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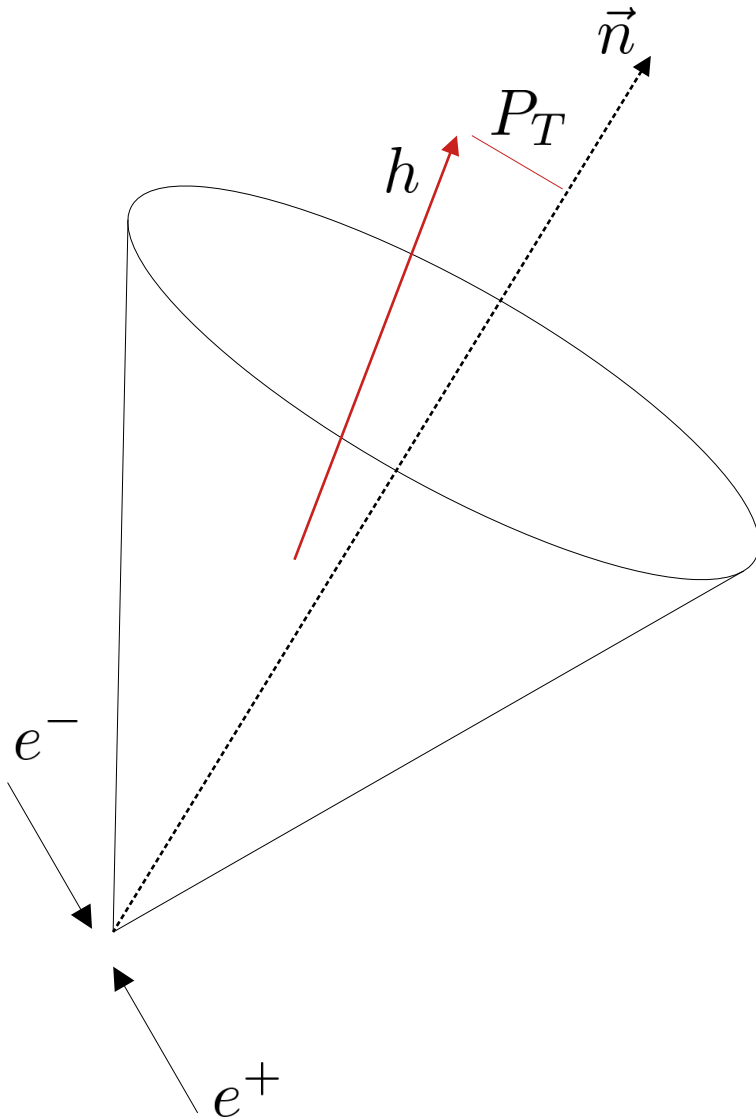
Istituto Nazionale di Fisica Nucleare  
SEZIONE DI TORINO

# Thrust Resummation and Rapidity Divergences

DIS2022: XXIX International Workshop on Deep Inelastic  
Scattering and Related Subjects

# The process $e^+ e^- \rightarrow h X$ (thrust)

The cross section is differential in:

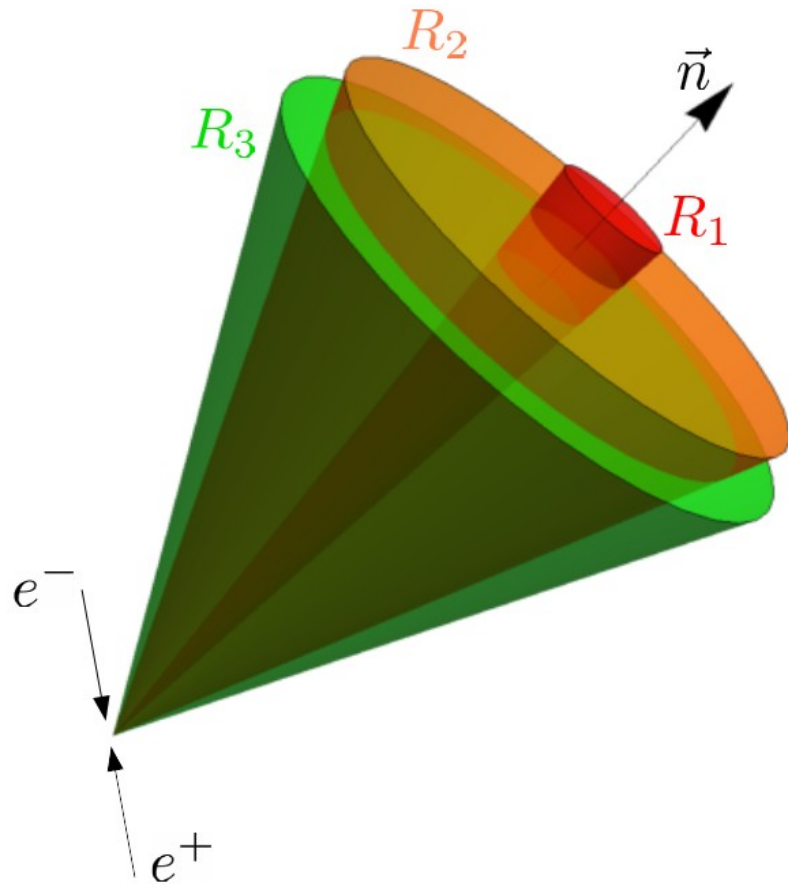
$$z_h = \frac{E}{Q/2}, \quad T = \frac{\sum_i |\vec{P}_{(\text{c.m.}),i} \cdot \hat{n}|}{\sum_i |\vec{P}_{(\text{c.m.}),i}|}, \quad P_T \text{ w.r.t } \vec{n}$$


