

TMDs, PDFs, and multiparton distributions of spin-1 hadrons and their relations

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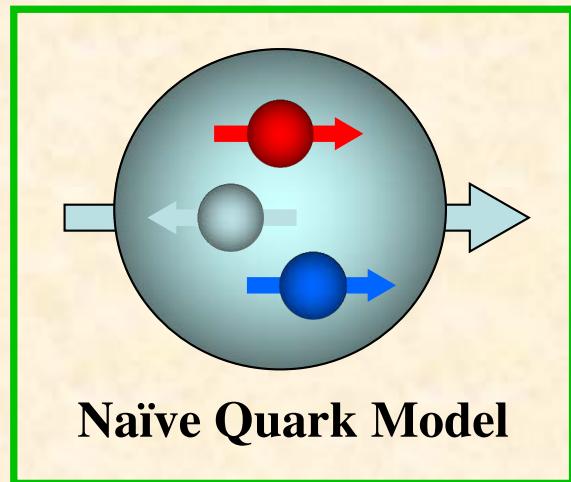
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References

- [1] SK and Qin-Tao Song, PRD 103 (2021) 014025.
- [2] SK and Qin-Tao Song, JHEP 09 (2021) 141.
- [3] SK and Qin-Tao Song, PLB 826 (2022) 136908.

Nucleon spin

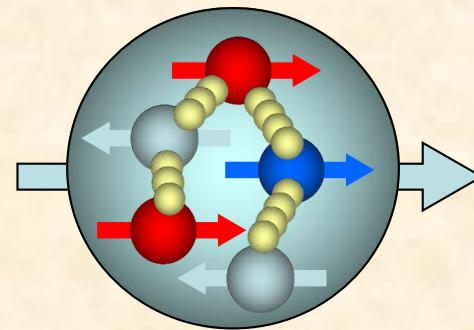


Naïve Quark Model

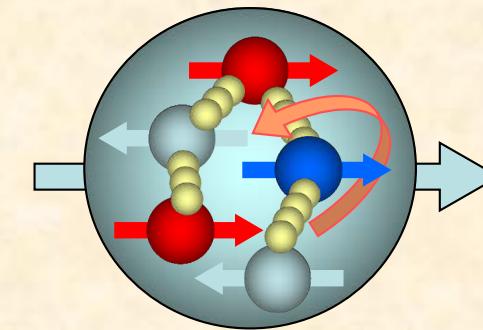
“old” standard model

Almost none of nucleon spin
is carried by quarks!

→ Nucleon spin puzzle!?



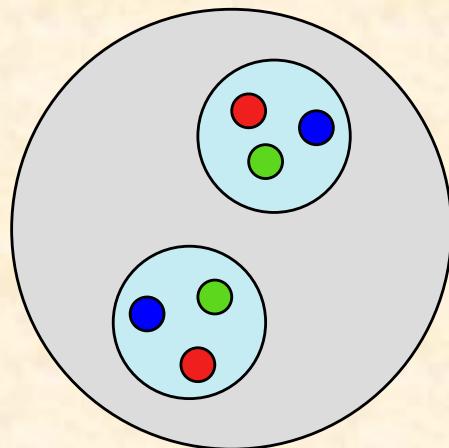
Sea-quarks and gluons?



Orbital angular momenta ?

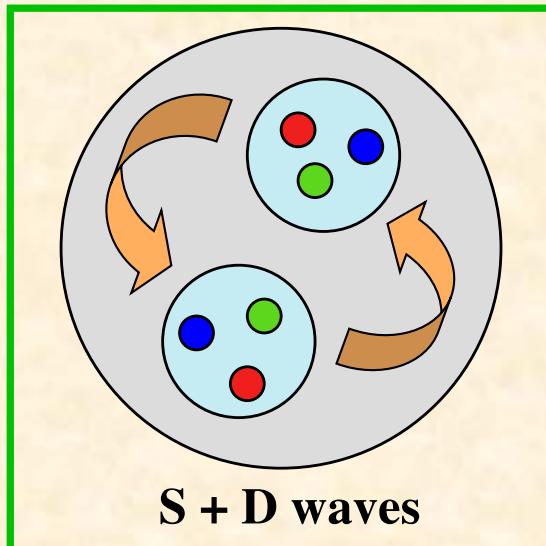
Tensor structure b_1 (e.g. deuteron)

Tensor-structure puzzle!?

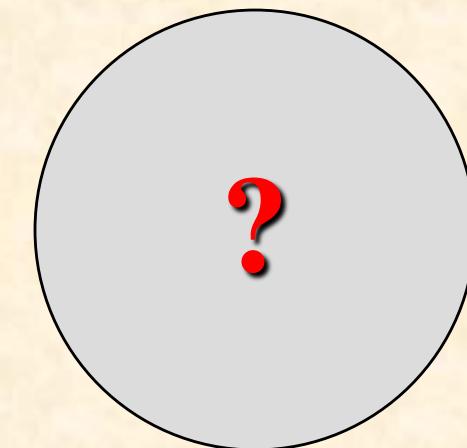


only S wave

$$b_1 = 0$$



S + D waves
standard model $b_1 \neq 0$



$b_1^{\text{experiment}}$
 $\neq b_1^{\text{"standard model"}}$

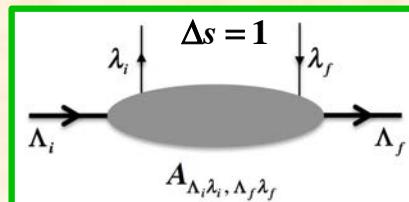
Gluon transversity $\Delta_T g$

Helicity amplitude $A(\Lambda_i, \lambda_i, \Lambda_f, \lambda_f)$, conservation $\Lambda_i - \lambda_i = \Lambda_f - \lambda_f$

Longitudinally-polarized quark in nucleon: $\Delta q(x) \sim A\left(+\frac{1}{2} + \frac{1}{2}, +\frac{1}{2} + \frac{1}{2}\right) - A\left(+\frac{1}{2} - \frac{1}{2}, +\frac{1}{2} - \frac{1}{2}\right)$

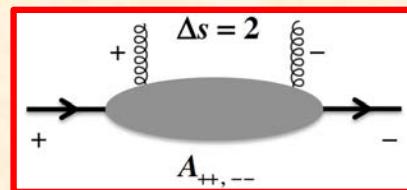
Quark transversity in nucleon:

$\Delta_T q(x) \sim A\left(+\frac{1}{2} + \frac{1}{2}, -\frac{1}{2} - \frac{1}{2}\right), \quad \lambda_i = +\frac{1}{2} \rightarrow \lambda_f = -\frac{1}{2}$ quark spin flip ($\Delta s = 1$)

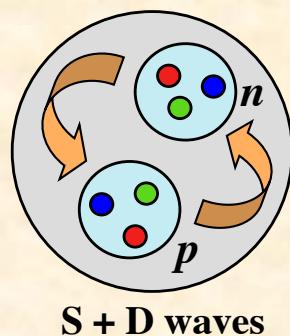


Gluon transversity in deuteron:

$\Delta_T g(x) \sim A(+1+1, -1-1)$,



$A\left(+\frac{1}{2} + 1, -\frac{1}{2} - 1\right)$ not possible for nucleon

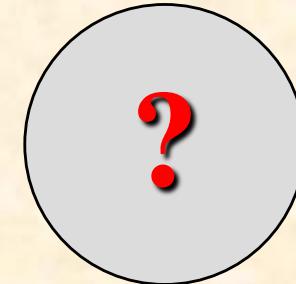


Note: Gluon transversity does not exist for spin-1/2 nucleons.

$b_1 (\delta_T q, \delta_T g) \neq 0 \Leftrightarrow \text{still } \Delta_T g = 0$



What would be the mechanism(s)
for creating $\Delta_T g \neq 0$?



Twist-2 TMDs for spin-1/2 nucleons and spin-1 hadrons

Twist-2 TMDs

Quark \ Hadron	U (γ^+)		L ($\gamma^+ \gamma_5$)		T ($i\sigma^{i+} \gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					$[h_1^\perp]$
L			g_{1L}		$[h_{1L}^\perp]$	
T		f_{1T}^\perp	g_{1T}		$[h_1], [h_{1T}^\perp]$	
LL	f_{1LL}					$[h_{1LL}^\perp]$
LT	f_{1LT}			g_{1LT}		$[h_{1LT}], [h_{1LT}^\perp]$
TT	f_{1TT}			g_{1TT}		$[h_{1TT}], [h_{1TT}^\perp]$

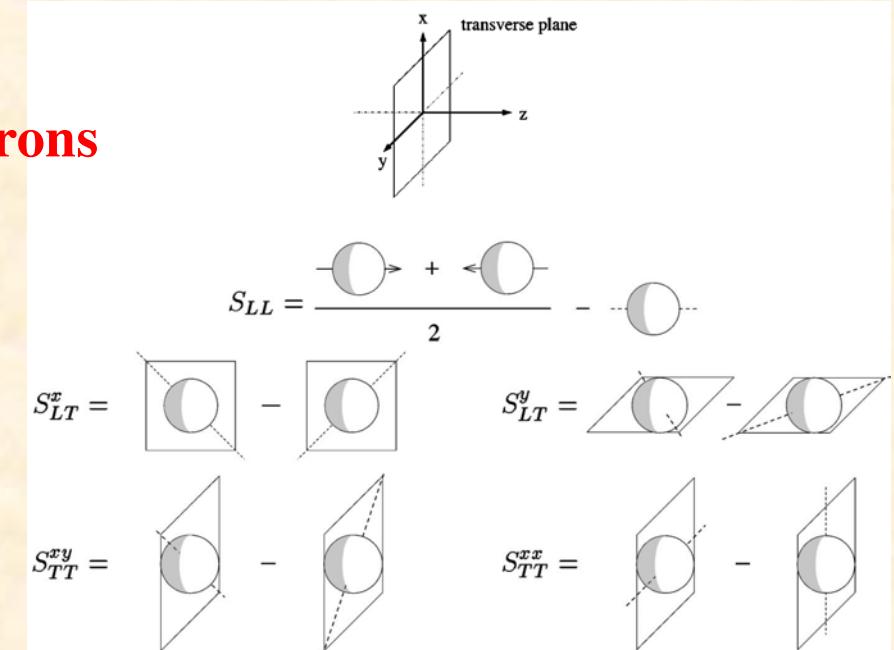
Twist-2 collinear PDFs $[\dots] = \text{chiral odd}$

Quark \ Hadron	U (γ^+)		L ($\gamma^+ \gamma_5$)		T ($i\sigma^{i+} \gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					
L			$g_{1L}(g_1)$			
T					$[h_1]$	
LL	$f_{1LL}(b_1)$					
LT						*1
TT						

Bacchetta-Mulders, PRD 62 (2000) 114004.

Spin-1/2 nucleon
(also spin-1 hadrons)

Spin-1 hadrons



*1 Because of the time-reversal invariance, the collinear PDF $h_{1LT}(x)$ vanishes. However, since the time-reversal invariance cannot be imposed in the fragmentation functions, we should note that the corresponding fragmentation function $H_{1LT}(z)$ should exist as a collinear fragmentation function. (see our PRD paper for the details)

TMDs and PDFs for spin-1 hadrons up to twist 4

Note: Higher-twist effects are sizable at a few $\text{GeV}^2 Q^2$
in tensor-polarized structure functions,
W. Cosyn, Yu-Bing Dong, SK, M. Sargsian,
PRD 95 (2017) 074036.

SK and Qin-Tao Song,
PRD 103 (2021) 014025.

