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F Hautmann

Perturbative hysteresis and emergent resummation scales

based on

V Bertone, G Bozzi and F Hautmann

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- an observation on the treatment of theoretical uncertainties in the extraction of parton distribution functions (PDFs) from fits to data
- a practical method to take these uncertainties into account by using a "resummation scale" technique

Hysteresis effects in the perturbative solution of renormalization group equations

Consider
a generic renormalised quantity R, function of the strong coupling α_s and renormalisation scale μ, of the form

$$\frac{d \ln R}{d \ln \mu}(\mu, \alpha_s(\mu)) = \gamma(\alpha_s(\mu)),\qquad (1)$$

where the anomalous dimension γ can be expanded in powers of α_s as follows

$$\gamma(\alpha_s(\mu)) = \frac{\alpha_s(\mu)}{4\pi}\sum_{n=0}^{\infty}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \gamma_n.\qquad (2)$$

Introducing the evolution operator G connecting R at any two given scales μ_1 and μ_2,

$$R(\mu_1, \alpha_s(\mu_1)) = G(\mu_1, \mu_2)R(\mu_2, \alpha_s(\mu_2)),\qquad (3)$$

the effects we examine cause the identity $G(\mu_1, \mu_0)G(\mu_0, \mu_2) = G(\mu_1, \mu_2)$ to be violated for an arbitrary scale μ_0 as a result of the expansions in α_s

$$G(\mu_1, \mu_0)G(\mu_0, \mu_2) \neq G(\mu_1, \mu_2)\qquad (4)$$

due to formally subleading terms in the α_s expansion.

"perturbative hysteresis"

- *Examples of the hysteresis behavior in Eq.(4) have recently been studied in the context of transverse momentum dependent (TMD) Sudakov resummation by analytic methods [Billis, Tackmann, Talbert, JHEP 03 (2020) 182; Ebert, JHEP 02 (2022) 136] and angular-ordered parton branching methods [Keersmaekers, Lelek, van Kampen & H, Nucl. Phys. B 949 (2019) 114795].*

- *In this talk we move from the observation that hysteresis effects show up not only in Sudakov but also in single-logarithmic resummations, and investigate the case of QCD running coupling and PDF evolution.*
- *We will see that hysteresis corresponds to additional theory uncertainties (besides renormalization/factorization scales) which can be taken into account via introduction of "resummation scales".*

Contents

- *Hysteresis in running coupling and PDF evolution*

- *The g-function formalism (extended from soft-gluon resummations)*

- *"Emergent resummation scales" in coupling and PDF*

- *Phenomenological applications: the case of the F2 structure function*

The case of QCD running coupling

RGE for running coupling:

$$R = \alpha_s/4\pi = a_s \quad \text{and} \quad \gamma = -8\pi\beta/\alpha_s, \text{ where } \beta \text{ is the QCD beta function}$$

LL: closed-form. From NLL on: no closed-form, either numerical or analytic approximation.

- Analytic g-function formalism:

$$a_s^{N^k LL}(\mu) = a_s(\mu_0) \sum_{l=0}^{k} a_s^l(\mu_0) g_{l+1}^{(\beta)}(\lambda), \qquad (5)$$

with

$$\lambda = a_s(\mu_0)\beta_0 \ln\left(\frac{\mu_{\text{Res}}}{\mu_0}\right), \qquad (6)$$

where $\mu_{\text{Res}} = \kappa\mu$ is the "resummation" scale with $\kappa \sim 1$. The g-functions necessary up to NLL read

$$g_1^{(\beta)}(\lambda) = \frac{1}{1-\lambda},$$
$$g_2^{(\beta)}(\lambda) = \frac{1}{(1-\lambda)^2}\left[-\frac{\beta_1}{\beta_0}\ln(1-\lambda) - \beta_0 \ln\kappa\right]. \qquad (7)$$

The functional form of the $g_i^{(\beta)}$ for $i > 2$ is straightforwardly obtained from the corresponding $N^{i-1}LL$ expansion of the running coupling.

Eq.(5) gives rise to hysteresis: see Fig.1

- Numerical strategy to evaluate theory uncertainties due to missing subleading orders in the beta function:

To be specific, by displacing the scale μ by a factor ξ, we obtain

$$\bar{\beta}(\mu) = a_s(\xi\mu)\beta_0\left(1 + a_s(\xi\mu)\left[\frac{\beta_1}{\beta_0} - 2\beta_0 \ln\xi\right]\right) + \mathcal{O}(\alpha_s^3). \qquad (8)$$

This effectively defines a new β-function that differs from the original one by subleading corrections. The difference between the solution obtained with the original $\beta(\mu)$ and the one in Eq. (8) gives an estimate of the effect of higher-order corrections, much as variations of the resummation scale do for the analytic solution. In fact, it can be shown that at NLL accuracy the β-function generated by the analytic solution in Eq. (5) can be recast in the same form as Eq. (8) provided that $\kappa = \xi$.

