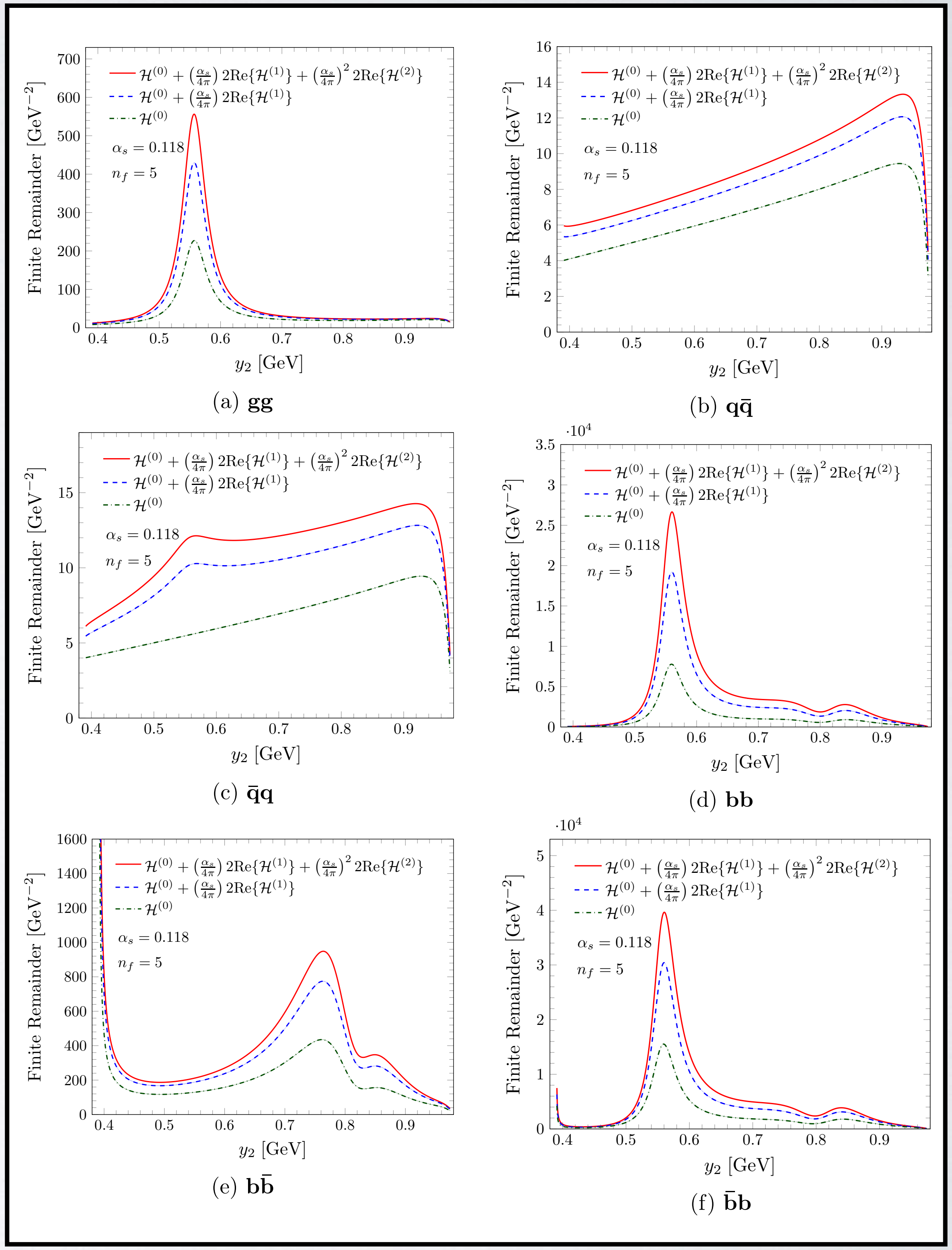
analytic subtraction of IR/UV poles

UV counter-terms required to renormalise Yukawa coupling

$pp \to Hb\bar{b}$

channel	Re $\mathcal{H}^{(2),1}$	Re $\mathcal{H}^{(2),n_f}$	Re $\mathcal{H}^{(2),n_f^2}$
gg	156680.6267	−41215.80337	405.9379563
q$\bar{\text{q}}$	0.09391314268	−0.02045942258	−0.004225713438
$\bar{\text{q}}$q	0.3494872243	−0.08069122736	−0.004225713438
b$\bar{\text{b}}$	48640.80398	−26530.01855	2458.442153