- **Recent data** (BELLE collab. 2019) that must be interpreted in the framework of factorization theorems.
- **Non-standard process**, as it is not covered by standard TMD factorization.
- The effects associated with the thrust intertwine those due to transverse momentum dependence. The final result should possess **features typical of both event-shape observables and TMD cross sections**.
- Naively, it is the **cleanest way to access a TMD** (FF). However, relating it with those encountered in standard TMD factorization is subtle and non-trivial.

# Kinematic regions of $e^+ e^- \rightarrow h X$ (thrust)

Depending on where the hadron is located within the jet the underlying kinematics can be remarkably different, resulting in different factorization theorems

## Three Regions:



The hadron is detected very close to the **axis** of the jet:

- Extremely small  $P_T$
- Soft radiation affects significantly the transverse deflection of the hadron from the thrust axis

**TMD FF + non-pert. SOFT contribution**

The hadron is detected in the **central region** of the jet:

- Most common scenario
- Majority of experimental data fall into this case

**TMD FF**

The hadron is detected near the **boundary** of the jet:

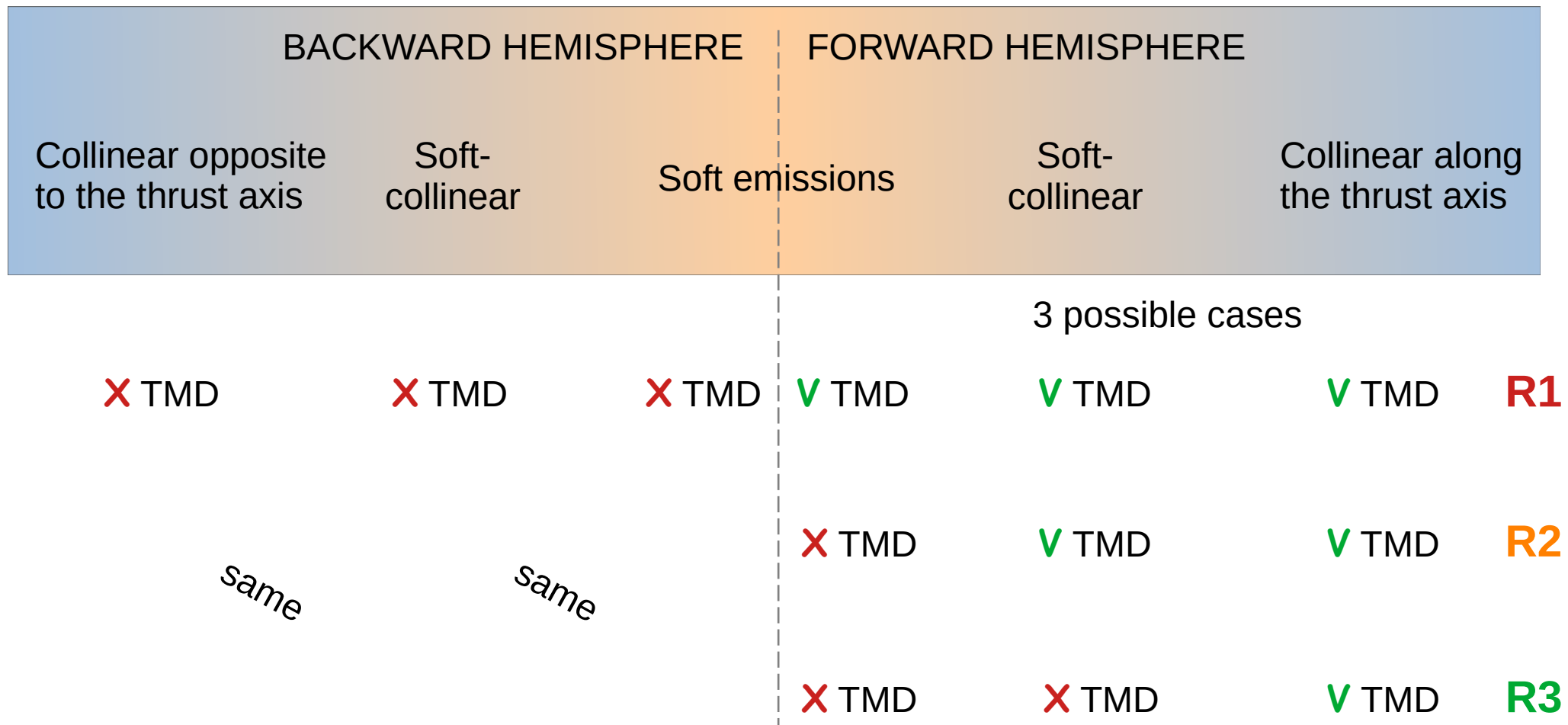
- Moderately small  $P_T$
- The hadron  $P_T$  causes the spread of the jet affecting the topology of the final state (i.e. the value of thrust)

**Generalized FJF**



Each region is determined *uniquely* by which kind of radiation contributes to the TMD effects.

