Proton mass decomposition in QCD*

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(Dated: July 15, 2022)

Understanding the proton mass in quantum chromodynamics (QCD) is a very important and timely topic in hadronic physics. Different decompositions (sum rules) of the proton mass, which are related to the QCD energy-momentum tensor, have been proposed in the literature. In this presentation write-up, we briefly review and compare these sum rules.

I. INTRODUCTION

Key questions in hadronic physics are concerned with the decomposition of global properties of the proton, such as its mass and spin, in terms of individual contributions from quarks and gluons. The focus of this write-up is on the understanding of the proton mass in QCD. Several sum rules of the proton mass have been put forward in the literature [1–7]. Those sum rules are typically related to the QCD energy-momentum tensor (EMT) $T^{\mu\nu}$ — either its energy (density) component T^{00} or its trace $T^{\mu}{}_{\mu}$, including the anomalous contribution [8, 9].

Here we concentrate on the following four mass sum rules: (i) a decomposition of T^{00} by Ji into four terms [1, 2]; (ii) a two-term decomposition of T^{00} by Lorcé [3]; (iii) a two-term decomposition of $T^{\mu}{}_{\mu}$ by Hatta, Rajan, Tanaka [4, 5]; (iv) a three-term decomposition of T^{00} [6, 7]. We briefly review these sum rules and point out the most important similarities and differences.

The community largely agrees that, from a physics point of view, at most a few meaningful mass sum rules can be identified. On the other hand, different opinions/results still exist in relation to two crucial questions:

1. What is the correct form of the (renormalized) operators associated with the individual terms of the mass sum rules?

2. What is the physical meaning and significance of (some of) the terms of the various decompositions?

The present contribution addresses those questions to some extent, while we refer the reader to [6, 7, 10] for more details.

II. QCD ENERGY-MOMENTUM TENSOR AND ITS RENORMALIZATION

Let us first recall the symmetric (and gauge invariant) EMT in QCD, which can be decomposed into a quark and gluon contribution according to

$$T^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu}\,, \text{ with} \tag{1}$$

$$T_q^{\mu\nu} = \frac{i}{4}\,\overline{\psi}\,\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}\,\psi\,, \quad T_g^{\mu\nu} = -F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{g^{\mu\nu}}{4}F^{\alpha\beta}F_{\alpha\beta}\,, \tag{2}$$

where in $T_q^{\mu\nu}$ a summation over quark flavors is understood, the notation $a^{\{\mu}b^{\nu\}}$ stands for $a^\mu b^\nu + a^\nu b^\mu$, and $\overleftrightarrow{D}^\mu = \overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu - 2igA_a^\mu\,T_a$ is the covariant derivative. For both $T_q^{\mu\nu}$ and $T_g^{\mu\nu}$, renormalization of the parameters of the QCD Lagrangian is understood. The total EMT is conserved and therefore not renormalized (beyond renormalization at the level of the Lagrange density), but the individual quark and gluon parts of the EMT must be renormalized.

We repeat that the component T^{00} describes the energy density of a system, while the spatial terms T^{jj} of the main diagonal of the EMT are typically considered as pressure contributions [11, 12]. Note that the notion of pressure

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* Presented at DIS2022: XXIX International Workshop on Deep-Inelastic Scattering and Related Subjects, Santiago de Compostela, Spain, May 2-6 2022

is not generally accepted for a hadronic system like the proton; see for instance [13]. However, there seems to be an agreement that the components T^{jj} have a meaning that differs from T^{00}, which is sufficient for our discussion of the mass sum rules. We will briefly return to this point below.

A very important quantity is the trace of the QCD EMT which is given by [8, 9]

$$T^{\mu}{}_{\mu} = (T_R)^{\mu}{}_{\mu} = (T^{\mu}{}_{\mu})_R = (m\overline{\psi}\psi)_R + \gamma_m\,(m\overline{\psi}\psi)_R + \frac{\beta}{2g}(F^{\alpha\beta}F_{\alpha\beta})_R\,, \tag{3}$$

with the anomalous dimension γ_m for the quark mass, the strong coupling g, and the QCD beta function β. The subscript 'R' indicates renormalization. We point out that the quark-mass term actually does not require renormalization, that is, $m\overline{\psi}\psi = (m\overline{\psi}\psi)_R$. A classical treatment of the trace of the EMT in Eq. (1) would provide $m\overline{\psi}\psi$ only. The additional terms on the r.h.s. of Eq. (3) are therefore quantum effects and referred to as the trace anomaly.