$\bar{\text{b}}$b	−141130.5373	42183.03094	3711.445449
bb/$\bar{\text{b}}\bar{\text{b}}$	−53679.25708	1988.662899	894.7895467

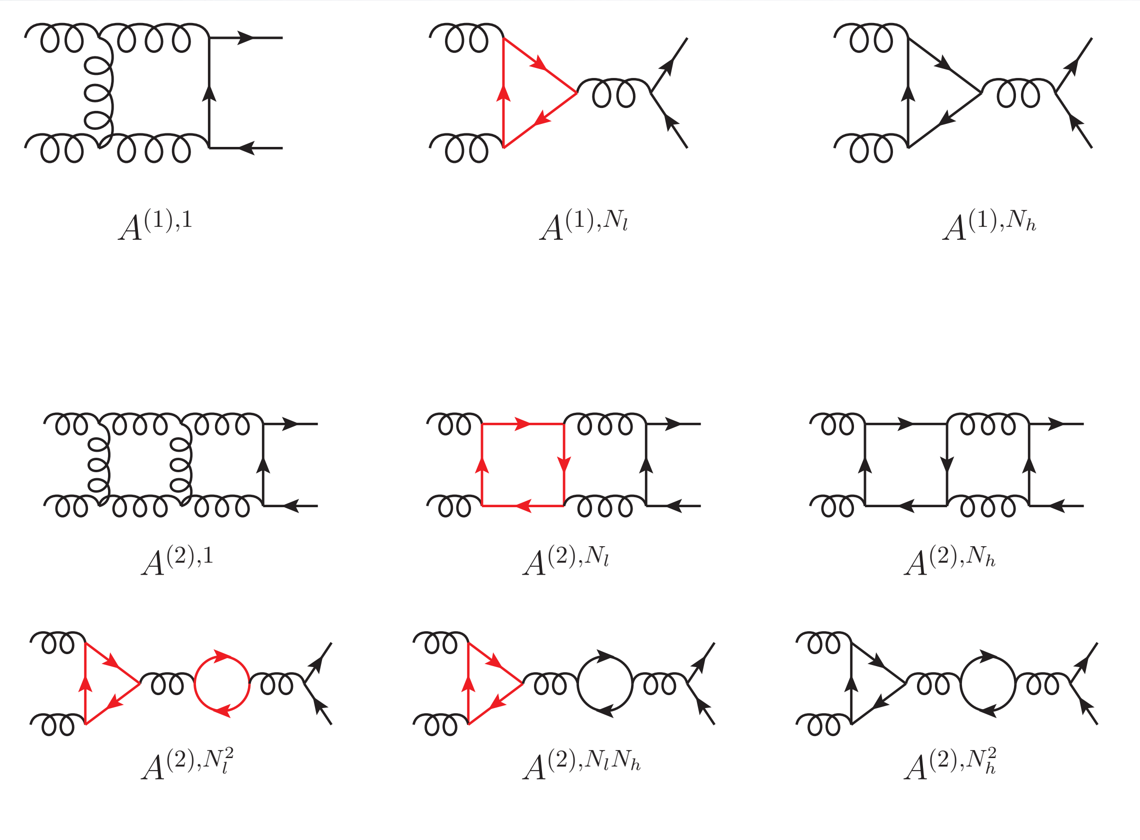
$$p_1 = \frac{y_1\sqrt{s}}{2}(1\,,1\,,0\,,0)\ ,$$
$$p_2 = \frac{y_2\sqrt{s}}{2}(1\,,\cos\theta\,,-\sin\theta\sin\phi\,,-\sin\theta\cos\phi)\ ,$$
$$p_3 = \frac{\sqrt{s}}{2}(-1\,,0\,,0\,,-1)\ ,$$
$$p_4 = \frac{\sqrt{s}}{2}(-1\,,0\,,0\,,1)\ ,$$