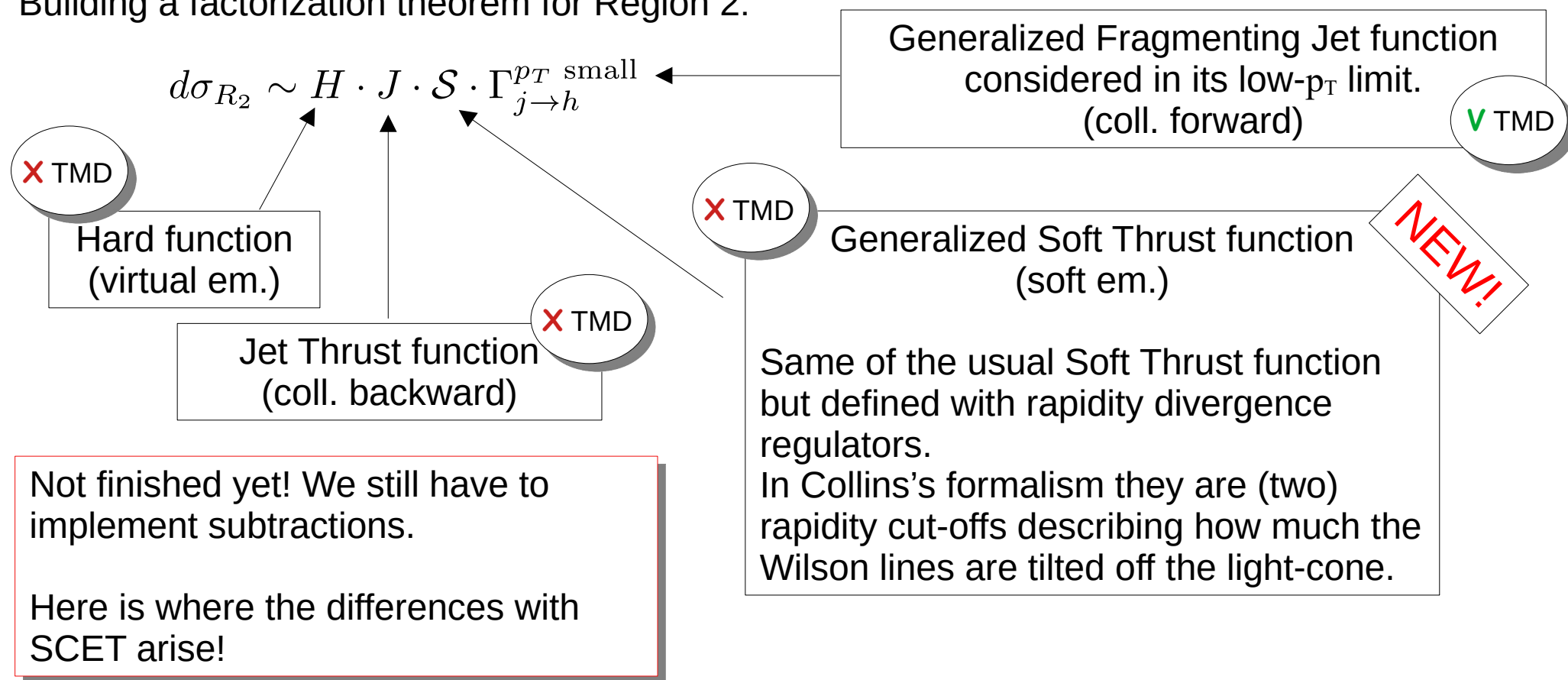
- Virtual emissions, clearly, cannot produce any TMD effect
- Real emissions can be emitted in each of the two hemispheres associated with the thrust axis. Their direction crucially determines if they are able to produce TMD effects:



In this talk → **Region 2**

- Most interesting, as the majority of data is expected to fall into this case.
- Most debated, no agreement with SCET result (cf. *Makris et al.* [hep-ph:2009.11871])
- Most difficult, unexpected issues rising from factorization (absence of non-perturbative TMD soft contributions, unusual treatment of rapidity divergences).

Building a factorization theorem for Region 2:



Our interpretation:

$$d\sigma_{R_2} \sim \frac{H \cdot J \cdot \mathcal{S} \cdot \Gamma_{j \rightarrow h}^{p_T \text{ small}}}{\mathcal{Y}_L \cdot \Upsilon_R^{p_T \text{ small}}}$$

CRUCIAL!!