TMDs and their sum rules for spin-1 hadrons

see Appendix I and PRD paper
for the details

Twist-2 TMDs

Quark \ Hadron	U (γ^+)	L ($\gamma^+ \gamma_5$)	T ($i\sigma^{i+} \gamma_5 / \sigma^{i+}$)			
Hadron	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					$[h_1^\perp]$
L			g_{1L}		$[h_{1L}^\perp]$	
T		f_{1T}^\perp	g_{1T}		$[h_1], [h_{1T}^\perp]$	
LL	f_{1LL}					$[h_{1LL}^\perp]$
LT	f_{1LT}		g_{1LT}		$[h_{1LT}], [h_{1LT}^\perp]$	
TT	f_{1TT}		g_{1TT}		$[h_{1TT}], [h_{1TT}^\perp]$	

Twist-3 TMDs

Quark \ Hadron	$\gamma^i, 1, i\gamma_5$		$\gamma^+ \gamma_5$		σ^{ij}, σ^{+-}	
Hadron	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_e^\perp			g_e^\perp		$[h_e]$
L		f_{eL}^\perp	g_{eL}^\perp		$[h_{eL}]$	
T		f_{er}, f_{eT}^\perp	g_{er}, g_{eT}^\perp		$[h_{er}], [h_{eT}^\perp]$	
LL	f_{eLL}^\perp			g_{eLL}^\perp		$[h_{eLL}]$
LT	f_{eLT}, f_{eT}^\perp			g_{eLT}, g_{eT}^\perp		$[h_{eLT}], [h_{eT}^\perp]$
TT	f_{eTT}, f_{eT}^\perp			g_{eTT}, g_{eT}^\perp		$[h_{eTT}], [h_{eT}^\perp]$

New TMDs

Time-reversal invariance in collinear correlation functions (PDFs)

$$\int d^2 k_T \Phi_{T\text{-odd}}(x, k_T^2) = 0$$

Sum rules for the TMDs of spin-1 hadrons

$$\begin{aligned} \int d^2 k_T h_{1LT}(x, k_T^2) &= 0, & \int d^2 k_T g_{LT}(x, k_T^2) &= 0, \\ \int d^2 k_T h_{LL}(x, k_T^2) &= 0, & \int d^2 k_T h_{3LT}(x, k_T^2) &= 0 \end{aligned}$$

For example, in the twist-4

$$\begin{aligned} \int d^2 k_T h_{3LT}(x, k_T^2) &\equiv \int d^2 k_T \left[h'_{3LT}(x, k_T^2) - \frac{k_T^2}{2M^2} h_{3LT}(x, k_T^2) \right] = 0 \\ \Phi^{[\sigma^{i-}]} &= \frac{M^2}{P^{+2}} \left[h_{3LL}^\perp(x, k_T^2) S_{LL} \frac{k_T^i}{M} + h'_{3LT}(x, k_T^2) S_{LT}^i - h_{3LT}^\perp(x, k_T^2) \frac{k_T^i S_{LT} \cdot k_T}{M^2} \right. \\ &\quad \left. - h'_{3TT}(x, k_T^2) \frac{S_{TT}^{ij} k_T j}{M} + h_{3TT}^\perp(x, k_T^2) \frac{k_T \cdot S_{TT} \cdot k_T}{M^2} \frac{k_T^i}{M} \right] \end{aligned}$$

Twist-4 TMDs

Quark \ Hadron	γ^-		$\gamma^- \gamma_5$		σ^{i-}	
Hadron	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_3					$[h_3^\perp]$
L			g_{3L}		$[h_{3L}^\perp]$	
T		f_{3T}^\perp	g_{3T}		$[h_{3T}], [h_{3T}^\perp]$	
LL	f_{3LL}					$[h_{3LL}^\perp]$
LT	f_{3LT}		g_{3LT}		$[h_{3LT}], [h_{3LT}^\perp]$	
TT	f_{3TT}				g_{3TT}	$[h_{3TT}^\perp]$

New fragmentation functions (FFs) for spin-1 hadrons

see arXiv:2201.05397

Corresponding fragmentation functions exist for the spin-1 hadrons

simply by changing function names and kinematical variables.

TMD distribution functions: $f, g, h, e ; x, k_T, S, T, M, n, \gamma^+, \sigma^{i+}$
 \downarrow

TMD fragmentation functions: $D, G, H, E ; z, k_T, S_h, T_h, M_h, \bar{n}, \gamma^-, \sigma^{i-}$

Collinear FFs, twist 2

Quark	U (γ^+)		L ($\gamma^+ \gamma_5$)		T ($i\sigma^{i+} \gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	D_1					
L			G_{IL}			
T					$[H_1]$	
LL	D_{ILL}					
LT						$[H_{ILT}]$
TT						

TMD FFs, twist 2 [] = chiral odd

Quark	U (γ^+)		L ($\gamma^+ \gamma_5$)		T ($i\sigma^{i+} \gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	D_1					$[H_1^\perp]$
L			G_{IL}		$[H_{IL}^\perp]$	
T		D_{IT}^\perp	G_{IT}		$[H_1], [H_{IT}^\perp]$	
LL	D_{ILL}					$[H_{ILL}^\perp]$
LT	D_{ILT}			G_{ILT}		$[H_{ILT}], [H_{ILT}^\perp]$
TT	D_{ITT}			G_{ITT}		$[H_{ITT}], [H_{ITT}^\perp]$

Collinear FFs, twist 3

Quark	$\gamma^i, 1, i\gamma_5$		$\gamma^i \gamma_5$		σ^{ij}, σ^{+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	$[E]$					
L					$[H_L]$	
T			G_T			
LL	$[E_{LL}]$					$[H_{LL}]$
LT	D_{LT}			G_{LT}		
TT						

Collinear FFs, twist 4

Quark	γ^-		$\gamma^- \gamma_5$		σ^{i-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	D_3					
L				G_{3L}		
T						$[H_{3T}]$
LL	D_{3LL}					
LT						$[H_{3LT}]$
TT						

TMD FFs, twist 3

Quark	$\gamma^i, 1, i\gamma_5$		$\gamma^i \gamma_5$		σ^{ij}, σ^{+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	D_1			G^1		$[H]$
L			D_L^\perp	G_L^\perp		$[H_L]$
T		D_{IT}^\perp, D_T^\perp	E_T, E_{IT}^\perp	G_T, G_{IT}^\perp		$[H_T], [H_{IT}^\perp]$
LL	D_{ILL}		E_{LL}		G_{LL}^\perp	$[H_{LL}]$
LT	D_{ILT}		E_{LT}, E_{IT}^\perp	G_{LT}, G_{IT}^\perp		$[H_{LT}], [H_{IT}^\perp]$
TT	D_{ITT}		E_{TT}, E_{IT}^\perp	G_{TT}, G_{IT}^\perp		$[H_{TT}], [H_{IT}^\perp]$

TMD FFs, twist 4

Quark	γ^-		$\gamma^- \gamma_5$		σ^{i-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	D_3					$[H_3^\perp]$
L				G_{3L}^\perp		$[H_{3L}^\perp]$
T			D_{3T}^\perp	G_{3T}		$[H_{3T}], [H_{3T}^\perp]$
LL	D_{3LL}					$[H_{3LL}^\perp]$
LT	D_{3LT}				G_{3LT}	$[H_{3LT}], [H_{3LT}^\perp]$
TT	D_{3TT}				G_{3TT}	$[H_{3TT}], [H_{3TT}^\perp]$

New TMD FFs

PDFs for spin-1 hadrons

Twist-2 PDFs

Quark \ Hadron	U (γ^+)		L ($\gamma^+ \gamma_5$)		T ($i\sigma^{i+} \gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					
L			$g_{1L}(g_1)$			
T					$[h_1]$	
LL	$f_{1LL}(b_1)$					
LT						*1
TT						

*1: $h_{1LT}(x)$, *2: $g_{LT}(x)$, *3: $h_{LL}(x)$, *4: $h_{3LT}(x)$

Because of the time-reversal invariance, the collinear PDF vanishes. However, since the time-reversal invariance cannot be imposed in the fragmentation functions, we should note that the corresponding fragmentation function should exist as a collinear fragmentation function.

[] = chiral odd

Twist-3 PDFs

Quark \ Hadron	$\gamma^i, 1, i\gamma_5$		$\gamma^+ \gamma_5$		σ^{ij}, σ^{-+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	$[e]$					
L					$[h_L]$	
T			g_T			
LL	$[e_{LL}]$					*3
LT	f_{LT}			*2		
TT						

Twist-4 PDFs

Quark \ Hadron	γ^-		$\gamma^- \gamma_5$		σ^{i-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_3					
L				g_{3L}		
T						$[h_{3T}]$
LL	f_{3LL}					
LT						*4
TT						

New collinear PDFs

Twist-2 relation and sum rule for PDFs of spin-1 hadrons

(analogous to the Wandzura-Wilczek relation
and the Burkhardt-Cottingham sum rule)

SK and Qin-Tao Song,
JHEP 09 (2021) 141.

PDFs for spin-1 hadrons

Twist-2 PDFs

Quark \ Hadron	U (γ^+)		L ($\gamma^+ \gamma_5$)		T ($i\sigma^{i+} \gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					
L			$g_{1L}(g_1)$			
T					$[h_1]$	
LL	$f_{1LL}(b_1)$					
LT						*1
TT						

We derived analogous relations to Wandzura-Wilczek relation and Burkhardt-Cottingham sum rule for f_{LT} and f_{1LL} .

SK and Qin-Tao Song (2021)

Twist-3 PDFs

[] = chiral odd

Quark \ Hadron	$\gamma^i, 1, i\gamma_5$		$\gamma^+ \gamma_5$		σ^{ij}, σ^{+-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	[e]					
L					$[h_L]$	
T			g_T			
LL	[e_{LL}]					*3
LT	f_{LT}			*2		
TT						

Twist-4 PDFs

Quark \ Hadron	γ^-		$\gamma^- \gamma_5$		σ^{i-}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_3					
L				g_{3L}		
T					$[h_{3T}]$	
LL	f_{3LL}					
LT						*4
TT						

Wandzura-Wilczek and Burkhardt-Cottingham relations for g_1 and g_2

Structure functions: $\int \frac{d(P^+ \xi^-)}{2\pi} e^{ixP^+ \xi^-} \langle P, S | \bar{\psi}(0) \gamma^\mu \gamma_5 \psi(\xi) | P, S \rangle_{\xi^+ = \xi^- = 0} = 2M_N \left[g_{1L}(x) \bar{n}^\mu S \cdot n + g_T(x) S_T^\mu + g_{3L}(x) \frac{M_N^2}{(P^+)^2} n^\mu S \cdot n \right]$

$$S^\mu = S_L \frac{P^+}{M} \bar{n}^\mu - S_L \frac{M_N}{2P^+} n^\mu + S_T^\mu, \quad P^\mu = P^+ \bar{n}^\mu + \frac{M_N^2}{2P^+} n^\mu, \quad S \cdot n = S_L \frac{P^+}{M_N}$$

$$g_1(x) = \frac{1}{2} [g_{1L}(x) + g_{1L}(-x)], \quad g_1(x) + g_2(x) = \frac{1}{2} [g_T(x) + g_T(-x)]$$

Operators: $R^{\sigma\{\mu_1 \dots \mu_{n-1}\}} = i^{n-1} \bar{\psi} \gamma^\sigma \gamma_5 D^{\{\mu_1} \dots D^{\mu_{n-1}\}} \psi = R^{\{\sigma\mu_1 \dots \mu_{n-1}\}} + R^{\{\sigma\{\mu_1 \dots \mu_{n-1}\}} = \text{twist 2} + \text{twist 3}}$

J. Kodaira and K. Tanaka,
Prog. Theor. Phys. 101 (1999) 191.

$$R^{\{\sigma\mu_1 \dots \mu_{n-1}\}} = \frac{1}{n} [S^\sigma P^{\{\mu_1} P^{\mu_2} \dots P^{\mu_{n-1}\}} + S^{\mu_1} P^{\{\sigma} P^{\mu_2} \dots P^{\mu_{n-1}\}} + S^{\mu_2} P^{\{\mu_1} P^{\sigma} \dots P^{\mu_{n-1}\}} + \dots]$$

$$R^{\{\sigma\{\mu_1 \dots \mu_{n-1}\}} = \frac{1}{n} [(n-1) S^\sigma P^{\{\mu_1} P^{\mu_2} \dots P^{\mu_{n-1}\}} - S^{\mu_1} P^{\{\sigma} P^{\mu_2} \dots P^{\mu_{n-1}\}} - S^{\mu_2} P^{\{\mu_1} P^{\sigma} \dots P^{\mu_{n-1}\}} - \dots]$$

$$\langle P, S | R^{\{\sigma\mu_1 \dots \mu_{n-1}\}} | P, S \rangle = \frac{2}{n} a_n M_N [S^\sigma P^{\mu_1} \dots P^{\mu_{n-1}} + P^{\mu_1} S^\sigma \dots P^{\mu_{n-1}} + \dots]$$

$$\langle P, S | R^{\{\sigma\{\mu_1 \dots \mu_{n-1}\}} | P, S \rangle = \frac{2}{n} d_n M_N [(S^\sigma P^{\mu_1} - P^\sigma S^{\mu_1}) P^{\mu_2} \dots P^{\mu_{n-1}} + (S^\sigma P^{\mu_2} - P^\sigma S^{\mu_2}) P^{\mu_1} \dots P^{\mu_{n-1}} + \dots]$$

$$\frac{1}{2M_N (P^+)^{n-1}} n_{\mu_1} \dots n_{\mu_{n-1}} \langle P, S | R^{\{\sigma\mu_1 \dots \mu_{n-1}\}} | P, S \rangle = \bar{n}^\sigma (S \cdot n) \int_{-1}^1 dx x^{n-1} g_{1L}(x) + S_T^\sigma \int_{-1}^1 dx x^{n-1} g_T(x)$$

$$= \frac{1}{2M_N (P^+)^{n-1}} n_{\mu_1} \dots n_{\mu_{n-1}} \langle P, S | R^{\{\sigma\mu_1 \dots \mu_{n-1}\}} | P, S \rangle + \frac{1}{2M_N (P^+)^{n-1}} n_{\mu_1} \dots n_{\mu_{n-1}} \langle P, S | R^{\{\sigma\{\mu_1 \dots \mu_{n-1}\}} | P, S \rangle$$

$$\rightarrow \int_{-1}^1 dx x^{n-1} g_{1L}(x) = a_n, \quad \int_{-1}^1 dx x^{n-1} g_T(x) = \frac{1}{n} a_n + \frac{n-1}{n} d_n$$

$$\rightarrow \int_0^1 dx x^{n-1} g_1(x) = \int_{-1}^1 dx x^{n-1} \frac{1}{2} g_{1L}(x) = \frac{1}{2} a_n, \quad \int_0^1 dx x^{n-1} [g_1(x) + g_2(x)] = \int_{-1}^1 dx x^{n-1} \frac{1}{2} g_T(x) = \frac{1}{2n} a_n + \frac{n-1}{2n} d_n$$

$$\rightarrow \int_0^1 dx x^{n-1} g_2(x) = \int_0^1 dx x^{n-1} \left[-g_1(x) + \int_x^1 \frac{dy}{y} g_1(y) \right] + \frac{n-1}{2n} d_n$$