In recent work, the renormalization of the individual contributions $T_i^{\mu\nu}$ $(i = q, g)$ to the EMT was investigated in detail [4, 5]. As an important outcome of these studies, it was found that the trace of the (renormalized) quark and gluon contributions to the EMT read

$$(T_{q,R})^{\mu}{}_{\mu} = (1+y)(m\overline{\psi}\psi)_R + x\,(F^{\alpha\beta}F_{\alpha\beta})_R\,, \tag{4}$$

$$(T_{g,R})^{\mu}{}_{\mu} = (\gamma_m - y)(m\overline{\psi}\psi)_R + \left(\frac{\beta}{2g} - x\right)(F^{\alpha\beta}F_{\alpha\beta})_R\,. \tag{5}$$

The parameters x and y are related to the finite terms of certain renormalization constants. Choosing specific values for x and y corresponds to the choice of a renormalization scheme. These parameters may be fixed in minimal-subtraction-type schemes [4, 5], but in the present context other schemes seem equally (if not more) natural. In fact, we proposed the so-called D1 and D2 schemes which are specified as follows [6, 7]:

- D1 scheme: $x = 0$, $y = \gamma_m$, implying that Eqs. (4) and (5) become diagonal.
- D2 scheme: $x = y = 0$, implying that the entire trace anomaly arises from the trace of the renormalized gluon part of the EMT.

III. MASS SUM RULES

To discuss the decompositions of the proton mass in QCD, we consider matrix elements of the EMT. For the full EMT one finds

$$\langle\, T^{\mu\nu}\,\rangle \equiv \langle P|\,T^{\mu\nu}\,|P\rangle = 2P^{\mu}P^{\nu}\,, \tag{6}$$

where $P^{\mu} = (P^0, \boldsymbol{P})$ (with $P^2 = M^2$) is the 4-momentum of the proton. Equation (6), which holds in this form for the normalization $\langle P'|P\rangle = 2P^0(2\pi)^3\delta^{(3)}(\boldsymbol{P'} - \boldsymbol{P})$, readily implies that the matrix elements of both the EMT trace and, in the proton rest frame, the component T^{00} are related to the proton mass according to

$$M = \mathcal{N}\,\langle\, T^{\mu}{}_{\mu}\,\rangle = \mathcal{N}\,\langle\, T^{00}\,\rangle\big|_{\boldsymbol{P}=\boldsymbol{0}} = \left.\frac{\langle\, H_{\text{QCD}}\,\rangle}{\langle P|P\rangle}\right|_{\boldsymbol{P}=\boldsymbol{0}}\,, \tag{7}$$

where $\mathcal{N} = \frac{1}{2M}$ for the aforementioned normalization of $\langle P'|P\rangle$. Here we used the fact that T^{00} is the QCD Hamiltonian density, that is, $H_{\text{QCD}} = \int d^3\boldsymbol{x}\,T^{00}$. Note that the relation between M and the Hamiltonian H_{QCD} in Eq. (7) does not depend on the normalization of the proton state [1, 3].

In the following, we will also consider the matrix elements of the individual contributions to the EMT which are given by [14]

$$\langle\, T_{i,R}^{\mu\nu}\,\rangle = 2P^{\mu}P^{\nu}A_i(0) + 2M^2 g^{\mu\nu}\overline{C}_i(0)\,, \tag{8}$$

with the EMT form factors A_i and $\overline{C}_i$ evaluated at vanishing momentum transfer. Conservation of the total EMT provides the constraints

$$A_q(0) + A_g(0) = 1\,, \qquad \overline{C}_q(0) + \overline{C}_g(0) = 0\,, \tag{9}$$

which ensure that the sum of the quark and gluon contributions gives the r.h.s. of Eq. (6). These two constraints imply that the EMT in the forward limit is fully determined by two numbers only. Any mass decomposition of the proton can therefore have at most two independent terms as has been emphasized in [3].