Scale variations at the level of the beta function: see Fig. 2.
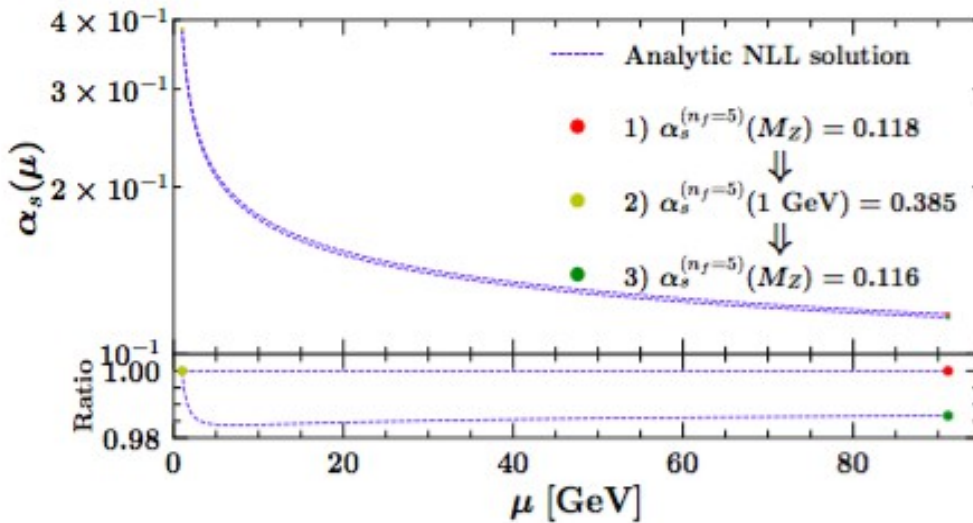
The case of QCD running coupling

FIG. 1. Perturbative hysteresis for the NLL evolution of the strong coupling α_s.

FIG. 2. Analytic and numerical evolution of the strong coupling α_s at NLL. The bands indicate the uncertainty computed by varying the factors κ and ξ in the range $[0.5:2]$.

- *Loop from mu_0 = M_Z to mu = 1 GeV back to mu_0 = M_Z: alpha_s, given by the NLL solution in Eq.(5), does not return to the original value.*
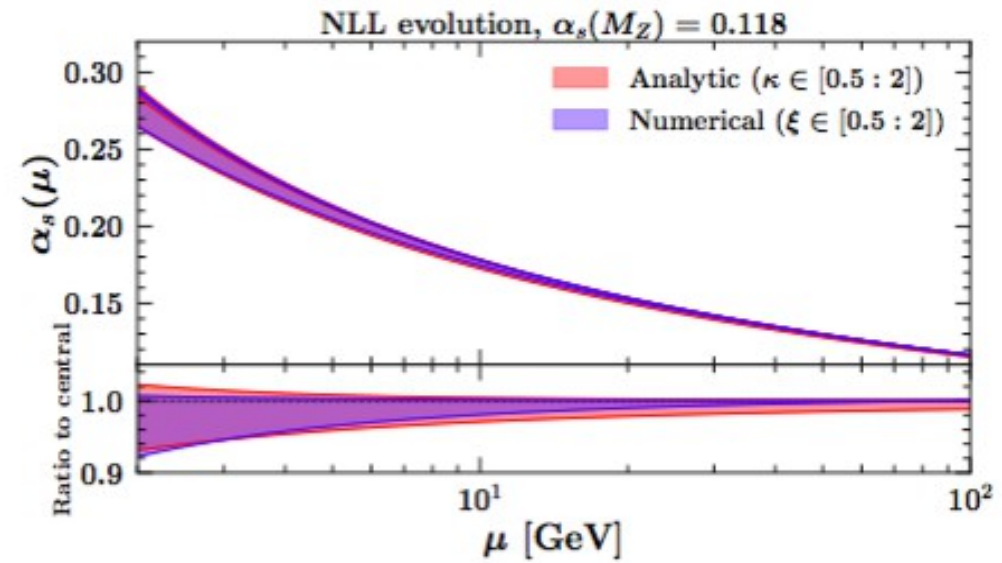This is the known hysteresis effect in RGE perturbative solutions.

- *Hysteresis is here traced back to RGE theory uncertainties due to uncalculated subleading orders in the evolution kernels.*
This can be evaluated by using the "emergent resummation scale" technique.

The case of PDF evolution

Next is the case of RGE solution for Melllin transform *f* of a PDF.