If we write $g_2(x) = g_2^{WW}(x) + \bar{g}_2(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(y) + \bar{g}_2(x)$

$$\rightarrow g_2^{WW}(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(y) \text{ (Wandzura-Wilczek relation)}, \quad \int_0^1 dx x^{n-1} \bar{g}_2(x) = \frac{n-1}{2n} d_n$$

$$\rightarrow \int_0^1 dx g_2(x) = 0 \text{ (Burkhardt-Cottingham sum rule)}$$

Note: Twist-3 operators $R^{\{\sigma\{\mu_1 \dots \mu_{n-1}\}}}$ are obtained by the Tayler expansion of $\xi_\mu \bar{\psi}(0) (\partial^\mu \gamma^\sigma - \partial^\sigma \gamma^\mu) \gamma_5 \psi(\xi)$, which needs to be investigated in details for finding the details of twist-3 terms.

Twist-2 relation and sum rule

- Twist-3 matrix element in terms of tensor-polarized PDFs

$$\langle P, T | \bar{\psi}(0)(\partial^\mu \gamma^\alpha - \partial^\alpha \gamma^\mu) \psi(\xi) | P, T \rangle = 2MS_{LT}^\alpha \int_0^1 dx e^{-ixP^\ast \xi} \left[-\frac{3}{2} f_{1LL}(x) + f_{LT}(x) - \frac{d}{dx} \{ x f_{LT}(x) \} \right]$$

see Appendix II for the details

- Twist-3 operator in terms of gluon field tensor

$$\xi_\mu [\bar{\psi}(0)(\gamma^\alpha \partial^\mu - \gamma^\mu \partial^\alpha) \psi(\xi)] = g \int_0^1 dt \bar{\psi}(0) \left\{ i \left(t - \frac{1}{2} \right) G^{\alpha\mu}(t\xi) - \frac{1}{2} \gamma_s \tilde{G}^{\alpha\mu}(t\xi) \right\} \xi_\mu \xi^\nu \psi(\xi)$$

- Matrix element of field tensor in terms of twist-3 multiparton distribution functions

$$\begin{aligned} & \int \frac{d(P \cdot \xi)}{2\pi} e^{ix_1 P \cdot \xi} \langle P, T | g \int_0^1 dt \bar{\psi}(0) \left\{ i \left(t - \frac{1}{2} \right) G^{\mu\nu}(t\xi) - \frac{1}{2} \gamma_s \tilde{G}^{\mu\nu}(t\xi) \right\} \xi_\mu \xi^\nu \psi(\xi) | P, T \rangle_{\xi^+ = \tilde{\xi}_r = 0} \\ &= -2MS_{LT}^\nu \mathcal{P} \int_0^1 dx_2 \frac{1}{x_1 - x_2} \left[\frac{\partial}{\partial x_1} \{ F_{G,LT}(x_1, x_2) + G_{G,LT}(x_1, x_2) \} + \frac{\partial}{\partial x_2} \{ F_{G,LT}(x_2, x_1) + G_{G,LT}(x_2, x_1) \} \right] \end{aligned}$$

Note: Twist-3 operators $R^{[\sigma \{ \mu_1 \dots \mu_{n-1} \}]}$ are obtained by the Tayler expansion of $\xi_\mu \bar{\psi}(0)(\partial^\mu \gamma^\sigma - \partial^\sigma \gamma^\mu) \psi(\xi)$, which needs to be investigated in details for finding the details of twist-3 terms.

$$x \frac{df_{LT}(x)}{dx} = -\frac{3}{2} f_{1LL}(x) - f_{LT}^{(HT)}(x), \quad \text{Higher-twist: } f_{LT}^{(HT)}(x) = -\mathcal{P} \int_0^1 dy \frac{1}{x-y} \left[\frac{\partial}{\partial x} \{ F_{G,LT}(x, y) + G_{G,LT}(x, y) \} + \frac{\partial}{\partial y} \{ F_{G,LT}(y, x) + G_{G,LT}(y, x) \} \right]$$

$$\rightarrow f_{LT}(x) = \frac{3}{2} \int_x^{\varepsilon(x)} \frac{dy}{y} f_{1LL}(y) + \int_x^{\varepsilon(x)} \frac{dy}{y} f_{LT}^{(HT)}(y), \quad \varepsilon(x) = \frac{i}{\pi} P \int_{-\infty}^{\infty} dy \frac{1}{y} e^{-ixy} = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Define $f^+(x) = f(x) + \bar{f}(x) = f(x) - f(-x)$, $f = f_{1LL}$, f_{LT} , $f_{LT}^{(HT)}$, $x > 0$

$$\rightarrow f_{LT}^+(x) = \frac{3}{2} \int_x^1 \frac{dy}{y} f_{1LL}^+(y) + \int_x^1 \frac{dy}{y} f_{LT}^{(HT)+}(y) \quad \rightarrow \text{Twist-2 relation: } f_{LT}^+(x) = \frac{3}{2} \int_x^1 \frac{dy}{y} f_{1LL}^+(y)$$

If we define $f_{2LT}(x) = \frac{2}{3} f_{LT}(x) - f_{1LL}(x)$,

$$f_{2LT}^+(x) = -f_{1LL}^+(x) + \int_x^1 \frac{dy}{y} f_{1LL}^+(y) + \frac{2}{3} \int_x^1 \frac{dy}{y} f_{LT}^{(HT)+}(y) \quad \rightarrow \text{Twist-2 relation: } f_{2LT}^+(x) = -f_{1LL}^+(x) + \int_x^1 \frac{dy}{y} f_{1LL}^+(y), \quad \text{Wandzura-Wilczek like}$$

$$\rightarrow \text{Sum rule: } \int_0^1 dx f_{2LT}^+(x) = 0, \quad \text{Burkhardt-Cottingham like}$$

If the parton-model sum rule without the tensor-polarized antiquark distributions $\int_0^1 dx f_{1LL}^+(x) = \frac{2}{3} \int_0^1 dx b_1^+(x) = 0$ is valid, $\rightarrow \text{Sum rule: } \int_0^1 dx f_{LT}^+(x) = 0$

Summary on the twist-2 relation and sum rule

$$g_2(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(y) \quad (\text{Wandzura-Wilczek relation}), \quad \int_0^1 dx g_2(x) = 0 \quad (\text{Burkhardt-Cottingham sum rule})$$

For tensor-polarized spin-1 hadrons, we obtained

$$f_{2LT}^+(x) = -f_{1LL}^+(x) + \int_x^1 \frac{dy}{y} f_{1LL}^+(y),$$

$$\int_0^1 dx f_{2LT}^+(x) = 0, \quad f_{2LT}(x) \equiv \frac{2}{3} f_{LT}(x) - f_{1LL}(x)$$

$$\int_0^1 dx f_{LT}^+(x) = 0 \quad \text{if} \int_0^1 dx f_{1LL}^+(x) = \frac{2}{3} \int_0^1 dx b_1^+(x) = 0$$

Existence of multiparton distribution functions: $F_{G,LT}(x_1, x_2)$, $G_{G,LT}(x_1, x_2)$, $H_{G,LL}^\perp(x_1, x_2)$, $H_{G,TT}(x_1, x_2)$

$$\int dx b_1^D(x) = \lim_{t \rightarrow 0} -\frac{5}{12} \frac{t}{M^2} F_Q(t) + \sum_i e_i^2 \int dx \delta_T \bar{q}_i(x) = 0 ?$$

F. E. Close and SK, PRD 42 (1990) 2377.

Relations from equation of motion for PDFs of spin-1 hadrons

(Equation-of-motion and Lorentz-invariance relations)

**SK and Qin-Tao Song,
PLB 826 (2022) 136908.**

Relations from equation of motion and Lorentz-invariance relation for spin-1 hadrons

Lorentz invariance = frame independence of twist-3 observables

see Appendix III
for works on spin-1/2 nucleon

We explain derivations on relations from equation of motion for quarks

- $x\mathbf{f}_{LT}(x) - \int_{-1}^{+1} dy [F_{D,LT}(x,y) + G_{D,LT}(x,y)] = 0, \quad x\mathbf{f}_{LT}(x) - \mathbf{f}_{1LT}^{(1)}(x) - \mathcal{P} \int_{-1}^{+1} dy \frac{F_{G,LT}(x,y) + G_{G,LT}(x,y)}{x-y} = 0$
- $x\mathbf{e}_{LL}(x) - 2 \int_{-1}^{+1} dy H_{D,LL}^\perp(x,y) - \frac{m}{M} f_{1LL}(x) = 0, \quad x\mathbf{e}_{LL}(x) - 2\mathcal{P} \int_{-1}^{+1} dy \frac{H_{G,LL}^\perp(x,y)}{x-y} - \frac{m}{M} \mathbf{f}_{1LL}(x) = 0$

and the Lorentz-invariance relation

- $\frac{d\mathbf{f}_{1LT}^{(1)}(x)}{dx} - \mathbf{f}_{LT}(x) + \frac{3}{2} \mathbf{f}_{1LL}(x) - 2\mathcal{P} \int_{-1}^{+1} dy \frac{F_{G,LT}(x,y)}{(x-y)^2} = 0, \quad \text{transverse-momentum moment of TMD: } f^{(1)}(x) = \int d^2 k_T \frac{\vec{k}_T^2}{2M^2} f(x, k_T^2)$

Twist-2 PDFs

Quark \ Hadron	U (γ^+)		L ($\gamma^+\gamma_5$)		T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					
L			$g_{1L}(g_1)$			
T					$[h_1]$	
LL	$f_{1LL}(b_1)$					
LT						
TT						

Twist-3 PDFs

Quark \ Hadron	$\gamma^i, 1, i\gamma_5$		$\gamma^+\gamma_5$		σ^{ij}, σ^{-+}	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	$[e]$					
L					$[h_L]$	
T			g_T			
LL	$[e_{LL}]$					
LT	f_{LT}					*1
TT						

Twist-3 TMDs

Quark \ Hadron	U (γ^+)		L ($\gamma^+\gamma_5$)		T ($i\sigma^{i+}\gamma_5 / \sigma^{i+}$)	
	T-even	T-odd	T-even	T-odd	T-even	T-odd
U	f_1					$[h_1^\perp]$
L			g_{1L}			$[h_{1L}^\perp]$
T		f_{1T}^\perp	g_{1T}			$[h_1], [h_{1T}^\perp]$
LL	f_{1LL}					$[h_{1LL}^\perp]$
LT	f_{1LT}				g_{1LT}	$[h_{1LT}], [h_{1LT}^\perp]$
TT	f_{1TT}				g_{1TT}	$[h_{1TT}], [h_{1TT}^\perp]$