✗ TMD

Soft-Collinear Thrust function  
(soft-coll. backward em.)

No correspondence with any usual object in thrust observables.  
It is defined as the Generalized Soft Thrust function but the Wilson lines associated with forward propagation are on the light-cone.

NEW!

Soft-Collinear Thrust factor  
considered in its low- $p_T$  limit.  
(soft-coll. forward em.)

It coincides with the “thrust-TMD collinear-soft function” of *Makris et al.* [hep-ph:2009.11871]

✓ TMD

The (unpolarized) TMD FF  
appears naturally as:

$$\tilde{D}_{j \rightarrow h}(z, b_T; y_1) = \frac{\Gamma_{j \rightarrow h}^{p_T \text{ small}}(z, b_T, 1 - T)}{\Upsilon_R^{p_T \text{ small}}(b_T, 1 - T; y_1)}$$

Finally:

$$d\sigma_{R_2} \sim H \cdot J \cdot \frac{\mathcal{S}}{\mathcal{Y}_L} \cdot \tilde{D}_{j \rightarrow h}$$

SCET interpretation:

$$d\sigma_{R_2} \sim \frac{H \cdot J \cdot \mathcal{S} \cdot \Gamma_{j \rightarrow h}^{p_T \text{ small}}}{\mathcal{Y}_L \cdot \mathcal{Y}_R}$$

CRUCIAL!!

✗ TMD

Soft-Collinear Thrust function  
(soft-coll. backward em.)

✗ TMD

Soft-Collinear Thrust function  
(soft-coll. forward em.)

The usual Soft Thrust  
function appears from:

$$S(1 - T) = \frac{\mathcal{S}(1 - T; y_1, y_2)}{\mathcal{Y}_L(1 - T; y_2) \cdot \mathcal{Y}_R(1 - T; y_1)}$$

Finally:

Multiply&Divide by  $\Upsilon_R^{p_T \text{ small}}$

$$d\sigma_{R_2} \sim H \cdot J \cdot \mathcal{S} \cdot \Gamma_{j \rightarrow h}^{p_T \text{ small}} = \underbrace{H \cdot J \cdot \mathcal{S} \cdot \Upsilon_R^{p_T \text{ small}}}_{\text{cf. Eq. (2.21) in Makris et al. [hep-ph:2009.11871]}} \cdot \tilde{D}_{j \rightarrow h}$$

cf. Eq. (2.21) in Makris et al. [hep-ph:2009.11871]

Our result differs from that obtained in the framework of SCET...  
...but now at least we know *where*!

Our interpretation (CSS):

$$d\sigma_{R_2} \sim H \cdot J \cdot \frac{S}{\mathcal{Y}_L} \cdot \tilde{D}_{j \rightarrow h}$$

- A TMD FF appears *naturally* as an ingredient of the factorization theorem. However, this is NOT the same TMD FF appearing in usual TMD factorization (SIDIS,  $e^+e^- \rightarrow h_1 h_2$ ).

The difference is in the non-perturbative regime (large  $b_T$ ):

$$\tilde{D}^{\text{usual}} = \tilde{D} \times \sqrt{M_S} \leftarrow$$

Soft  
Model

NEW!

- Rapidity divergences are regulated in two different *and overlapping* ways:
  - The Thrust (T)
  - The rapidity cut-off ( $y_1$ )

This redundancy results in a (fictitious) left-out dependence of the rapidity cut-off.

SCET interpretation:

$$d\sigma_{R_2} \sim H \cdot J \cdot S \cdot \Gamma_{j \rightarrow h}^{p_T \text{ small}}$$

- The factorization theorem does not include a TMD FF, but instead a Generalized FJF at low- $p_T$ . This is a consequence of having considered the same decomposition of radiation of Region 3 and, in fact, the two factorization theorems are almost identical.
- Rapidity divergences are regulated solely by the thrust, and hence there are no issues regarding the rapidity cut-off.



# Treatment of the rapidity cut-off

The left-out dependence on the rapidity cut-off is not interpreted as the signal of the breaking of factorization, but instead as the signal that **the two regulators, the thrust and the rapidity cut-off, must be related through some certain specific condition.**

$$T \longleftrightarrow y_1$$

Such condition is exposed by imposing the CS-invariance to the factorized cross-section:

$$\frac{d\sigma_{R_2}}{dy_1} = 0 \longrightarrow$$

$$\hat{G}_R(u, y_1) + \tilde{K}(b_T) = 0$$

← RG-invariant combination!