$gg \to t\bar{t}$

SB, Chaubey, Hartanto, Marzucca [2102.02516]

massive internal propagators have always challenged analytic methods



numerical solutions very successful even if computationally intensive
[Baernreuther, Czakon, Chen, Fiedler, Poncelet (2008-2018)]

analytic solutions for qq→tt and all non-elliptic sectors of gg→tt known
[Bonciani, Ferroglia, Gehrmann, Studerus, von Manteuffel, Di Vita, Laporta, Mastrolia, Primo, Schubert, Becchetti, Casconi, Lavacca (2009-2019)]

helicity amplitudes

include full top-quark decays efficiently

e.g. at one-loop [Melnikov, Schulze (2008)]

$$p^{\flat,\mu} = p^\mu - \frac{m^2}{2p.n} n^\mu$$

$$u_+(p,m) = \frac{(\not{p}+m)|n\rangle}{\langle p^\flat n\rangle}, \qquad u_-(p,m) = \frac{(\not{p}+m)|n]}{[p^\flat n]},$$
$$v_-(p,m) = \frac{(\not{p}-m)|n\rangle}{\langle p^\flat n\rangle}, \qquad v_+(p,m) = \frac{(\not{p}-m)|n]}{[p^\flat n]}.$$

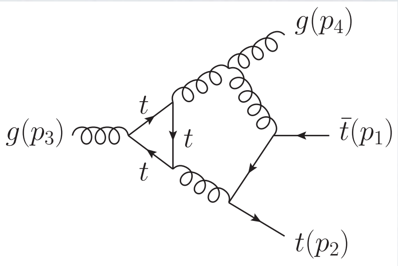
$$\begin{aligned} A^{(L)}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}; n_1, n_2) = m \frac{\Phi^{h_3 h_4}}{\langle 1^\flat n_1\rangle\langle 2^\flat n_2\rangle} \Bigg(& \langle n_1 n_2\rangle A^{(L),[1]}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}) \\ & + \frac{\langle n_1 3\rangle\langle n_2 4\rangle}{\langle 34\rangle} A^{(L),[2]}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}) \\ & + \frac{s_{34}\langle n_1 3\rangle\langle n_2 3\rangle}{\langle 3|14|3\rangle} A^{(L),[3]}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}) \\ & + \frac{s_{34}\langle n_1 4\rangle\langle n_2 4\rangle}{\langle 4|13|4\rangle} A^{(L),[4]}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}) \Bigg) \end{aligned}$$

obtain rational parametrisation in terms of only 2 variables (via Momentum twistors, m_t=1)

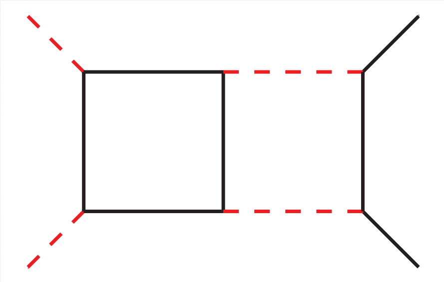
$$-\frac{s}{m_t^2} = \frac{(1-x)^2}{x}, \qquad \frac{t}{m_t^2} = y.$$