[] = chiral odd

Equation of motion for quarks I

$$0 = (iD_\mu \gamma^\mu - m)\psi = (iD^+ \gamma^- + iD^- \gamma^+ + iD_\alpha \gamma^\alpha - m)\psi, \quad \alpha = 1, 2 \text{ (transverse index)}$$

$$i\sigma^{+\alpha} \cdot (\text{this equation of motion}), \quad \text{use } \sigma^{\alpha\mu} \gamma^\rho = i(g^{\rho\mu} \gamma^\alpha - g^{\rho\alpha} \gamma^\mu) - \epsilon^{\alpha\mu\rho\sigma} \gamma_\sigma \gamma_5, \quad \epsilon^{0123} = +1$$

$$\rightarrow [i(\gamma^+ D^\alpha - \gamma^\alpha D^+) + i\epsilon_T^{\alpha\mu} \gamma_\mu \gamma_5 iD^+ - i\epsilon_T^{\alpha\mu} \gamma^+ \gamma_5 iD_{T\mu} + im\sigma^{+\alpha}] \psi = 0$$

$$0 = \int \frac{d\xi^-}{2\pi} e^{ixP^+ \xi^-} \left\langle P, T \middle| \bar{\psi}(0) [\gamma^+ iD^\alpha - \gamma^\alpha iD^+ + i\epsilon_T^{\alpha\mu} \gamma_\mu \gamma_5 iD^+ - i\epsilon_T^{\alpha\mu} \gamma^+ \gamma_5 iD_{T\mu} + im\sigma^{+\alpha}] \psi(\xi^-) \right\rangle P, T$$

$$\rightarrow 0 = P^+ \text{Tr}[\Phi_D^\alpha(x, P, T) \gamma^+] - P^+ \text{Tr}[\Phi_D^+(x, P, T) \gamma^\alpha] + i\epsilon_T^{\alpha\mu} P^+ \text{Tr}[\Phi_D^+(x, P, T) \gamma_\mu \gamma_5] - i\epsilon_T^{\alpha\mu} P^+ \text{Tr}[\Phi_{D\mu}(x, P, T) \gamma^+ \gamma_5] + im \text{Tr}[\Phi(x, P, T) \sigma^{+\alpha}]$$

"Equation of motion" expressed by multiparton correlation functions, P : momentum, T : tensor polarization

$$\begin{aligned} \text{Multiparton correlation functions: } (\Phi_X^\mu)_{ij}(y, x, P, T) &= \int \frac{d\xi_1^-}{2\pi} \frac{d\xi_2^-}{2\pi} e^{iyP^+ \xi_1^-} e^{i(x-y)P^+ \xi_2^-} \left\langle P, T \middle| \bar{\psi}_j(0) Y^\mu(\xi_2^-) W(0, \xi^-) \psi_i(\xi_1^-) \right\rangle P, T \\ X(Y^\mu) &= G(gG^{+\mu}), \quad A(gA^\mu), \quad D(iD^\mu), \quad D^\mu = \partial^\mu - igA^\mu, \quad W(0, \xi^-) = P \exp \left[-ig \int_0^{\xi^-} d\xi^- A^+(\xi) \right]_{\xi^+ = \tilde{\xi}_r = 0} \end{aligned}$$

$$\text{Collinear correlation function: } (\Phi_D^\mu)_{ij}(x, P, T) = \int_{-1}^{+1} dy (\Phi_D^\mu)_{ij}(y, x, P, T) = \frac{1}{P^+} \int \frac{d\xi^-}{2\pi} e^{ixP^+ \xi^-} \left\langle P, T \middle| \bar{\psi}_j(0) iD^\mu(\xi^-) W(0, \xi^-) \psi_i(\xi^-) \right\rangle P, T$$

$$\Phi_D^\alpha(x, P, T) = \int_{-1}^{+1} dy \Phi_D^\alpha(y, x, P, T), \quad \Phi_D^\alpha(y, x, P, T) = \frac{M}{2P^+} \left[S_{LT}^\alpha \mathbf{F}_{D,LT}(y, x) + i\epsilon_T^{\alpha\mu} S_{LT,\mu} \gamma_5 \mathbf{G}_{D,LT}(y, x) + S_{LL} \gamma^\alpha H_{D,LL}^\perp(y, x) + S_{TT}^{\alpha\mu} \gamma_\mu H_{D,TT}(y, x) \right] \bar{n}$$

Expression in terms of multiparton distribution functions.

$$F_{D,LT}(x, y) = F_{D,LT}(y, x), \quad G_{D,LT}(x, y) = -G_{D,LT}(y, x), \quad H_{D,LL}^\perp(x, y) = -H_{D,LL}^\perp(y, x), \quad H_{D,TT}(x, y) = -H_{D,TT}(y, x)$$

$$\Phi_D^+(x, P, T) = \int_{-1}^{+1} dy \Phi_D^+(y, x, P, T) = \int_{-1}^{+1} dy \delta(y - x) y \Phi(y, P, T) = x \Phi(x, P, T)$$

$$\Phi_{ij}(x, P, T) = \int \frac{d\xi^-}{2\pi} e^{ixP^+ \xi^-} \left\langle P, T \middle| \bar{\psi}_j(0) W(0, \xi^-) \psi_i(\xi) \right\rangle P, T \Big|_{\xi^+ = \tilde{\xi}_r = 0} = \frac{1}{2} \left[S_{LL} \bar{n} f_{1LL}(x) + \frac{M}{P^+} S_{LL} e_{LL}(x) + \frac{M}{P^+} S_{LT} \mathbf{f}_{LT}(x) + \frac{M^2}{(P^+)^2} S_{LL} \bar{n} f_{3LL}(x) \right]_{ij}$$

Refs. SK, Qin-Tao Song, PRD 103 (2021) 014025; JHEP 09 (2021) 141; PLB 826 (2022) 136908.

$$\rightarrow x \mathbf{f}_{LT}(x) - \int_{-1}^{+1} dy [F_{D,LT}(x, y) + G_{D,LT}(x, y)] = 0$$

$$\Phi_G^\alpha(x, P, T) = \frac{M}{2} i \left[S_{LT}^\alpha F_{G,LT}(x_1, x_2) + i\epsilon_T^{\alpha\mu} S_{LT,\mu} \gamma_5 G_{G,LT}(x_1, x_2) + S_{LL} \gamma^\alpha H_{G,LL}^\perp(x_1, x_2) + S_{TT}^{\alpha\mu} \gamma_\mu H_{G,TT}(x_1, x_2) \right] \bar{n}$$

$$F_{D,LT}(x, y) = \delta(x - y) f_{1LT}^{(1)}(x) + \mathcal{P}\left(\frac{1}{x - y}\right) F_{G,LT}(x, y), \quad G_{D,LT}(x, y) = \mathcal{P}\left(\frac{1}{x - y}\right) G_{G,LT}(x, y), \quad f^{(1)}(x) = \int d^2 k_T \frac{\vec{k}_T^2}{2M^2} f(x, k_T^2)$$

$$\rightarrow x \mathbf{f}_{LT}(x) - f_{1LT}^{(1)}(x) - \mathcal{P} \int_{-1}^{+1} dy \frac{F_{G,LT}(x, y) + G_{G,LT}(x, y)}{x - y} = 0, \quad \text{transverse-momentum moment of TMD: } f^{(1)}(x) = \int d^2 k_T \frac{\vec{k}_T^2}{2M^2} f(x, k_T^2)$$

see Appendix IV
for the details

Equation of motion for quarks II

$$0 = (iD_\mu \gamma^\mu - m)\psi = (iD^+ \gamma^- + iD^- \gamma^+ + iD_\alpha \gamma^\alpha - m)\psi, \quad \alpha = 1, 2 \text{ (transverse index)}$$

$$\gamma^+ \cdot (\text{this equation of motion}), \quad (iD^+ - i\sigma^{+\mu} iD_\mu - m\gamma^+) \psi = 0$$

$$0 = \int \frac{d\xi^-}{2\pi} e^{ixP^+ \xi^-} \left\langle P, T \left| \bar{\psi}(0) [iD^+ - i\sigma^{+\mu} iD_\mu - m\gamma^+] \psi(\xi^-) \right| P, T \right\rangle$$

$$\rightarrow 0 = P^+ \text{Tr} [\Phi_D^+(x, P, T)] - P^+ i \text{Tr} [\Phi_D^+(x, P, T) \sigma^{+-}] - P^+ i \text{Tr} [\Phi_{D\alpha}(x, P, T) \sigma^{+\alpha}] - m \text{Tr} [\Phi(x, P, T) \gamma^+]$$

$$\Phi_D^\alpha(x, P, T) = \int_{-1}^{+1} dy \Phi_D^\alpha(y, x, P, T), \quad \Phi_D^\alpha(y, x, P, T) = \frac{M}{2P^+} \left[S_{LT}^\alpha F_{D,LT}(y, x) + i\varepsilon_T^{\alpha\mu} S_{LT,\mu} \gamma_5 G_{D,LT}(y, x) + S_{LL} \gamma^\alpha H_{D,LL}^\perp(y, x) + S_{TT}^{\alpha\mu} \gamma_\mu H_{D,TT}(y, x) \right] \bar{n}$$

$$F_{D,LT}(x, y) = F_{D,LT}(y, x), \quad G_{D,LT}(x, y) = -G_{D,LT}(y, x), \quad H_{D,LL}^\perp(x, y) = -H_{D,LL}^\perp(y, x), \quad H_{D,TT}(x, y) = -H_{D,TT}(y, x)$$

$$\Phi_D^+(x, P, T) = x \Phi(x, P, T), \quad \Phi(x, P, T) = \frac{1}{2} \left[S_{LL} \bar{n} f_{1LL}(x) + \frac{M}{P^+} S_{LL} e_{LL}(x) + \frac{M}{P^+} S_{LT} f_{LT}(x) + \frac{M^2}{(P^+)^2} S_{LL} \bar{n} f_{3LL}(x) \right]$$

$$\rightarrow x e_{LL}(x) - 2 \int_{-1}^{+1} dy H_{D,LL}^\perp(x, y) - \frac{m}{M} f_{1LL}(x) = 0, \quad H_{D,LL}^\perp(x, y) = \mathcal{P} \left(\frac{1}{x-y} \right) H_{G,LL}^\perp(x, y)$$

$$\rightarrow x e_{LL}(x) - 2 \mathcal{P} \int_{-1}^{+1} dy \frac{H_{G,LL}^\perp(x, y)}{x-y} - \frac{m}{M} f_{1LL}(x) = 0$$

Lorentz-invariance relation for tensor-polarized PDFs

$$x f_{LT}(x) - f_{1LT}^{(1)}(x) - \mathcal{P} \int_{-1}^{+1} dy \frac{F_{G,LT}(x, y) + G_{G,LT}(x, y)}{x-y} = 0 \quad (1) \quad \text{from equation of motion I}$$

$$x \frac{df_{LT}(x)}{dx} = -\frac{3}{2} f_{1LL}(x) + \mathcal{P} \int_{-1}^{+1} dy \frac{1}{x-y} \left[\frac{\partial}{\partial x} \{F_{G,LT}(x, y) + G_{G,LT}(x, y)\} + \frac{\partial}{\partial y} \{F_{G,LT}(y, x) + G_{G,LT}(y, x)\} \right] \quad (2) \quad \text{from Wandzura-Wilczek-type studies}$$

$$\frac{d}{dx} \text{ of (1) and use (2): } f_{LT}(x) + x \frac{df_{LT}(x)}{dx} - \frac{f_{1LT}^{(1)}(x)}{dx} - \mathcal{P} \int_{-1}^{+1} dy \left[\frac{F_{G,LT}(y, x) - G_{G,LT}(y, x)}{(y-x)^2} + \frac{1}{y-x} \left\{ \frac{\partial F_{G,LT}(y, x)}{\partial x} - \frac{\partial G_{G,LT}(y, x)}{\partial x} \right\} \right] = 0$$

$$= f_{LT}(x) - \frac{3}{2} f_{1LL}(x) + \mathcal{P} \int_{-1}^{+1} dy \frac{1}{x-y} \left[\frac{\partial}{\partial x} \{F_{G,LT}(x, y) + G_{G,LT}(x, y)\} + \frac{\partial}{\partial y} \{F_{G,LT}(y, x) + G_{G,LT}(y, x)\} \right]$$

$$- \frac{df_{1LT}^{(1)}(x)}{dx} - \mathcal{P} \int_{-1}^{+1} dy \left[\frac{F_{G,LT}(y, x) - G_{G,LT}(y, x)}{(y-x)^2} + \frac{1}{y-x} \left\{ \frac{\partial F_{G,LT}(y, x)}{\partial x} - \frac{\partial G_{G,LT}(y, x)}{\partial x} \right\} \right]$$

$$= f_{LT}(x) - \frac{df_{1LT}^{(1)}(x)}{dx} - \frac{3}{2} f_{1LL}(x) + 2 \mathcal{P} \int_{-1}^{+1} dy \frac{F_{G,LT}(x, y)}{(x-y)^2}, \quad F_{G,LT}(y, x) = -F_{G,LT}(x, y), \quad G_{G,LT}(y, x) = G_{G,LT}(x, y)$$

$$\frac{df_{1LT}^{(1)}(x)}{dx} - f_{LT}(x) + \frac{3}{2} f_{1LL}(x) - 2 \mathcal{P} \int_{-1}^{+1} dy \frac{F_{G,LT}(x, y)}{(x-y)^2} = 0$$

Summary on relations from equation of motion and a Lorentz-invariance relation

- We derived relations among tensor-polarized PDFs and multiparton distribution functions by using the equation of motion for quarks and also showed a Lorentz-invariance relation.