- *g*-function formalism for (non-singlet) PDFs in terms of anomalous dimension gamma and beta function:

$$f^{N^kLL}(\mu) = g_0^{(\gamma),N^kLL}(\lambda) \exp\left[\sum_{l=0}^{k} a_s^l(\mu_0) g_{l+1}^{(\gamma)}(\lambda)\right] f(\mu_0). \tag{9}$$

The *g*-functions for the NLL evolution read

$$g_0^{(\gamma),NLL}(\lambda) = 1 + a_s(\mu_0)\frac{1}{\beta_0}\left(\gamma_1 - \frac{\beta_1}{\beta_0}\gamma_0\right)\frac{\lambda}{1-\lambda},$$

$$g_1^{(\gamma)}(\lambda) = -\frac{\gamma_0}{\beta_0}\ln(1-\lambda),$$

$$g_2^{(\gamma)}(\lambda) = -\frac{\gamma_0}{\beta_0^2}\frac{\beta_1\ln(1-\lambda) + \beta_0^2\ln\kappa}{1-\lambda}. \tag{10}$$

The procedure can be extended to N^kLL accuracy by including the appropriate $g_i^{(\gamma)}$'s, with $i \le k+1$, along with the $\mathcal{O}(a_s^k)$ corrections to $g_0^{(\gamma),N^kLL}$. The *g*-functions in Eq. (9) are written in terms of the λ variable given in Eq. (6) automatically allowing for resummation-scale variations. Such variations can be used to probe higher-order corrections to the anomalous dimensions.

See hysteresis effect in Fig.3.

- Numerical strategy to evaluate theory uncertainties due to missing subleading orders in the anomalous dimensions:

To estimate higher-order corrections in the case of the numerical solution, we shift the argument of α_s appearing in the expansion of the anomalous dimension by a factor ξ. This effectively defines a new anomalous dimension differing from the previous one by subleading terms. At NLL it reads

$$\bar{\gamma}(\mu) = a_s(\xi\mu)\gamma_0 + a_s^2(\xi\mu)\left[\gamma_1 - \beta_0\gamma_0\ln\xi\right]. \tag{11}$$

See impact of scale variations on PDF evolution in Fig.4,

for the cases of analytic and numerical solutions.
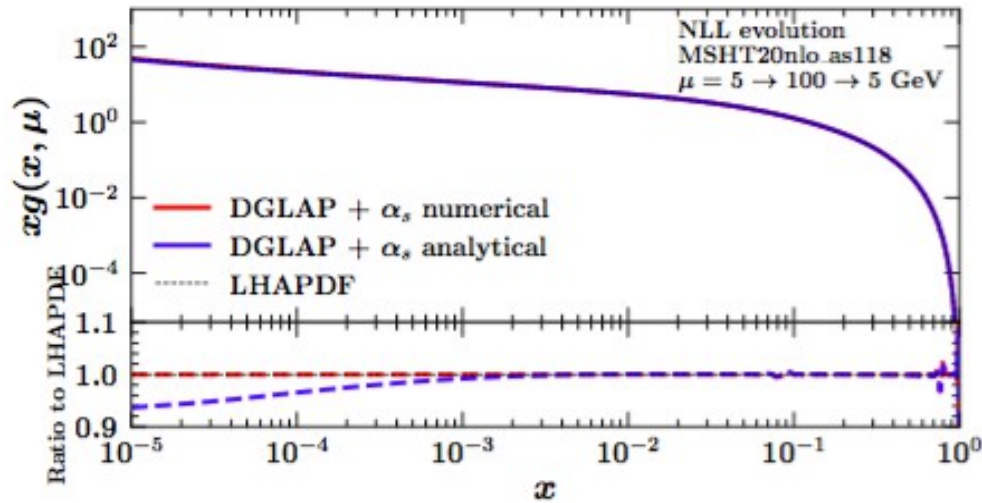
The case of PDF evolution

FIG. 3. Perturbative hysteresis for the NLL evolution of the gluon PDF.

FIG. 4. Analytic and numerical NLL evolution of the non-singlet combination $f_d - f_{\bar{d}} = f_{d^-}$ from $\mu_0 = 100$ GeV down to $\mu = 5$ GeV. The bands indicate the theoretical uncertainty computed by varying the factors κ and ξ in the range $[0.5 : 2]$.