master integrals

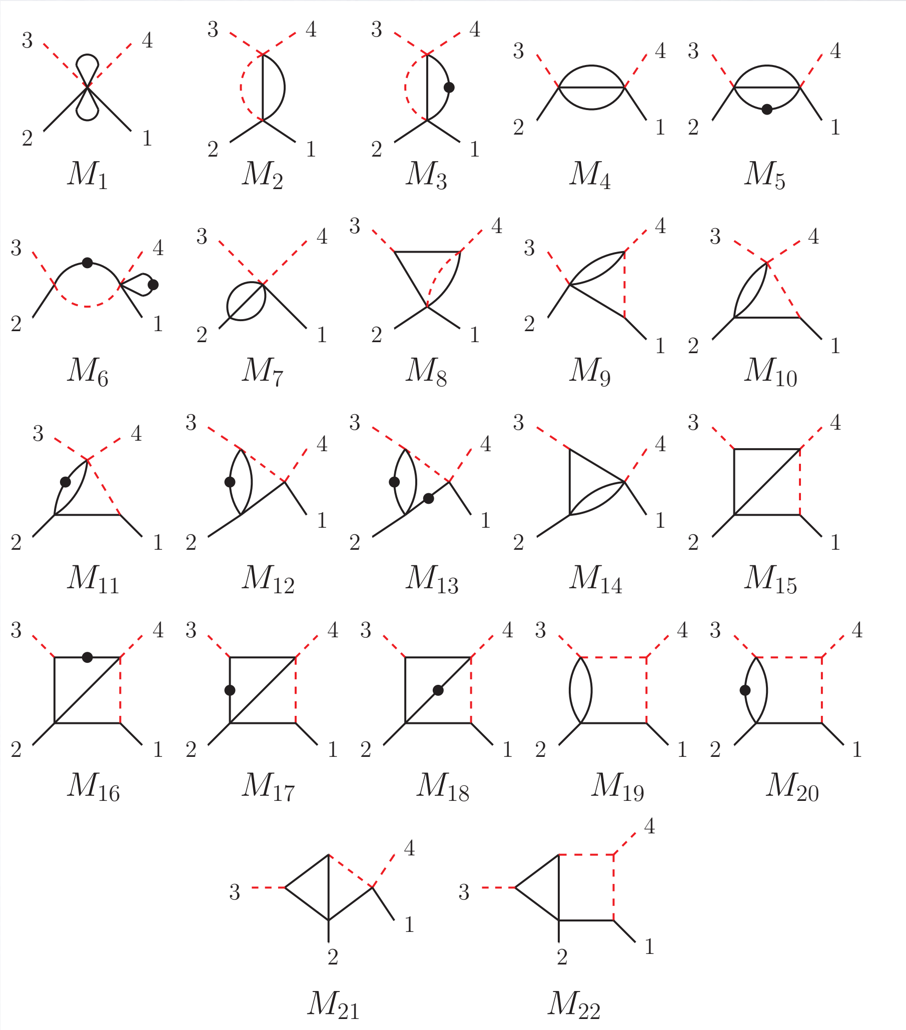
top-box integral recently computed analytically in terms of iterated integrals over **3 elliptic curves**



[Adams, Chaubey, Weinzierl (2018)]



one missing integral in the amplitude



derived canonical form DE obtained **iterated integral form** and boundary constants

analytic finite remainders

direct reconstruction of finite remainders

	monomials	monos. with rels.
amplitude	12025	11791
finite remainder	3586	3158

$$
\begin{aligned}
I(a_{3,3}^{(b)}, f, \dots) &= \int a_{3,3}^{(b)} I(f, \dots) \\
&= \int d\left(\psi_1^{(b)} \frac{x(-1+y)}{\pi(-1+x)^2}\right) \cdot I(f, \dots) \\
&= \left[\psi_1^{(b)} \frac{x(-1+y)}{\pi(-1+x)^2} I(f, \dots)\right]_{(0,1)}^{(x,y)} - I\left(\psi_1^{(b)} \frac{x(-1+y)}{\pi(-1+x)^2} f, \dots\right)
\end{aligned}
$$

additional function relations necessary to cancel IR poles - beyond shuffle relations

- test evaluations match with previous numerical results
- analytic continuation of iterated integrals needs further investigation

lots more to understand!