Relations from equation of motion for quarks

- $xf_{LT}(x) - \int_{-1}^{+1} dy [F_{D,LT}(x,y) + G_{D,LT}(x,y)] = 0, \quad xf_{LT}(x) - f_{1LT}^{(1)}(x) - \mathcal{P} \int_{-1}^{+1} dy \frac{F_{G,LT}(x,y) + G_{G,LT}(x,y)}{x-y} = 0$
- $xe_{LL}(x) - 2 \int_{-1}^{+1} dy H_{D,LL}^\perp(x,y) - \frac{m}{M} f_{1LL}(x) = 0, \quad xe_{LL}(x) - 2\mathcal{P} \int_{-1}^{+1} dy \frac{H_{G,LL}^\perp(x,y)}{x-y} - \frac{m}{M} f_{1LL}(x) = 0$

Lorentz-invariance relation

- $\frac{df_{1LT}^{(1)}(x)}{dx} - f_{LT}(x) + \frac{3}{2} f_{1LL}(x) - 2\mathcal{P} \int_{-1}^{+1} dy \frac{F_{G,LT}(x,y)}{(x-y)^2} = 0$

Future prospects and summary

Spin-1 deuteron experiments from the middle of 2020's

JLab



A Letter of Intent to Jefferson Lab PAC 44, June 6, 2016
Search for Exotic Gluonic States in the Nucleus

M. Jones, C. Keith, J. Maxwell*, D. Meekins
Thomas Jefferson National Accelerator Facility, Newport News, VA 23606
W. Detmold, R. Jaffe, R. Milner, P. Shanahan
Laboratory for Nuclear Science, MIT, Cambridge, MA 02139
D. Crabb, D. Day, D. Keller, O. A. Rondon
University of Virginia, Charlottesville, VA 22904
J. Pierce
Oak Ridge National Laboratory, Oak Ridge, TN 37831

**Proposal (approved),
Experiment: middle of 2020's**

Fermilab



The Transverse Structure of the Deuteron with Drell-Yan

D. Keller¹

¹University of Virginia, Charlottesville, VA 22904

**Proposal,
Fermilab-PAC: 2022
Experiment: 2020's**

NICA



Progress in Particle and Nuclear Physics 119 (2021) 103858

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Progress in Particle and Nuclear Physics

journal homepage: www.elsevier.com/locate/ppnp

Review

On the physics potential to study the gluon content of proton and deuteron at NICA SPD

A. Arbuzov^a, A. Bacchetta^{b,c}, M. Butenschoen^d, F.G. Celiberto^{b,d,f}, U. D'Alesio^{b,g}, M. Deka^a, I. Denisenko^a, M.G. Echevarria^a, A. Efremov^d, N.Ya. Ivanov^{a,h}, A. Guskov^{a,i,j}, A. Karpishkov^a, Ya. Klopot^{a,k,l}, B.A. Kniehl^d, A. Kotzinian^a, S. Kumano^a, J.P. Lansberg^a, Keh-Fei Liu^a, F. Murgia^a, M. Nefedov^a, B. Parsamyan^{a,k,l}, C. Pisano^{a,b}, M. Radici^a, A. Rymbekova^a, V. Saleev^a, A. Shipilova^a, Qin-Tao Song^a, O. Teryaev^a

Prog. Nucl. Part. Phys.
119 (2021) 103858,
Experiment: middle of 2020's

LHCspin

CERN-ESPP-Note-2018-111

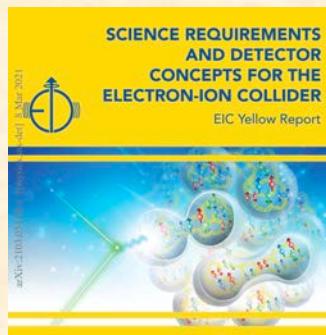
The LHCSpin Project

C. A. Aidala¹, A. Bacchetta^{2,3}, M. Boglione^{4,5}, G. Bozzi^{2,3}, V. Carassiti^{6,7}, M. Chiocca^{4,5}, R. Cimino⁸, G. Ciullo^{6,7}, M. Contalbrigo^{6,7}, U. D'Alesio^{9,10}, P. Di Nezza⁸, R. Engels¹¹, K. Grigoryev¹¹, D. Keller¹², P. Lenisa^{6,7}, S. Liuti¹², A. Metra¹³, P.J. Mulders^{14,15}, F. Murgia¹⁰, A. Nass¹¹, D. Pandieris¹⁶, L. L. Pappalardo^{6,7}, B. Pasquini^{2,3}, C. Pisano^{9,10}, M. Radici³, F. Rathmann¹¹, D. Reggiani¹⁷, M. Schlegel¹⁸, S. Scopetta^{19,20}, E. Steffens²¹, A. Vasilyev²²

arXiv:1901.08002,
Experiment: ~2028

**see Appendix V
for some history**

2030's EIC/EicC

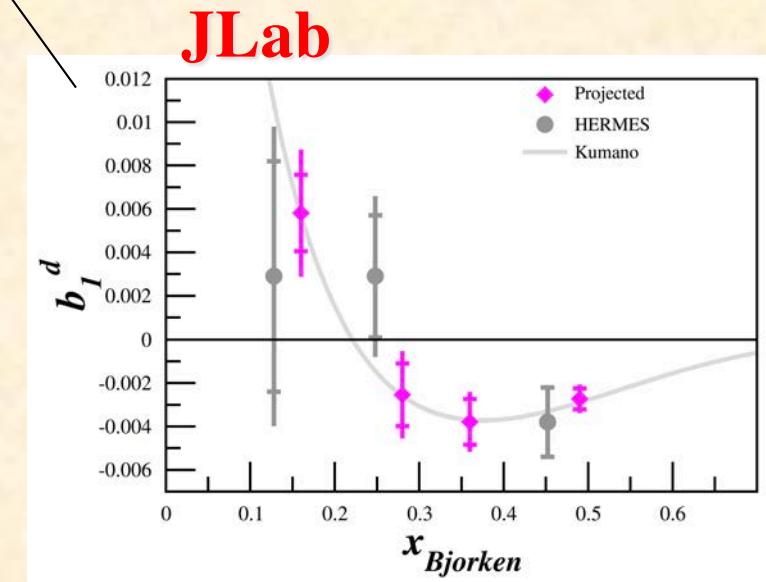
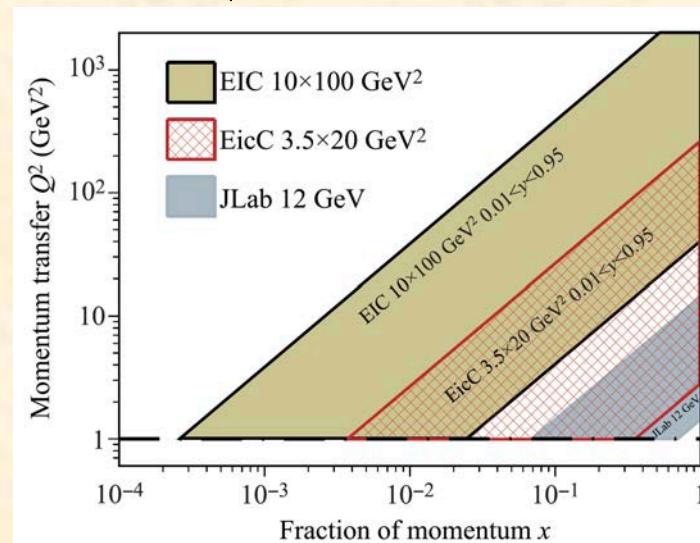
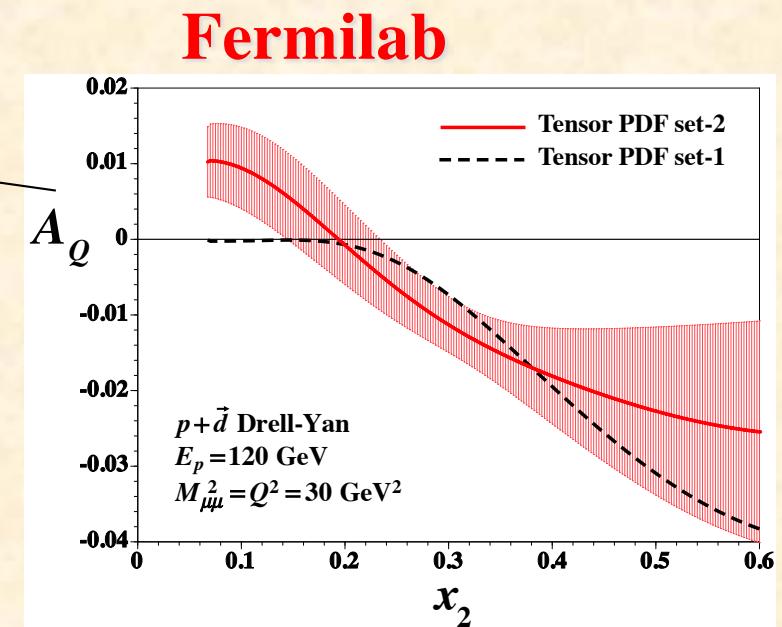
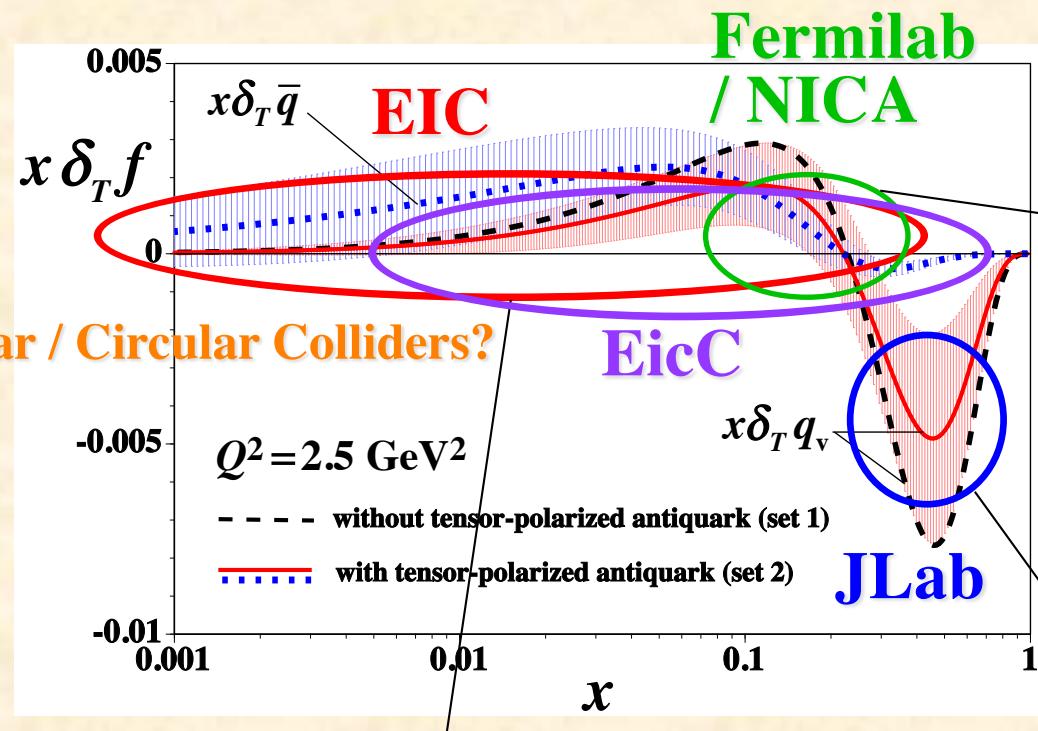


R. Abdul Khalek *et al.*
arXiv:2103.05419.

D. P. Anderle *et al.*,
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x regions of b_1 in 2020's and 2030's



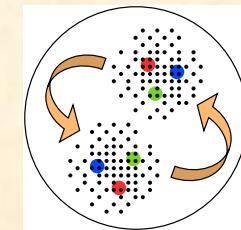
Summary

Spin-1 structure functions of the deuteron (additional spin structure to nucleon spin)

- Tensor structure in quark-gluon degrees of freedom
- Tensor-polarized structure function b_1 and PDFs, gluon transversity

Experiments at JLab, Fermilab, NICA, LHCspin/AMBER, EIC/EicC, ...