(right) **G-kernel.**

It rules the evolution of the Generalized Soft Thrust function together with its left counterpart:

$$\hat{G}_R(u, y_1) = -2 \lim_{y_2 \rightarrow -\infty} \frac{\partial \log \hat{\mathcal{S}}(u; y_1, y_2)}{\partial y_1}$$

$$\hat{G}_L(u, y_2) = 2 \lim_{y_1 \rightarrow +\infty} \frac{\partial \log \hat{\mathcal{S}}(u; y_1, y_2)}{\partial y_2}$$

**NEW!**

**Collins-Soper kernel.**

It rules the CS-evolution of TMDs and the evolution of the 2-h Soft Factor:

$$\tilde{K}(b_T) = \pm 2 \lim_{y_{2,1} \rightarrow \mp \infty} \frac{\partial \log \hat{\mathcal{S}}_{2-h}(b_T; y_1, y_2)}{\partial y_{1,2}}$$

$$\frac{\partial \log \tilde{D}(z, b_T; y_1)}{\partial y_1} = -\frac{1}{2} \tilde{K}(b_T)$$

Solving the condition gives  $y_1$  as a function of thrust and transverse momentum (implicitly encoded into  $u$  and  $b_T$ , respectively):

$$\bar{y}_1 = -L_u \frac{1 - 2\lambda_u}{2\lambda_u} \left( 1 - e^{\frac{2\beta_0}{\gamma_K^{[1]}} \tilde{K}|_{\mu_S}} \right) + \text{power suppr.} \quad \text{with} \quad \begin{aligned} L_u &= \log u e^{\gamma_E}, \\ \lambda_u &= \frac{\alpha_S(Q)}{4\pi} \beta_0 L_u, \\ \mu_S &= \frac{Q}{u} e^{-\gamma_E}. \end{aligned}$$

The final factorization theorem must be considered fixing the rapidity cut-off to the function  $\bar{y}_1$

$$\begin{aligned} \frac{d\sigma_{R_2}}{dz_h d^2\vec{P}_T dT} &= \sigma_B N_C \int \frac{d^2\vec{b}_T}{(2\pi)^2} e^{i \frac{\vec{P}_T}{z_h} \cdot \vec{b}_T} \int \frac{du}{2\pi i} e^{u(1-T)} \\ &\quad \times H \hat{J}(u) \frac{\hat{\mathcal{S}}(u, \bar{y}_1, y_2)}{\hat{\mathcal{Y}}_L(u, y_2)} \sum_j e_j^2 \tilde{D}_{h/j}(z_h, b_T, \bar{y}_1) \end{aligned}$$

Notice that the factorization theorem can only be trusted where  $\bar{y}_1$  is large and positive, i.e. at large  $u$  (two-jet limit) and small-moderate  $b_T$  (according to Region decomposition).

By using Evolution Equations the final result can be recasted as:

$$\begin{aligned}
& H(a_S(\mu), \log \mu/Q) \hat{J}(a_S(\mu), L_J) \frac{\hat{S}(a_S(\mu), L_S, \bar{y}_1, y_2; u)}{\hat{Y}_L(a_S(\mu), L_S + y_2)} \tilde{D}_{h/j}(z_h, a_S(\mu), L_b, \bar{y}_1; b_T) = \\
& = H(a_S(Q), 0) \hat{J}(a_S(\mu_J), 0) \frac{\hat{S}(a_S(\mu_S), 0, 0; u)}{\hat{Y}_L(a_S(\mu_S), 0)} \quad \left] \begin{array}{l} u\text{-dep. at} \\ \text{ref. scale} \end{array} \right. \\
& \times \exp \left\{ \int_{\mu_J}^Q \frac{d\mu'}{\mu'} \gamma_J(a_S(\mu'), L_J) + \frac{1}{2} \int_{\mu_S}^Q \frac{d\mu'}{\mu'} \gamma_S(a_S(\mu'), L_S) \right\} \quad \left] \begin{array}{l} u\text{-dep.} \\ \text{evolution} \end{array} \right. \\
& \times \tilde{D}_{h/j}(z_h, a_S(Q), L_b, ; b_T) \quad \left] \begin{array}{l} \text{TMD FF at} \\ \mu = Q; y_1 = 0 \end{array} \right. \\
& \times \exp \left\{ -\frac{1}{2} \left( \int_{\mu_S}^{\bar{\mu}_R} \frac{d\mu'}{\mu'} \left[ \hat{g}(a_S(\mu')) + \gamma_K(a_S(\mu')) \log \frac{\mu'}{\bar{\mu}_R} \right] + \right. \right. \\
& \quad \left. \left. + \log \frac{\bar{\mu}_R}{\mu_S} \tilde{K} \left( a_S(\mu_S), \log \left( \frac{b_T Q}{2u} \right); b_T \right) \right) \right\} \quad \left] \begin{array}{c} u\text{-}b_T \\ \text{correlation} \end{array} \right.
\end{aligned}$$

This result can be now computed at the desired log-accuracy, both in thrust ( $u$ ) and in transverse momentum ( $b_T$ ).

The rapidity cut-off induces a correlation between thrust and transverse momentum

# NLL-accuracy in Thrust

$$d\tilde{\sigma}_{R_2} \sim -\tilde{D}_{h/j}(z_h, a_S, L_b^*; b_T) \frac{1}{1-T} \times \\ e^{-\log(1-T) g_1(\lambda, \lambda_b^*, g_K) + g_2(\lambda, \lambda_b^*, g_K)} (g_1(\lambda, \lambda_b^*, g_K) + \lambda g_1'(\lambda, \lambda_b^*, g_K))$$

Where  $\lambda = -a_S \beta_0 \log(1-T)$ ,  $\lambda_b^* = 2 a_S \beta_0 \log(b_T^* Q/c_1)$  and  $g_k$  encodes the non-perturbative content of the CS-kernel.