We now have all the ingredients that are needed for the following discussion of the various mass sum rules. For presentation purposes, we will not follow the historical order.

A. Two-term decomposition of $T^{\mu}{}_{\mu}$

A two-term sum rule, proposed by Hatta, Rajan and Tanaka [4, 5], is based on the relation between the trace of the EMT and the mass in Eq. (7), as well as the decomposition of the trace into quark and gluon parts,

$$M = \overline{M}_q + \overline{M}_g = \mathcal{N}\left(\langle (T_{q,R})^{\mu}{}_{\mu}\rangle + \langle (T_{g,R})^{\mu}{}_{\mu}\rangle\right), \tag{10}$$

with the renormalized operators for the terms given in Eqs. (4) and (5). The two contributions of this sum rule can be readily related to the form factors of the EMT,

$$\overline{M}_i = M\big(A_i(0) + 4\overline{C}_i(0)\big). \tag{11}$$

B. Two-term decomposition of T^{00}

The two-term sum rule of Lorcé is a decomposition of the energy density T^{00} into quark and gluon contributions [3],

$$M = U_q + U_g = \mathcal{N}\left(\langle T^{00}_{q,R}\rangle + \langle T^{00}_{g,R}\rangle\right). \tag{12}$$

Expressed in terms of the EMT form factors, the parton energies read

$$U_i = M\big(A_i(0) + \overline{C}_i(0)\big), \tag{13}$$

showing that obviously $\overline{M}_i \neq U_i$, where the difference is due to the contribution from the $\overline{C}_i$. To shed more light on the underlying physics of this mismatch, we repeat that the T^{jj} may be interpreted as pressures. Equation (6) implies $\langle T^{xx}\rangle = \langle T^{yy}\rangle = \langle T^{zz}\rangle = 0$ for a proton at rest. This feature is intuitive since those (total) pressures must vanish for a mechanically stable system. But the individual pressure contributions $\langle T^{jj}_{i,R}\rangle$ do not vanish and, according to Eq. (8), are given by the form factors $\overline{C}_i$. From this perspective, the $\overline{M}_i$ represent admixtures of energy contributions (proportional to $A_i(0) + \overline{C}_i(0)$) and pressure contributions. In dimensional regularization, proper expressions for the renormalized operators of the two terms in Eq. (12) were obtained in [6],

$$T^{00}_{q,R} = (m\overline{\psi}\psi)_R + (\psi^\dagger\, i\boldsymbol{D}\cdot\boldsymbol{\alpha}\,\psi)_R\,, \qquad T^{00}_{g,R} = \frac{1}{2}(E^2 + B^2)_R\,. \tag{14}$$

The total quark energy $T^{00}_{q,R}$ in Eq. (14) consists of the quark-mass term and what is typically called quark kinetic plus potential energy. We emphasize that, for two reasons, the contributions in Eq. (12) have a clean interpretation as parton energies: First, they are exclusively given by the 00-component of the EMT. Second, the interpretation is supported by the "simple" and intuitive form of the corresponding renormalized operators in Eq. (14).

C. Three-term decomposition of T^{00}

The quark-mass term in $T^{00}_{q,R}$ has a clear interpretation, is scale-independent and has been studied frequently in different approaches including lattice QCD. This provides a strong motivation for a three-term decomposition of the proton mass [6, 7],

$$M = M_q + M_m + M_g = \mathcal{N}\left(\langle \mathcal{H}_q\rangle + \langle \mathcal{H}_m\rangle + \langle \mathcal{H}_g\rangle\right), \tag{15}$$

with the renormalized operators

$$\mathcal{H}_q = (\psi^\dagger\, i\boldsymbol{D}\cdot\boldsymbol{\alpha}\,\psi)_R\,, \qquad \mathcal{H}_m = (m\overline{\psi}\psi)_R\,, \qquad \mathcal{H}_g = \frac{1}{2}(E^2 + B^2)_R\,. \tag{16}$$

Since $\mathcal{H}_q + \mathcal{H}_m = T^{00}_{q,R}$ and $\mathcal{H}_g = T^{00}_{g,R}$, this three-term sum rule can be considered a refinement of the two-term decomposition of Lorcé. Historically, the three-term sum rule was not obtained as presented here but rather through revisiting the expressions of the renormalized operators associated with the decomposition of T^{00} [6, 7].