- New signature beyond “standard” hadron physics?



- TMDs up to twist 4
- Higher-twist effects could be sizable at Q^2 of a few GeV 2
 - Our relations (WW-like, BC-like, from eq. of motion, Lorentz invariance) could become valuable for future experimental analyses.
- Not discussed: GPDs, GDAs (Generalized Distribution amplitudes = timelike GPDs), ...

There are various experimental projects on the polarized spin-1 deuteron in 2020's and 2030', and “exotic” hadron structure could be found by focusing on the spin-1 nature.

Appendix

TMD correlation functions for spin-1 hadrons

Appendix I

Spin vector: $S^\mu = S_L \frac{P^+}{M} \bar{n}^\mu - S_L \frac{M}{2P^+} n^\mu + S_T^\mu$

Tensor: $T^{\mu\nu} = \frac{1}{2} \left[\frac{4}{3} S_{LL} \frac{(P^+)^2}{M^2} \bar{n}^\mu \bar{n}^\nu + \frac{P^+}{M} \bar{n}^{\{\mu} S_{LT}^{\nu\}} - \frac{2}{3} S_{LL} (\bar{n}^{\{\mu} n^{\nu\}} - g_T^{\mu\nu}) + S_{TT}^{\mu\nu} - \frac{M}{2P^+} n^{\{\mu} S_{LT}^{\nu\}} + \frac{1}{3} S_{LL} \frac{M^2}{(P^+)^2} n^\mu n^\nu \right]$

Tensor part (twist-2): [Bacchetta, Mulders, PRD 62 \(2000\) 114004](#)

$$\Phi(k, P, T) = \left(\frac{A_{13}}{M} I + \frac{A_{14}}{M^2} P + \frac{A_{15}}{M^2} k + \frac{A_{16}}{M^3} \sigma_{\rho\sigma} P^\rho k^\sigma \right) k_\mu k_\nu T^{\mu\nu} + \left[A_{17} \gamma_\nu + \left(\frac{A_{18}}{M} P^\rho + \frac{A_{19}}{M} k^\rho \right) \sigma_{\nu\rho} + \frac{A_{20}}{M^2} \epsilon_{\nu\rho\sigma} P^\rho k^\sigma \gamma^\tau \gamma_5 \right] k_\mu T^{\mu\nu}$$

Tensor part (twist-2, 3, 4): n^μ dependent terms are added for up to twist 4.

[For the spin-1/2 nucleon: [Goeke, Metzand, Schlegel, PLB 618 \(2005\) 90](#); [Metz, Schweitzer, Teckentrup, PLB 680 \(2009\) 141](#).]

[Kumano-Song-2021](#), for the details see PRD 103 (2021) 014025

$$\Phi(k, P, T | n) = \left(\frac{A_{13}}{M} I + \frac{A_{14}}{M^2} P + \frac{A_{15}}{M^2} k + \frac{A_{16}}{M^3} \sigma_{\rho\sigma} P^\rho k^\sigma \right) k_\mu k_\nu T^{\mu\nu} + \left[A_{17} \gamma_\nu + \left(\frac{A_{18}}{M} P^\rho + \frac{A_{19}}{M} k^\rho \right) \sigma_{\nu\rho} + \frac{A_{20}}{M^2} \epsilon_{\nu\rho\sigma} P^\rho k^\sigma \gamma^\tau \gamma_5 \right] k_\mu T^{\mu\nu}$$

[Bacchetta
-Mulders](#)

$$\begin{aligned} & + \left(\frac{B_{21}M}{P \cdot n} k_\mu + \frac{B_{22}M^3}{(P \cdot n)^2} n_\mu \right) n_\nu T^{\mu\nu} + i \gamma_5 \epsilon_{\mu\rho\sigma} P^\rho \left(\frac{B_{23}}{(P \cdot n)M} k^\tau n^\sigma k_\nu + \frac{B_{24}M}{(P \cdot n)^2} k^\tau n^\sigma n_\nu \right) T^{\mu\nu} \\ & + \left[\frac{B_{25}}{P \cdot n} \not{n} k_\mu k_\nu + \left(\frac{B_{26}M^2}{(P \cdot n)^2} \not{n} + \frac{B_{28}}{P \cdot n} P + \frac{B_{30}}{P \cdot n} k \right) k_\mu n_\nu + \left(\frac{B_{27}M^4}{(P \cdot n)^3} \not{n} + \frac{B_{29}M^2}{(P \cdot n)^2} P + \frac{B_{31}M^2}{(P \cdot n)^2} k \right) n_\mu n_\nu + \frac{B_{32}M^2}{P \cdot n} \gamma_\mu n_\nu \right] T^{\mu\nu} \\ & - \left[\epsilon_{\mu\rho\sigma} \gamma^\tau P^\rho \left(\frac{B_{34}}{P \cdot n} n^\sigma k_\nu + \frac{B_{33}}{P \cdot n} k^\sigma n_\nu + \frac{B_{35}M^2}{(P \cdot n)^2} n^\sigma n_\nu \right) + \epsilon_{\lambda\rho\sigma} k^\lambda \gamma^\tau P^\rho n^\sigma \left(\frac{B_{36}}{P \cdot n M^2} k_\mu k_\nu + \frac{B_{37}}{(P \cdot n)^2} k_\mu n_\nu + \frac{B_{38}M^2}{(P \cdot n)^3} n_\mu n_\nu \right) \right] \gamma_5 T^{\mu\nu} \\ & + \epsilon_{\mu\rho\sigma} k^\tau P^\rho n^\sigma \left(\frac{B_{39}}{(P \cdot n)^2} k_\nu + \frac{B_{40}M^2}{(P \cdot n)^3} n_\nu \right) \not{n} \gamma_5 T^{\mu\nu} \\ & + \sigma_{\rho\sigma} \left[P^\rho k^\sigma \left(\frac{B_{41}}{(P \cdot n)M} k_\mu n_\nu + \frac{B_{42}M}{(P \cdot n)^2} n_\mu n_\nu \right) + P^\rho n^\sigma \left(\frac{B_{43}}{(P \cdot n)M} k_\mu k_\nu + \frac{B_{44}M}{(P \cdot n)^2} k_\mu n_\nu + \frac{B_{45}M^3}{(P \cdot n)^3} n_\mu n_\nu \right) \right] T^{\mu\nu} \\ & + \sigma_{\rho\sigma} \left[k^\rho n^\sigma \left(\frac{B_{46}}{(P \cdot n)M} k_\mu k_\nu + \frac{B_{47}M}{(P \cdot n)^2} k_\mu n_\nu + \frac{B_{48}M^3}{(P \cdot n)^3} n_\mu n_\nu \right) \right] T^{\mu\nu} + \sigma_{\mu\sigma} \left[n^\sigma \left(\frac{B_{49}M}{P \cdot n} k_\nu + \frac{B_{50}M^3}{(P \cdot n)^2} n_\nu \right) + \left(\frac{B_{51}M}{P \cdot n} P^\sigma + \frac{B_{52}M}{P \cdot n} k^\sigma \right) n_\nu \right] T^{\mu\nu} \end{aligned}$$

New terms
in our paper
(2021)

From this correlation function, new tensor-polarized TMDs are defined in twist-3 and 4 in addition to twist-2 ones.

Terms associated with
 $\not{n} = \frac{1}{\sqrt{2}}(1, 0, 0, -1)$

Collinear PDFs for spin-1 hadrons

Appendix II

Tensor polarization:

$$T^{\mu\nu} = \frac{1}{2} \left[\frac{4}{3} S_{LL} \frac{(P^+)^2}{M^2} \bar{n}^\mu \bar{n}^\nu - \frac{2}{3} S_{LL} (\bar{n}^\mu n^\nu + \bar{n}^\nu n^\mu - g_T^{\mu\nu}) + \frac{1}{3} S_{LL} \frac{M^2}{(P^+)^2} n^\mu n^\nu + \frac{P^+}{M} (\bar{n}^\mu S_{LT}^\nu + \bar{n}^\nu S_{LT}^\mu) - \frac{M}{2P^+} (n^\mu S_{LT}^\nu + n^\nu S_{LT}^\mu) + S_{TT}^{\mu\nu} \right]$$

Collinear correlation function:

$$\Phi(x, P, T) = \frac{1}{2} \left[f_{LL}(x) S_{LL} \bar{n} + \frac{M}{P^+} e_{LL}(x) S_{LL} + \frac{M}{P^+} f_{LT}(x) S_{LT} \right], \text{ up to twist-3}$$

Matrix element of vector operator:

$$\langle P, T | \bar{\psi}(0) \gamma^\mu \psi(\xi^-) | P, T \rangle = \int_{-1}^1 dx e^{-ixP^+ \xi^-} P^+ \text{Tr} [\Phi_{ij}(x, P, T) (\gamma^\mu)_{ji}] = \int_{-1}^1 dx e^{-ixP^+ \xi^-} 2P^+ \left[f_{LL}(x) S_{LL} \bar{n}^\mu + \frac{M}{P^+} S_{LT}^\mu f_{LT}(x) \right]$$

$\alpha = 1, 2 = \text{transverse}$:

$$\langle P, T | \bar{\psi}(0) (\partial^\mu \gamma^\alpha - \partial^\alpha \gamma^\mu) \psi(\xi) | P, T \rangle = 2MS_{LT}^\alpha \int_{-1}^1 dx e^{-ixP^+ \xi^-} \left[-\frac{3}{2} f_{LL}(x) + f_{LT}(x) - \frac{d}{dx} \{ x f_{LT}(x) \} \right]$$

$$\bar{\psi}(0) (\partial^\mu \gamma^\alpha - \partial^\alpha \gamma^\mu) \psi(\xi) = \bar{\psi}(0) (D^\mu \gamma^\alpha - D^\alpha \gamma^\mu) \psi(\xi) - \bar{\psi}(0) \gamma^\mu \psi(\xi) ig \int_0^1 dt t \xi_\rho G^{\rho\alpha}(t\xi)$$

In the Fock-Schwinger gauge: $\xi_\mu A^\mu(\xi) = 0$, we have $A^\nu(\xi) = \int_0^1 dt t \xi_\mu G^{\mu\nu}(t\xi)$, $G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - i g [A^\mu, A^\nu]$

$$\bar{\psi}(0) (D^\mu \gamma^\alpha - D^\alpha \gamma^\mu) \psi(\xi) = -\frac{i}{2} \bar{\psi}(0) \sigma^{\alpha\mu} \tilde{D} \psi(\xi) - \frac{i}{2} \bar{\psi}(0) \tilde{D} \sigma^{\alpha\mu} \psi(\xi) - \frac{1}{2} g \int_0^1 dt \xi_\nu G^{\rho\nu}(t\xi) \bar{\psi}(0) \gamma_\rho \sigma^{\alpha\mu} \psi(\xi) + \frac{i}{2} \bar{\partial}_\rho \{ \bar{q}(0) W(0, \xi) \gamma^\rho \sigma^{\alpha\mu} q(\xi) \}$$

$$-\frac{1}{2} \xi_\mu g \int_0^1 dt \xi_\nu G^{\rho\nu}(t\xi) \bar{\psi}(0) \gamma_\rho \sigma^{\alpha\mu} \psi(\xi) = \frac{1}{2} g \int_0^1 dt \bar{\psi}(0) \left\{ -i \xi_\mu G^{\alpha\mu}(t\xi) \xi + \xi_\mu \tilde{G}^{\alpha\mu}(tx) \xi \gamma_5 - \left[\xi^2 \tilde{G}^{\alpha\sigma}(t\xi) - \xi_\mu x^\alpha \tilde{G}^{\mu\sigma}(tx) \right] \gamma_\sigma \gamma_5 \right\} \psi(x)$$

$$\bar{\partial}_\rho \{ \bar{q}(0) W(0, \xi) \gamma^\rho \sigma^{\alpha\mu} q(\xi) \} = \bar{\psi}(0) \tilde{D} \sigma^{\alpha\mu} \psi(\xi) + \bar{\psi}(0) \tilde{D} \sigma^{\alpha\mu} \psi(\xi) - ig \int_0^1 dt \xi^\nu G_{\rho\nu}(t\xi) \bar{\psi}(0) \gamma^\rho \sigma^{\alpha\mu} \psi(\xi)$$