Notice how  $g_k$  enters into the game in a very different way with respect to the usual TMD factorization.

For this reason, this process can be considered as a **new tool** for the investigation of the large-distance behavior of the CS-kernel (trending topic!)

The TMD FF is expressed as usual (here at NLL in  $b_T$ ):

$$\tilde{D}_{j \rightarrow h}(z_h, a_S, L_b^*; b_T) = \sum_l \left( \tilde{C}_{j \rightarrow l} \otimes d_{l \rightarrow h}(z_h) \right)^{\text{NLO}} \\ \times e^{L_b^* g_1^{\text{TMD}}(\lambda_b^*) + g_2^{\text{TMD}}(\lambda_b^*)} M_D(z_h, b_T, j \rightarrow h) e^{-\frac{1}{2} g_K(b_T) \log \frac{Q}{M_h}}$$

# Phenomenology

Many sources of theoretical errors!

- Factorization approximations,  $P_T \ll P^+$ ;  $M_h \ll Q$
- Crossing over into Region 1
- Crossing over into Region 3
- Contributions from higher order topologies
- Matching with Fixed Order (unapproximated, full QCD)
- Solution for the rapidity cut-off large and positive  $e^{-\bar{y}_1} \ll 1$
- Collinear FFs uncertainties (LHAPDF)
- Arbitrariness in modelling the non-perturbative TMD functions
- Arbitrariness in modelling the non-perturbative function(s) associated with thrust

Errors affected by:  
- log-accuracy (thrust)  
-  $g_k$  function

The size can be determined only “a posteriori” (after fit)



# Phenomenology: preliminary FIT of BELLE data @ NLL

BELLE collab. Phys.Rev.D 99 (2019) 11, 112006

Many sources of theoretical errors!

- Factorization approximations,  $P_T \ll P^+$ ;  $M_h \ll Q$   $\longrightarrow$   $P_T \lesssim 0.2 P^+$ , pions.
  - Crossing over into Region 1
  - Crossing over into Region 3
- }  $\longrightarrow$  Data selection algorithm  
*Boglione, Simonelli*  
JHEP 02 (2022) 013
- Contributions from higher order topologies  $\longrightarrow$   $T \gtrsim 0.9$
  - Matching with Fixed Order (unapproximated, full QCD)
  - Solution for the rapidity cut-off large and positive  $e^{-\bar{y}_1} \ll 1 \longrightarrow$  Checked “a posteriori”
  - Collinear FFs uncertainties (LHAPDF)
  - Arbitrariness in modelling the non-perturbative TMD functions
  - Arbitrariness in modelling the non-perturbative function(s) associated with thrust

We are also *assuming* that the non-perturbative effects associated with thrust are **negligible** with respect to the TMD non-perturbative effects.

# Phenomenology: preliminary FIT of BELLE data @ NLL

With this simplifications, all the non-perturbative information is encoded into **two functions**:

- The TMD FF model (POWER LAW):

$$M_D(z_h, b_T) = \frac{2^{2-p} (M b_T)^{p-1}}{\Gamma(p-1)} K_{p-1}(M b_T) \stackrel{\text{F.T.}}{\Leftarrow} \frac{\Gamma(p)}{\pi \Gamma(p-1)} M^{2(p-1)} \left( M^2 + \frac{P_T^2}{z_h^2} \right)^{-p}$$

It depends on a single free parameter  $z_0$ , through:

$$\left\{ \begin{array}{l} p = \frac{1}{2} \left( \frac{3}{1-R} - 1 \right) \\ M = \frac{W}{z_h} \sqrt{\frac{3}{1-R}} \end{array} \right. \quad \text{With:} \quad \left\{ \begin{array}{l} R = 1 - (1-z)^{z_0/1-z_0}, \\ W = \frac{M_h}{R^2} \end{array} \right.$$

In total **3** free parameters:  
 $z_0, g_R, p_1$

- The  $g_K$ -function, depending on two free parameters  $g_r$  and  $p_1$ :

$$g_K(b_T) = a(b_T) \left( \frac{b_T}{b_{\text{MAX}}} \right)^{c(b_T)} \quad \text{With:} \quad \begin{aligned} a(b_T) &= g_r + \tanh p_1 - \tanh \left[ p_1 \left( \frac{b_T^2}{b_{\text{MAX}}^2} - 1 \right) \right] \\ c(b_T) &= 2 \left( 1 - \tanh \left( \frac{b_T^2}{b_{\text{MAX}}^2} \right) \right) \end{aligned}$$