outlook

- techniques for **multi-scale two-loop amplitudes** reaching maturity
- **modular arithmetic** has played a key role in reducing complexity
- **fast and reliable analytic expressions** for many processes
- looking forward to phenomenological applications

a lot of progress since LH SMWG 2019! [2003.01700]

$pp \to 3\,\text{jets}$	$\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$	NNLO_{QCD}

Czakon et al. [2106.05331]

$pp \to \gamma\gamma + j$	NLO_{QCD} NLO_{EW}	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$
$pp \to \gamma\gamma\gamma$	NNLO_{QCD}	

Kallweit et al. [2010.04681]
Chawdhry et al. [1911.00479]

$pp \to V + 2j$	$\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ NLO_{EW}	NNLO_{QCD}
$pp \to V + b\bar{b}$	NLO_{QCD}	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$

$pp \to H + 2j$	$\text{NLO}_{\text{HTL}} \otimes \text{LO}_{\text{QCD}}$ $\text{N}^3\text{LO}^{(\text{VBF}^*)}_{\text{QCD}}$ (incl.) $\text{NNLO}^{(\text{VBF}^*)}_{\text{QCD}}$ $\text{NLO}^{(\text{VBF})}_{\text{EW}}$	$\text{NNLO}_{\text{HTL}} \otimes \text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ $\text{NNLO}^{(\text{VBF})}_{\text{QCD}} + \text{NLO}^{(\text{VBF})}_{\text{EW}}$

$pp \to t\bar{t} + j$	NLO_{QCD} (w/ decays) NLO_{EW}	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays)

first (leading colour) amplitudes

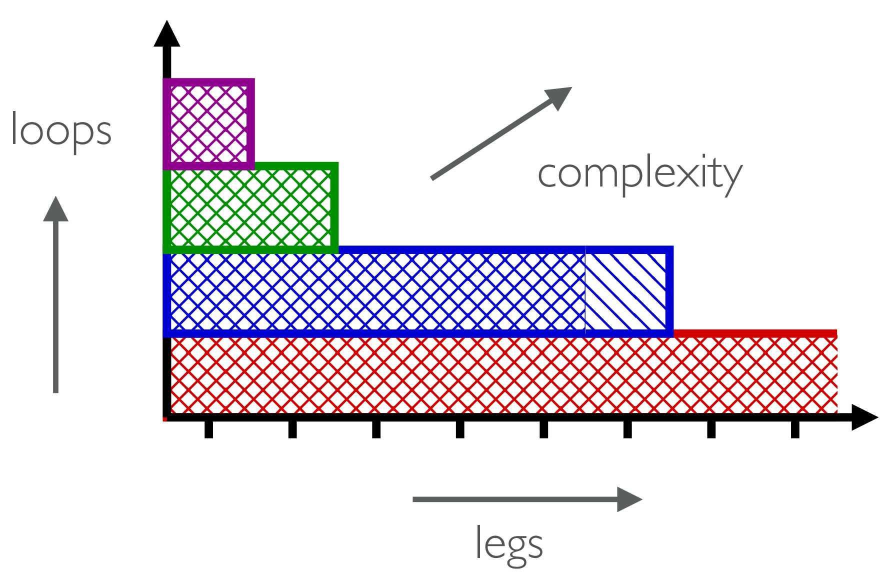
Chawdhry et al. [2105.05331]

Backup

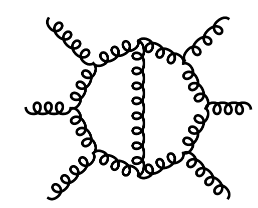
growing complexity

loops	1	2	3	4	5
diagrams	5	30	450	$50,000$	1.5×10^6
year	1973	1974	1980/1993	1997/2005	2016