$$\xi_\mu \{ \bar{\psi}(0) (\partial^\mu \gamma^\alpha - \partial^\alpha \gamma^\mu) \psi(\xi) \} = g \int_0^1 dt \bar{\psi}(0) \left\{ i \left(t - \frac{1}{2} \right) G^{\alpha\mu}(t\xi) - \frac{1}{2} \tilde{G}^{\alpha\mu}(t\xi) \gamma_5 \right\} \xi_\mu \xi^\nu \psi(x) + \frac{1}{2} g \int_0^1 dt \bar{\psi}(0) \left[\xi_\mu \xi^\alpha \tilde{G}^{\mu\sigma}(t\xi) - \xi^2 \tilde{G}^{\alpha\sigma}(t\xi) \right] \gamma_\sigma \gamma_5 \psi(x)$$

$$- \frac{i}{2} \xi_\mu \bar{\psi}(0) \sigma^{\alpha\mu} (\tilde{D} - m) \psi(\xi) - \frac{i}{2} \xi_\mu \bar{\psi}(0) (\tilde{D} + m) \sigma^{\alpha\mu} \psi(\xi) + \frac{i}{2} \xi_\mu \bar{\partial}_\rho \{ \bar{\psi}(0) W(0, \xi) \gamma^\rho \sigma^{\alpha\mu} \psi(\xi) \}$$

$$\simeq g \int_0^1 dt \bar{\psi}(0) \left\{ i \left(t - \frac{1}{2} \right) G^{\alpha\mu}(t\xi) - \frac{1}{2} \tilde{G}^{\alpha\mu}(t\xi) \gamma_5 \right\} \xi_\mu \xi^\nu \psi(x)$$

$$\text{Multiparton correlation function: } (\Phi_G^v)_{ij}(x_1, x_2) = \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} e^{ix_1 P^+ \xi_1^-} e^{ix_2 P^+ \xi_2^-} \langle P, T | \bar{\psi}_j(0) g G^{+\nu}(\xi_2^-) \psi_i(\xi_1^-) | P, T \rangle$$

Express Φ_G^v in terms of possible Lorentz vectors and multiparton distribution functions with the conditions Hermiticity, parity invariance, and time-reversal invariance

$$\Phi_G^v(x_1, x_2) = \frac{M}{2} \left[i S_{LT}^v F_{G,LT}(x_1, x_2) - \epsilon_\perp^{\alpha\mu} S_{LT\mu} \gamma_5 G_{G,LT}(x_1, x_2) + i S_{LL} \gamma^\alpha H_{G,LL}^\perp(x_1, x_2) + i S_{TT}^{\alpha\mu} \gamma_\mu H_{G,TT}(x_1, x_2) \right] \bar{n}$$

$$(\Phi_G^v)_{ij}(\eta) : S_{LT}^v F_{G,LT}(x_1, x_2) = -\frac{i}{2M} g \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} e^{ix_1 P^+ \xi_1^-} e^{ix_2 P^+ \xi_2^-} \langle P, T | \bar{\psi}(0) \eta n_\mu G^{\mu\nu}(\xi_2^-) \psi(\xi_1^-) | P, T \rangle$$

$$(\Phi_G^v)_{ij}(i\gamma_5 \eta) : S_{LT}^v G_{G,LT}(x_1, x_2) = \frac{i}{2M} g \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} e^{ix_1 P^+ \xi_1^-} e^{ix_2 P^+ \xi_2^-} \langle P, T | \bar{\psi}(0) i\gamma_5 \eta n_\mu \tilde{G}^{\mu\nu}(\xi_2^-) \psi(\xi_1^-) | P, T \rangle$$

$$\int \frac{d(P \cdot \xi)}{2\pi} e^{ix_1 P^+ \xi} \langle P, T | g \int_0^1 dt \bar{\psi}(0) \left\{ i \left(t - \frac{1}{2} \right) G^{\mu\nu}(t\xi) - \frac{1}{2} \gamma_5 \tilde{G}^{\mu\nu}(t\xi) \right\} \xi_\mu \xi^\nu \psi(x) | P, T \rangle_{\xi^*=\tilde{\xi}_r=0} = -2MS_{LT}^v \mathcal{P} \int_0^1 dx_2 \frac{1}{x_1 - x_2} \left[\frac{\partial}{\partial x_1} \{ F_{G,LT}(x_1, x_2) + G_{G,LT}(x_1, x_2) \} + \frac{\partial}{\partial x_2} \{ F_{G,LT}(x_2, x_1) + G_{G,LT}(x_2, x_1) \} \right]$$

Note: Twist-3 operators $R^{\{\sigma(\mu_1 \dots \mu_{n-1})\}}$ are obtained by the Tayler expansion of $\xi_\mu \bar{\psi}(0) (\partial^\mu \gamma^\sigma - \partial^\sigma \gamma^\mu) \psi(\xi)$, which needs to be investigated in details for finding the details of twist-3 terms.

Relations from equation of motion and Lorentz-invariance relation for spin-1/2 nucleons

Appendix III

References on related works

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- K. Kanazawa, Y. Koike, A. Metz, D. Pitonyak, and M. Schlegel, Phys. Rev. D 93 (2016) 054024;
- A. V. Belitsky, Int. J. Mod. Phys. A 32 (2017) 1730018;
- A. Rajan, M. Engelhardt, and S. Liuti, Phys. Rev. D 98 (2018) 074022.

We may miss some of your works.

Relations among multiparton distribution functions

Appendix IV

Multiparton correlation functions

$$(\Phi_A^\mu)_{\bar{j}}(x_1, x_2, P, T) = \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} e^{i(x_1-x_2)p^*\xi_1} e^{i(x_2-x_1)p^*\xi_2} \langle P, T | \bar{\psi}_j(0) Y^\mu(\xi_2^-) \psi_i(\xi_1^-) | P, T \rangle, \quad X(Y^\mu) = G(gG^{+\mu}), A(gA^\mu), D(iD^\mu), D^\mu = \partial^\mu - igA^\mu$$

$$\Phi_D^\alpha(x_1, x_2, P, T) = \frac{M}{2P^+} [S_{LT}^\alpha F_{D,LT}(x_1, x_2) + i\varepsilon_T^\alpha S_{LT,\mu} \gamma_\mu G_{D,LT}(x_1, x_2) + S_{LL}^\alpha \gamma^\alpha H_{D,LL}^\perp(x_1, x_2) + S_{TT}^\alpha \gamma_\mu H_{D,TT}(x_1, x_2)] \bar{n}$$

$$\Phi_G^\alpha(x_1, x_2, P, T) = \frac{M}{2} i [S_{LT}^\alpha F_{G,LT}(x_1, x_2) + i\varepsilon_T^\alpha S_{LT,\mu} \gamma_\mu G_{G,LT}(x_1, x_2) + S_{LL}^\alpha \gamma^\alpha H_{G,LL}^\perp(x_1, x_2) + S_{TT}^\alpha \gamma_\mu H_{G,TT}(x_1, x_2)] \bar{n}$$

k_T -weighted correlation function

$$(\Phi_D^\alpha)_{\bar{j}}(x, P, T) = \int d^2 k_T k_T^\alpha \Phi_{\bar{j}}^{[C]}(x, k_T, P, T) = \int d^2 k_T k_T^\alpha \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} e^{i(x^+ p^* \xi^- - \vec{k}_T \cdot \vec{\xi}_T)} \langle P, T | \bar{\psi}_j(0) W^{[C]}(0, \xi) \psi_i(\xi) | P, T \rangle_{k^* = x^+ p^*, \xi^* = 0} = \int \frac{d\xi^-}{2\pi} e^{i(x^+ p^* \xi^-)} \langle P, T | \bar{\psi}_j(0) i \partial_\xi^\alpha(\xi) W^{[C]}(0, \xi) \psi_i(\xi) | P, T \rangle_{k^* = x^+ p^*, \xi^* = \xi_T = 0}$$

$$W^{[\pm]}(0, \xi) = U^{-}[0, \pm\infty^-] U^T[0_T, \infty_T] U^T[\infty_T, \vec{\xi}_T] U^{-}[\pm\infty^-, \xi^-]$$

$$\partial_\xi^\alpha(\xi) W^{[\pm]}(0, \xi)_{\xi^* = \xi_T = 0} = U^{-}[0, \xi^-] D_\tau^\alpha(\xi)_{\xi^* = \xi_T = 0} + ig U^{-}[0, \pm\infty^-] \int_{\pm\infty^-}^\xi d\eta^- U^{-}[\pm\infty^-, \eta^-] G^{+\alpha}(\eta^-) U^-[\eta^-, \xi^-], \quad \alpha = \text{transverse} = 1, 2$$

$$(\Phi_{\bar{j}}^{[\pm]\alpha})_{\bar{j}}(x, P, T) = \int \frac{d\xi^-}{2\pi} e^{i(x^+ p^* \xi^-)} \left[\langle P, T | \bar{\psi}_j(0) i \partial_\tau^\alpha(\xi) W^{[C]}(0, \xi) \psi_i(\xi) | P, T \rangle_{k^* = x^+ p^*, \xi^* = \xi_T = 0} - \langle P, T | \bar{\psi}_j(0) \int_{\pm\infty^-}^\xi d\eta^- g G^{+\alpha}(\eta^-) \psi_i(\xi) | P, T \rangle_{k^* = x^+ p^*, \xi^* = \xi_T = 0} \right]$$

$$G^{+\alpha} = \partial_- A^\alpha(y^-) \quad \text{for } A^+ = 0 \quad \rightarrow \int_{\pm\infty^-}^\xi d\eta^- G^{+\alpha}(\eta^-) = A^\alpha(\xi^-) - A^\alpha(\pm\infty), \quad iD_\tau^\alpha(\xi^-) = i\partial_\tau^\alpha(\xi^-) + gA_\tau^\alpha(\xi^-)$$

$$= \int \frac{d\xi^-}{2\pi} e^{i(x^+ p^* \xi^-)} \left[\langle P, T | \bar{\psi}_j(0) i \partial_\tau^\alpha(\xi) \psi_i(\xi) | P, T \rangle + \langle P, T | \bar{\psi}_j(0) g G^\alpha(\xi^- = \pm\infty) \psi_i(\xi) | P, T \rangle \right]_{k^* = x^+ p^*, \xi^* = \xi_T = 0}$$

Average of $\Phi_{\bar{j}}^{[\pm]\alpha}$ and $\Phi_{\bar{j}}^{[\pm]\alpha}$:

$$(\Phi^\alpha)_{\bar{j}}(x, P, T) = \frac{(\Phi_{\bar{j}}^{[\pm]\alpha})_{\bar{j}}(x, P, T) + (\Phi_{\bar{j}}^{[-\alpha]})_{\bar{j}}(x, P, T)}{2} = \int \frac{d\xi^-}{2\pi} e^{i(x^+ p^* \xi^-)} \left[\langle P, T | \bar{\psi}_j(0) i \partial_\tau^\alpha(\xi) \psi_i(\xi) | P, T \rangle + \langle P, T | \bar{\psi}_j(0) g \left\{ \frac{A^\alpha(\infty^-) + A^\alpha(-\infty^-)}{2} \right\} \psi_i(\xi) | P, T \rangle \right]$$

$$\Phi(x, k_T, T)_{\text{twist-2}} = \frac{1}{2} \left[f_{LL}(x, k_T^2) S_{LL} \bar{n} - f_{LT}(x, k_T^2) \frac{k_T \cdot S_{LT}}{M} \bar{n} + f_{IT}(x, k_T^2) \frac{k_{T\perp} \cdot S_{IT}^{\perp\mu} k_{T\perp}}{M^2} \bar{n} + g_{LL}(x, k_T^2) \frac{\mathcal{E}_{T, \text{mv}} S_{IT}^{\perp\mu} k_T^\nu}{M} \gamma_S \bar{n} + g_{IT}(x, k_T^2) \frac{\mathcal{E}_{T, \text{mv}} S_{IT}^{\perp\mu} k_T^\nu k_T^\rho}{M^2} \gamma_S \bar{n} \right.$$

$$\left. + h_{LL}^\perp(x, k_T^2) S_{LL} \frac{k_T^2}{M} \sigma^{\mu\bar{\mu}} \bar{n}_\mu + h_{LT}(x, k_T^2) S_{LT}^\perp \sigma^{\mu\bar{\mu}} \bar{n}_\mu + h_{LT}^\perp(x, k_T^2) \frac{S_{LT}^i k_T^j - S_{LT}^j k_T^i / 2}{M^2} \sigma^{\mu\bar{\mu}} \bar{n}_\mu + h_{IT}(x, k_T^2) \frac{S_{IT}^i k_T^j - S_{IT}^j k_T^i / 2}{M} \sigma^{\mu\bar{\mu}} \bar{n}_\mu + h_{IT}^\perp(x, k_T^2) \frac{(S_{IT}^i k_T^j - S_{IT}^j k_T^i / 2) k_T^j}{M^3} \sigma^{\mu\bar{\mu}} \bar{n}_\mu \right]$$

$$\Phi_{\bar{j}}^\mu(x, T)_{\text{twist-2}} = \int d^2 k_T k_T^\mu \Phi(x, k_T, T)_{\text{twist-2}} = \frac{M}{2} \left[f_{LT}^{(1)}(x) S_{LT}^\mu \bar{n} + g_{LT}^{(1)}(x) \varepsilon_T^\mu S_{LT,\mu} \gamma_S \bar{n} - h_{LL}^{(1)}(x) S_{LL} \sigma^{\mu\bar{\mu}} \bar{n}_\mu + h_{IT}^{(1)}(x) S_{IT}^{\perp\mu} \bar{n}_\mu \right]$$

$$= \frac{M}{4} \left[f_{LT}^{(+1)(1)}(x) + f_{LT}^{(-1)(1)}(x) \right] S_{LT}^\mu \bar{n}_\mu \equiv \frac{M}{2} f_{LT}^{(1)}(x) S_{LT}^\mu \bar{n}_\mu, \quad \tilde{\Phi}^\alpha = T\text{-even}, \quad \text{only } f_{LT}^{(1)}(x) \text{ is T-even, transverse-momentum moments of TMDs: } f^{(1)}(x) = \int d^2 k_T \frac{\tilde{k}_T^2}{2M^2} f(x, k_T^2)$$