# Phenomenology: preliminary FIT of BELLE data @ NLL

$$\chi^2 = 0.971$$

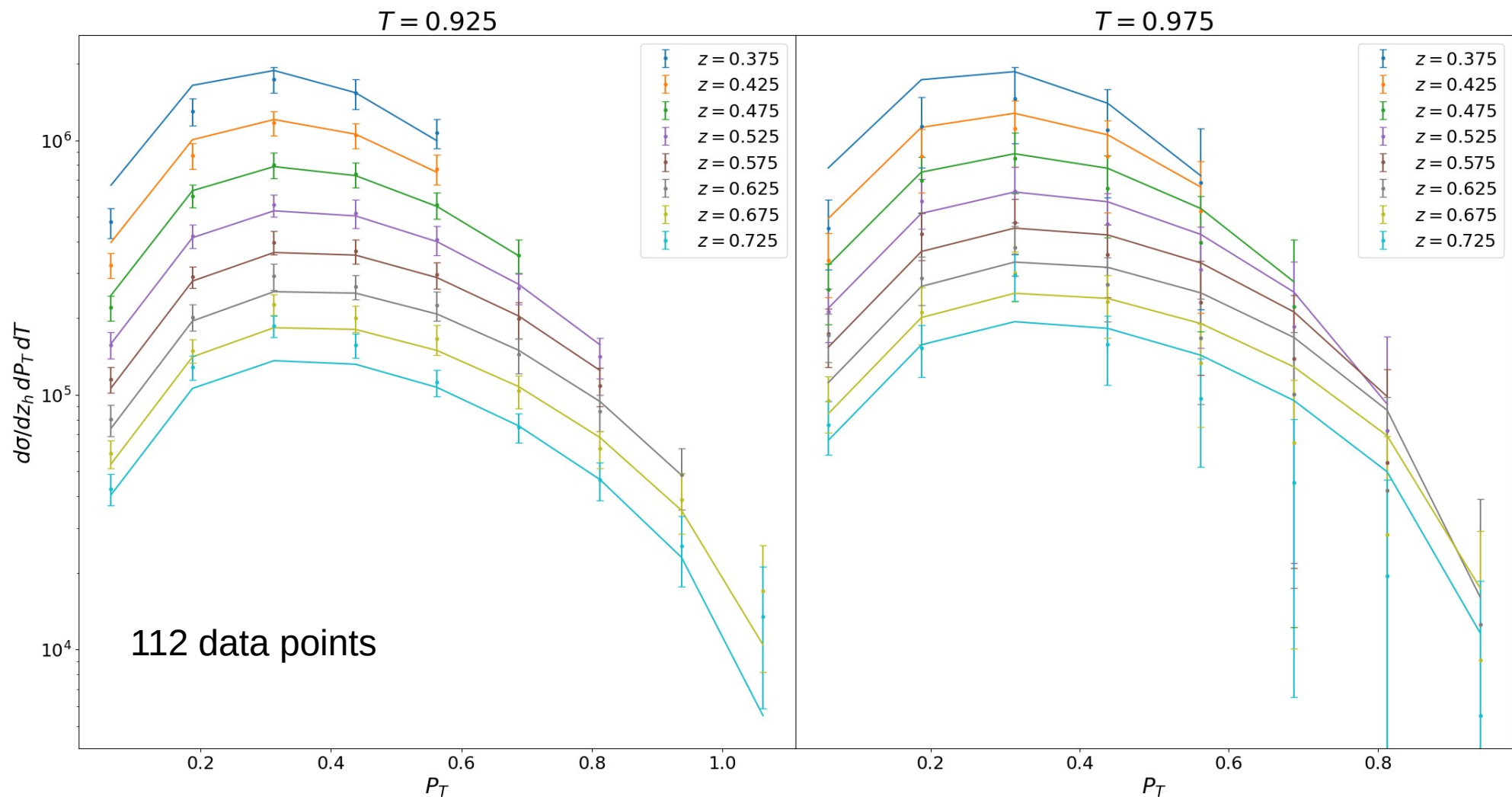
$$z_0 = 0.631 \pm 0.005$$

$$p_1 = 3.614 \pm 0.773$$

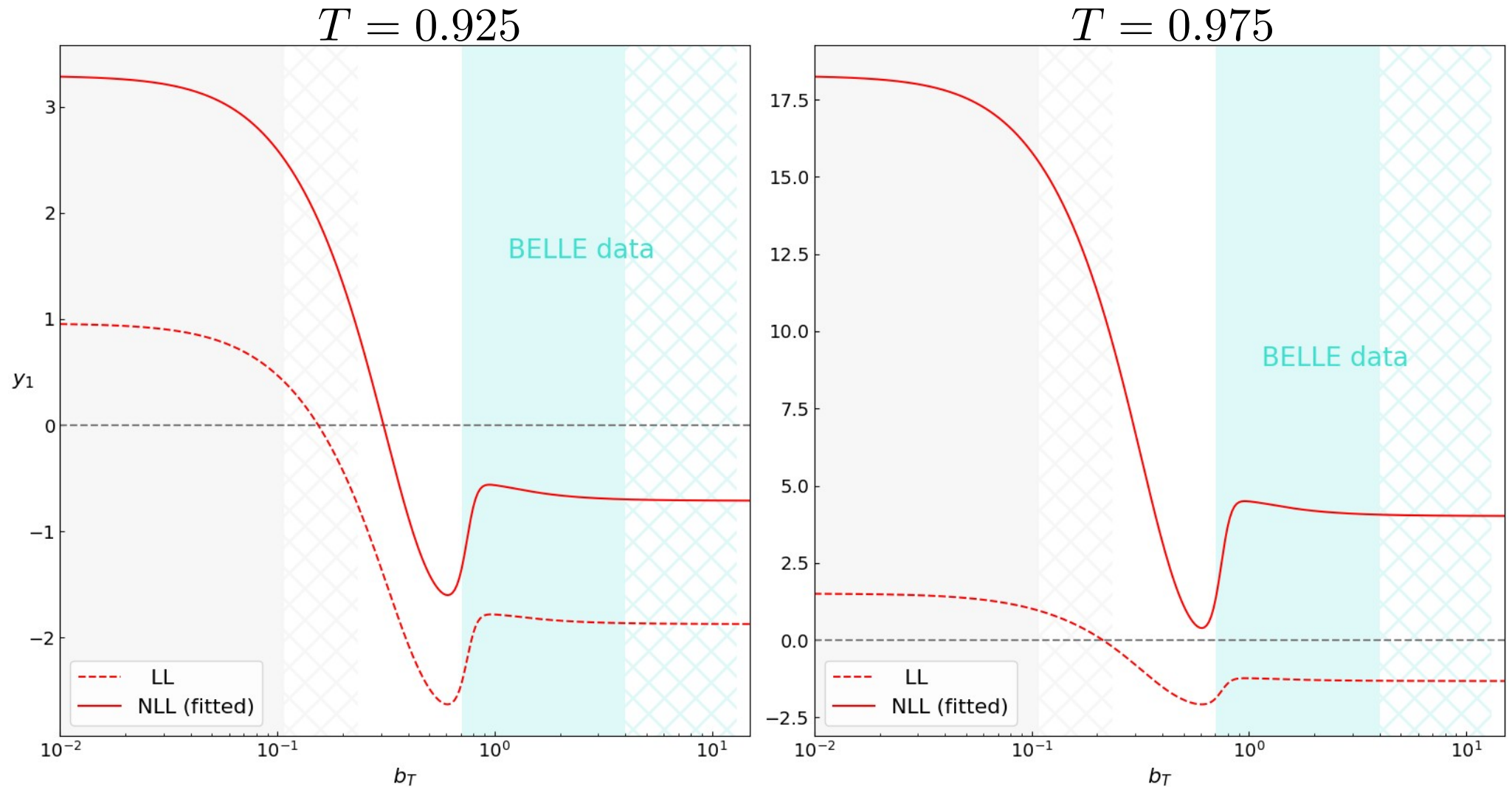
$$g_r = 0.704 \pm 0.010$$

In this analysis:

$$b_{\text{MAX}} = 0.7 \text{ GeV}^{-1}$$



# Is the rapidity cut-off large and positive?



- LL is not enough to have the errors associated with the largeness of  $y_1$  under control.
- NLL seems enough for  $T = 0.975$ , but not satisfactory for  $T = 0.925$ . Here  $y_1$  becomes negative around  $b_T \sim 0.3 \text{ GeV}^{-1}$ .

# Conclusions and Future perspectives

Summarizing...

- We have set the theoretical framework to treat the process  $e^+e^- \rightarrow h X$ , where the transverse momentum of the detected hadron is measured w.r.t. the thrust axis.
- The result for Region 2 obtained in our formalism (CSS) is different from that obtained in SCET. In particular, the differences are in the treatment of the soft-collinear radiation in the hemisphere where the hadron is detected.
- In Region 2, the rapidity regulator used in TMD factorization correlates the thrust and the transverse momentum. This is a completely **new feature** for a factorization theorem.
- Moreover, the TMD FF of Region 2 differs from that appearing (e.g.) in SIDIS at large distances. This is due to the absence of TMD non-perturbative contributions associated with soft radiation.
- The last two thrust bins of BELLE data, associated with Region 2, can be successfully fitted at **NLL-accuracy** in thrust, by using 3 free parameters for the (unpolarized, pion) TMD FF. This is the first time that the thrust behavior of such data is described within a consistent theoretical formalism.



# Conclusions and Future perspectives

In the future...

- Refining of the fit, taking into account all the error sources.
- Going to NNLL in thrust seems a necessary step to reduce the errors associated to the size of the rapidity cut-off.
- Comparison of the result obtained from Region 2 with the extraction of the TMD FF obtained from the standard TMD processes or from Region 1. This would allow to access the **soft model**, shedding light on its characteristic large-distance behavior.
- Extension of this theoretical framework to other non-standard TMD observables.

THANK YOU FOR YOUR ATTENTION!