D. Four-term decomposition of T^{00}

In pioneering papers by Ji a four-term mass sum rule related to T^{00} was proposed [1, 2]. The starting point of this approach is the decomposition of the EMT into a traceless part ($\overline{T}^{\mu\nu}$) and trace part ($\hat{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu}\, T^{\alpha}_{\ \alpha}$),

$$T^{\mu\nu} = \overline{T}^{\mu\nu} + \hat{T}^{\mu\nu} . \tag{17}$$

By definition, the traceless term is given by $\overline{T}^{\mu\nu} = T^{\mu\nu} - \hat{T}^{\mu\nu}$. Employing Eq. (17) is motivated by the fact that both terms on the r.h.s. are conserved. With the help of Eq. (6) one finds that the traceless and trace part of the EMT provide three quarters and one quarter of the proton mass, respectively. This result can be considered a consequence of the virial theorem; see [10] and references therein. Further decomposing $\overline{T}^{00}$ and $\hat{T}^{00}$ into quark and gluon contributions then provides four terms. Re-shuffling of a quark-mass-related term between the traceless and trace part, aimed at obtaining simple-looking operators, leads to the four-term decomposition of Ji [1, 2],

$$M = M_{q[\text{Ji}]} + M_m + M_{g[\text{Ji}]} + M_a = \mathcal{N} \left(\langle \mathcal{H}_{q[\text{Ji}]} \rangle + \langle \mathcal{H}_m \rangle + \langle \mathcal{H}_{g[\text{Ji}]} \rangle + \langle \mathcal{H}_a \rangle \right) , \tag{18}$$

with the renormalized operators

$$\mathcal{H}_{q[\text{Ji}]} = (\psi^\dagger \, i\boldsymbol{D}\!\cdot\!\boldsymbol{\alpha}\, \psi)_{R[\text{Ji}]} \, , \; \mathcal{H}_m = (m\overline{\psi}\psi)_R \, , \; \mathcal{H}_{g[\text{Ji}]} = \frac{1}{2}(E^2 + B^2)_{R[\text{Ji}]} \, , \; \mathcal{H}_a = \frac{1}{4}\left(\gamma_m \, (m\overline{\psi}\psi)_R + \frac{\beta}{2g} (F^{\alpha\beta} F_{\alpha\beta})_R \right) . \tag{19}$$

(Actually, in the original mass decomposition presented in [1, 2], the term proportional to γ_m in $\mathcal{H}_a$ was part of $\mathcal{H}_m$.) The operators in Eq. (19) and in Eq. (16) differ by the so-called anomaly contribution $\mathcal{H}_a$. The presentation in the original papers [1, 2] suggests that this mismatch is due to the fact that only the classical trace has been subtracted when constructing the traceless operators. Another interpretation is that the meaning of the $\mathcal{H}_q$ and $\mathcal{H}_g$ is different in the two approaches. Here we take this latter perspective and therefore have added a subscript in Eqs. (18), (19) to distinguish them from the ones in Eqs. (15), (16). But this still leaves unclear how the renormalized operators in (19) have been derived — for instance, despite the nomenclature, the E and B fields in Eq. (19) do not have the canonical relation to the components of the field-strength tensor — and how the terms $\mathcal{H}_{q[\text{Ji}]}$ and $\mathcal{H}_{g[\text{Ji}]}$ can be interpreted. Note that generally, by construction, the quark and gluon energies in Eq. (19) also contain pressure terms, which can make their interpretation as energy contributions questionable [3]. (Above we have elaborated on the interpretation of the parton energies $\mathcal{H}_q$ and $\mathcal{H}_g$; see the discussion after Eq. (14).)

Concerning the form of the renormalized operators we also mention the results of a very recent work [15]. Using dimensional regularization, in that work the proton mass was decomposed into three terms: $\mathcal{H}_m$, $\mathcal{H}_a$, and the sum of the quark and gluon "parton energies" which we denote here as $(\mathcal{H}_q{+}\mathcal{H}_g)_{[\text{JLS}]}$. The operator for the latter contribution still contains the parameter ε of dimensional regularization. Expanding in powers of ε and keeping the contributions which survive as $\varepsilon \to 0$, we find [10]

$$(\mathcal{H}_q + \mathcal{H}_g)_{[\text{JLS}]} = \mathcal{H}_q + \mathcal{H}_g - \mathcal{H}_a \, , \tag{20}$$

so that, upon summation of all the terms contributing to the mass sum rule, the result of [15] actually confirms our earlier finding in [6, 7]. One could therefore consider the question of the form of the renormalized operators as settled.