Lightcone gauge $A^+ = 0$, $G^{+\alpha} = \partial_- A^\alpha(y^-) \rightarrow A^\alpha(y^-) = - \int_{-\infty}^y dz^- G^{+\alpha}(z^-) + A^\alpha(\infty) = - \int_{-\infty}^y dz^- \theta(z^- - y^-) G^{+\alpha}(z^-) + A^\alpha(\infty)$, $A^\alpha(y^-) = \int_{-\infty}^y dz^- G^{+\alpha}(z^-) + A^\alpha(-\infty) = \int_{-\infty}^y dz^- \theta(y^- - z^-) G^{+\alpha}(z^-) + A^\alpha(-\infty)$

$$A^\alpha(y^-) = \frac{A^\alpha(\infty) + A^\alpha(-\infty)}{2} - \frac{1}{2} \int_{-\infty}^y dz^- \varepsilon(z^- - y^-) G^{+\alpha}(z^-), \quad \varepsilon(z^- - y^-) = \theta(z^- - y^-) - \theta(y^- - z^-)$$

$$(\Phi_A^\mu)_{\bar{j}}(x_1, x_2, P, T) = \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} e^{i(x_1-x_2)p^* \xi_1} e^{i(x_2-x_1)p^* \xi_2} \langle P, T | \bar{\psi}_j(0) g A^\mu(\xi_2^-) \psi_i(\xi_1^-) | P, T \rangle$$

$$= \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} e^{i(x_1-x_2)p^* \xi_1} e^{i(x_2-x_1)p^* \xi_2} \left[\langle P, T | \bar{\psi}_j(0) g \left[\frac{A^\alpha(\infty^-) + A^\alpha(-\infty^-)}{2} - \frac{1}{2} \int_{-\infty^-}^\xi d\eta^- \varepsilon(\eta^- - \xi^-) G^{+\alpha}(\eta^-) \right] \psi_i(\xi_1^-) | P, T \rangle \right]$$

$$(\Phi_{A(\pm\infty^-)}^\alpha)_{\bar{j}}(x_1, P, T) = \frac{1}{P^+} \int \frac{d\xi_1}{2\pi} e^{i(x_1-x_2)p^* \xi_1} \langle P, T | \bar{\psi}_j(0) g A^\alpha(\xi_2^- = \pm\infty^-) \psi_i(\xi_1^-) | P, T \rangle$$

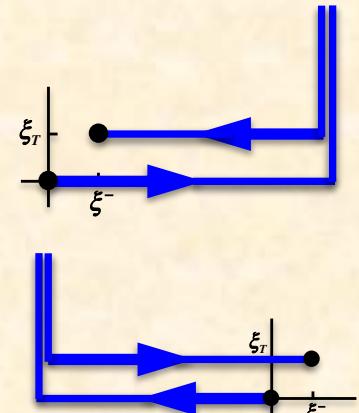
$$\varepsilon(\eta^- - \xi^-) = \frac{i}{\pi} \mathcal{P} \int_{-\infty}^\infty d\omega \frac{1}{2\pi} e^{-i\omega(\eta^- - \xi^-)}, \quad \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} e^{i(x_1-x_2)p^* \xi_1} e^{i(x_2-x_1)p^* \xi_2} \int_{-\infty^-}^\infty d\eta^- \varepsilon(\eta^- - \xi^-) = 2i\mathcal{P} \frac{1}{(x_1 - x_2)P^+} \int \frac{d\xi_1}{2\pi} e^{i(x_1-x_2)p^* \xi_1} \int_{-\infty^-}^\infty \frac{d\eta^-}{2\pi} e^{i(x_2-x_1)p^* \eta^-}$$

$$= \delta(x_1 - x_2) \frac{(\Phi_{A(\infty^-)}^\alpha)_{\bar{j}}(x_1, P, T) + (\Phi_{A(-\infty^-)}^\alpha)_{\bar{j}}(x_1, P, T)}{2} - \frac{1}{2} 2i\mathcal{P} \frac{1}{(x_1 - x_2)P^+} \int \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} e^{i(x_1-x_2)p^* \xi_1} e^{i(x_2-x_1)p^* \xi_2} \langle P, T | \bar{\psi}_j(0) g G^{+\alpha}(\xi_2^-) \psi_i(\xi_1^-) | P, T \rangle$$

$$= \delta(x_1 - x_2) \frac{(\Phi_{A(\infty^-)}^\alpha)_{\bar{j}}(x_1, P, T) + (\Phi_{A(-\infty^-)}^\alpha)_{\bar{j}}(x_1, P, T)}{2} - \mathcal{P} \frac{i}{(x_1 - x_2)P^+} (\Phi_G^\alpha)_{\bar{j}}(x_1, x_2, P, T)$$

$$\Phi_D^\alpha(x_1, x_2, P, T) = \delta(x_2 - x_1) \frac{1}{P^+} \tilde{\Phi}^\alpha(x, P, T) - \mathcal{P} \frac{i}{(x_1 - x_2)P^+} \Phi_G^\alpha(x_1, x_2, P, T)$$

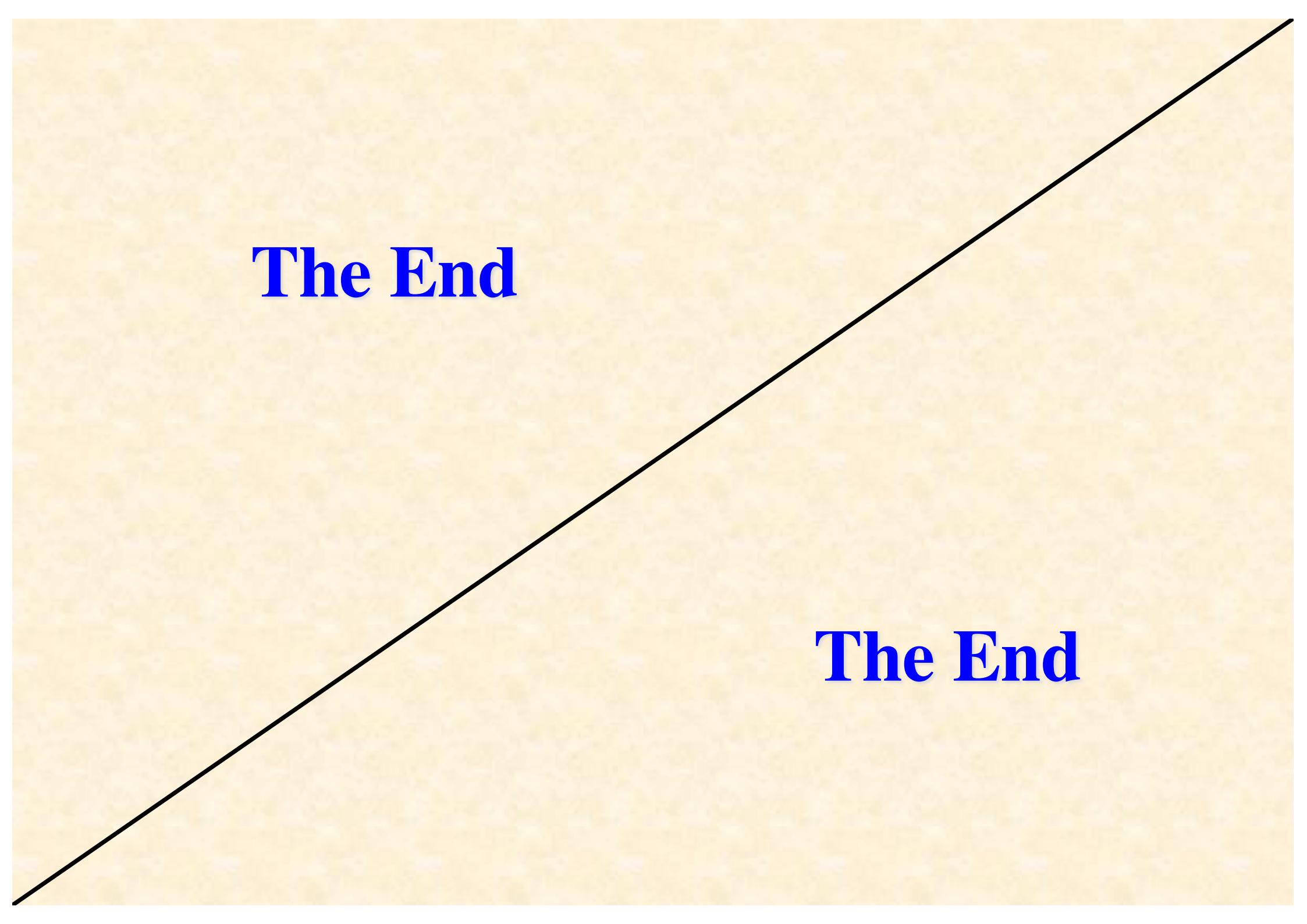
$$\rightarrow F_{D,LT}(x_1, x_2) = \delta(x_1 - x_2) f_{LT}^{(1)}(x_1) + \mathcal{P} \left(\frac{1}{x_1 - x_2} \right) F_{D,LT}(x_1, x_2), \quad G_{D,LT}(x_1, x_2) = \mathcal{P} \left(\frac{1}{x_1 - x_2} \right) G_{D,LT}(x_1, x_2), \quad H_{D,LL}^\perp(x_1, x_2) = \mathcal{P} \left(\frac{1}{x_1 - x_2} \right) H_{D,LL}^\perp(x_1, x_2), \quad H_{D,TT}(x_1, x_2) = \mathcal{P} \left(\frac{1}{x_1 - x_2} \right) H_{D,TT}(x_1, x_2)$$



Experimental projects and personal works on spin-1 physics

Appendix V

Year	Personal studies of SK with collaborators
1980	1988: EMC spin puzzle on proton
1990	1989: Hoodbhoy-Jaffe-Manohar on b_{1-4} [1983: Frankfurt-Strikman]
2000	1998: Courant's BNL report on polarized-deuteron acceleration at RHIC
2005	2005: HERMES measurement on b_1
2010	2011: JLab proposal on b_1
2016	2016: JLab LoI on gluon transversity
2020	2021: NICA paper on deuteron 2022: Fermilab proposal on deuteron
2025 ~	2025 ~ : JLab, Fermilab, NICA, LHCspin, ...
2030	2030's : EIC/EicC, ...
	Deuteron spin-1 physics will be developed significantly in 2020's and 2030's.
	1990: Close and SK on b_1 sum rule
	1999: Hino and SK on formalism of p-d Drell-Yan
	2008: SK on projection operators for b_{1-4}
	2010: SK on determination of tensor-polarized PDFs
	2016: SK and Song on tensor-polarized PDFs in p-d Drell-Yan
	2017: Cosyn, Dong, SK, Sargsian, on convolution estimate on b_1
	2020: SK and Song on gluon transversity in p-d Drell-Yan,
	2021: on TMDs and PDFs up to twist 4, twist-2 relation and sum rule for PDFs
	2022 : on relations from eq. of motion and Lorentz-invariance relation



The End

The End