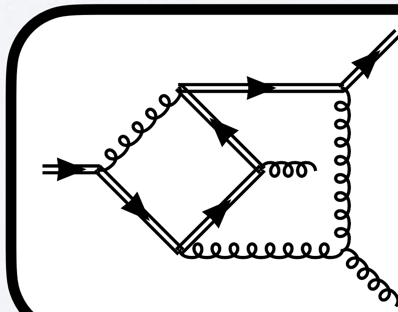
Diagrams contributiing to QCD β function up to 5 loops



more scales = more complicated



algebraic complexity
e.g. six-gluon scattering



analytic complexity
e.g. pp⟶tt

algebraic algorithms for multi-loop amplitudes

integration-by-parts identities

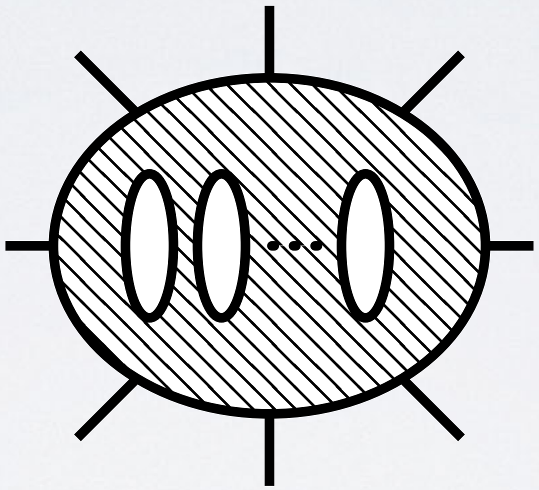
$$\int d^dk \frac{\partial}{\partial k^\mu} \frac{n^\mu(k)}{k^2(k-q_1)^2(k-q_2)^2\ldots} = 0$$

linear relations amongst Feynman integrals (in the same 'family' - i.e. different powers on propagators)

Unitarity compatible IBPs,
Maximal unitarity,
Numerical unitarity

on-shell trees*→MI

[Gluza, Kadja, Kosower (2010)][Kosower, Larsen (2011)] [Ita (2015)][Larsen, Zhang (2015)][Abreu, Dormans, Febres-Cordero, Ita, Jaquier, Page, Zeng, Sotnikov (2017-]



new efficient methods using computational algebraic geometry

Integrand reduction via polynomial division

on-shell trees*→integrand basis

[Mastrolia, Ossola (2011)][SB, Frellesvig, Zhang (2012)][Zhang (2012)] [Mastrolia, Mirabella, Ossola, Peraro (2012)]

$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

IBPs

$$A = \sum_i (\text{rational})_i (\text{integral})_i$$

*trees in higher dimensions amplitudes in dim. reg.

some remarks on master integrals

the choice of master integrals makes a huge difference to the subsequent evaluation (numerical or analytic)

‘UT’

universal transcendental weight

$1+\epsilon\log(x)+\epsilon^2\log^2(x)$ ✓

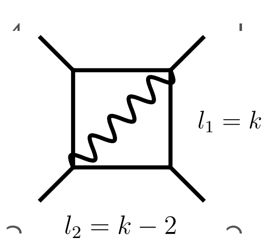
$1+\epsilon(1+\log(x))+\epsilon^2\log^2(x)$ ✗

‘dlog’

$$\int\frac{dx}{x+c}=\int d(\log(x+c))$$

- IBPs valid in d=4-2**ε***
- make expansion around d=4 easy
- expose divergence structure
- simple differential equations [Henn]
- often simpler amplitude representations

local integrands



$$=I_4^{4-2\epsilon}\left[\mathrm{tr}\left(1l_1l_23\right)\right]=I_4^{6-2\epsilon}[1]$$

quasi-finite
[Panzer, von Manteuffel, Schabinger]

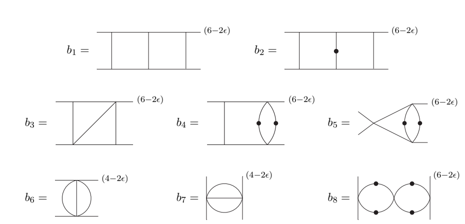


Figure 3. A minimal quasi-finite basis for the planar massless double box integral family.

* IBPs in d=4 are possible but require careful treatment