IV. SUMMARIZING COMPARISON AND NUMERICAL RESULTS

Obviously, the four sum rules presented in this write-up correspond to (quite) distinct pictures of the nucleon mass, with the closest relation between the two-term and three-term decompositions of T^{00}. Maximum "matching" between the different approaches is obtained in the D2 scheme; see the expressions of the operators in [7]. Specifically, in that scheme the trace of the quark contribution to the EMT agrees with the quark mass term,

$$\overline{M}_q^{\text{D2}} = M_m \, , \tag{21}$$

and the difference between the three-term and four-term decompositions of T^{00} is in the gluon sector only, because

$$M_q^{\text{D2}} = M_{q[\text{Ji}]} \, , \qquad M_g^{\text{D2}} = M_{g[\text{Ji}]} + M_a \, . \tag{22}$$

Let us repeat that, in our view, the physical meaning of $M_{g[\text{Ji}]}$ in the context of a decomposition of the nucleon mass in the rest frame is unclear, and M_a must be added to get a meaningful quantity.

Numerical results for the four sum rules are shown in Fig. 1. Let us briefly discuss two cases. Using the two-term sum rule based on the EMT trace, one would conclude that the bulk of the proton mass is related to gluon fields. On the other hand, according to the three-term decompostion of T^{00}, about 8% of the proton mass is due to quark masses, while the rest can be attributed to quark and gluon energies.

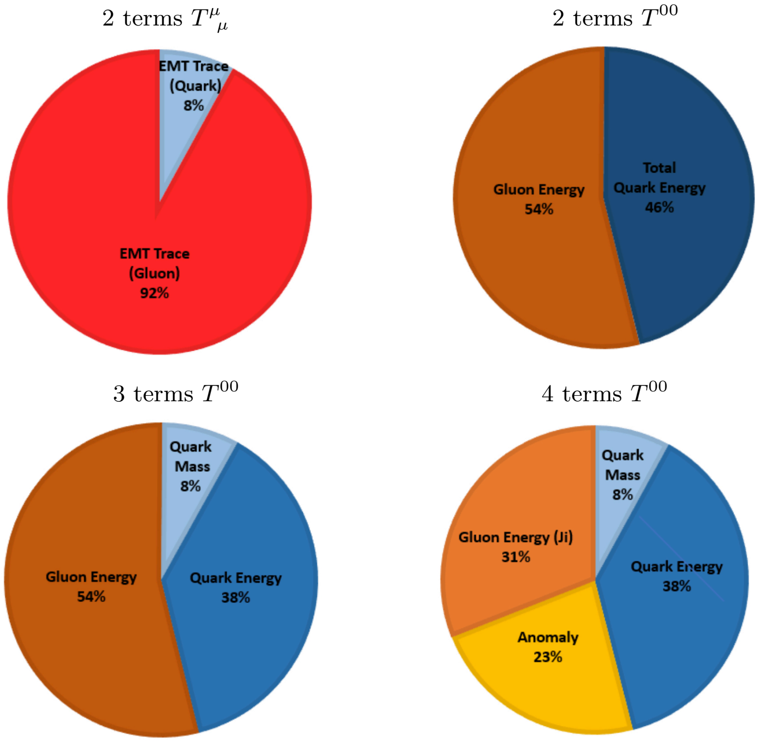

FIG. 1: Numerical results for the four mass sum rules at the scale $\mu = 2\,\text{GeV}$. We have used the D2 scheme and numerical input corresponding to Scenario A of [7], in which up, down and strange quarks are included in the quark-mass term.

Acknowledgments

The work of A. M. has been supported by the National Science Foundation under grant number PHY-2110472 and by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics, within the framework of the TMD Topical Collaboration. The work of B. P. and S. R. is part of a project that has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement STRONG - 2020 - No 824093.

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