- *Gluon PDF evolved from 5 GeV to 100 GeV and then back to 5 GeV, compared to the starting 5 GeV distribution in analytic (blue) and numerical (red) cases.*
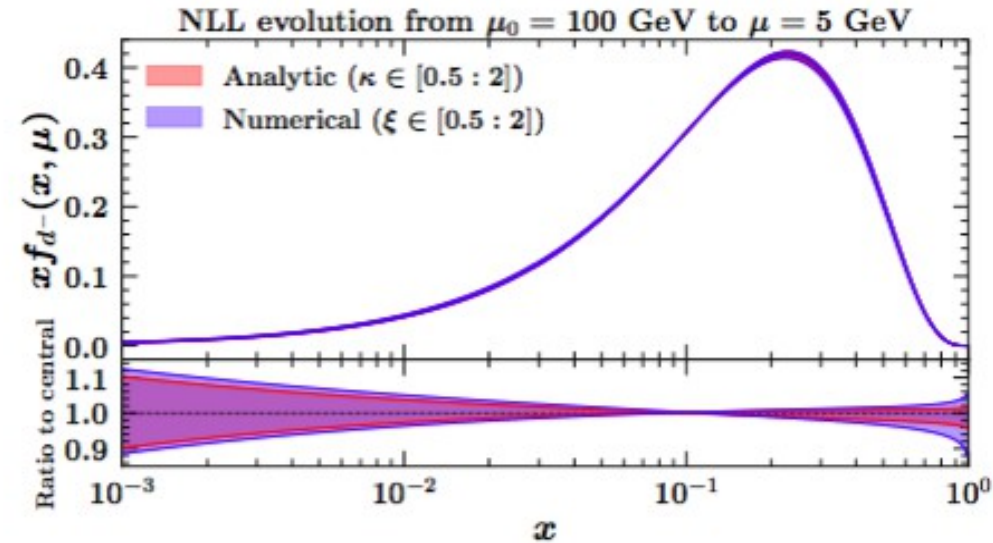
- *Resummation scale variations in PDF evolution through kappa and xi parameters.*

Implications for the structure function F2

- *What is the impact of RGE theory uncertainties on F2 predictions?*
  compare "resummation scale" uncertainties with the standard ones
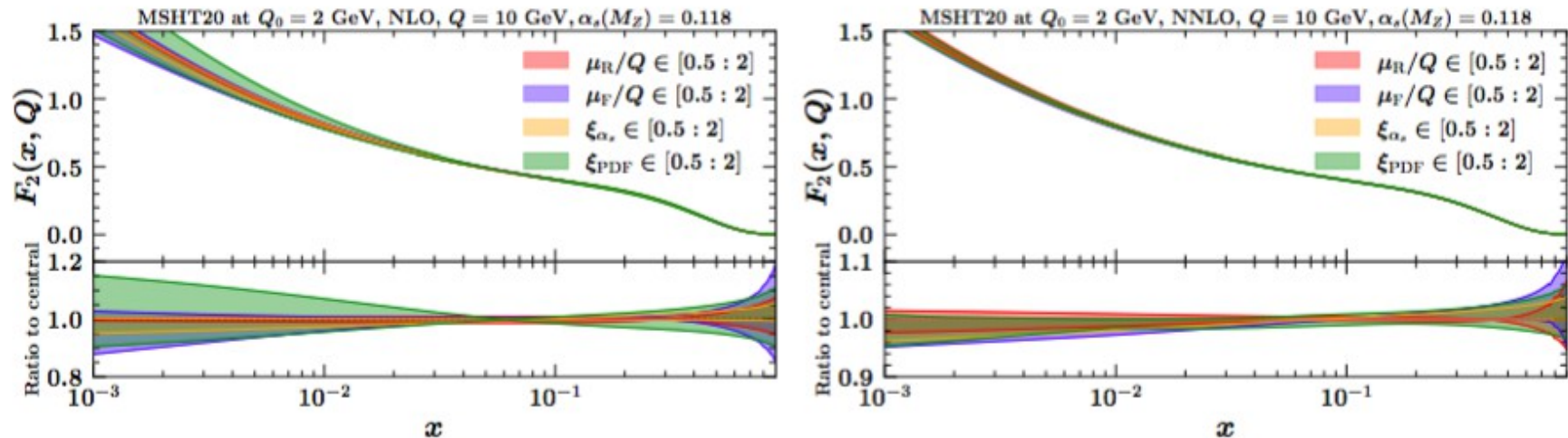 *from renormalization and factorization scales*

FIG. 5. The x-dependence of the structure function F_2 at NLO and NNLO in perturbation theory, with the uncertainty bands associated with variations of renormalisation and factorisation scales, μ_R and μ_F, and resummation scales ξ_{α_s} and ξ_{PDF}.

• resummation scale uncertainties found to be generally non-negligible with respect to renormalization and factorization scale uncertainties

• left panel (NLO) illustrates that xi_PDF dominates low-x region while mu_F dominates at largest x; band size reduced in right panel (NNLO)

• the importance of resummation scale effects, relative to mu_F and mu_R, increases with Q: eventually relevant also in the large-x region

Implications for the structure function F2

🔘 *Relative variation Delta-F2 / F2 as a function of Q, from resummation scales and renormalization/factorization scales*
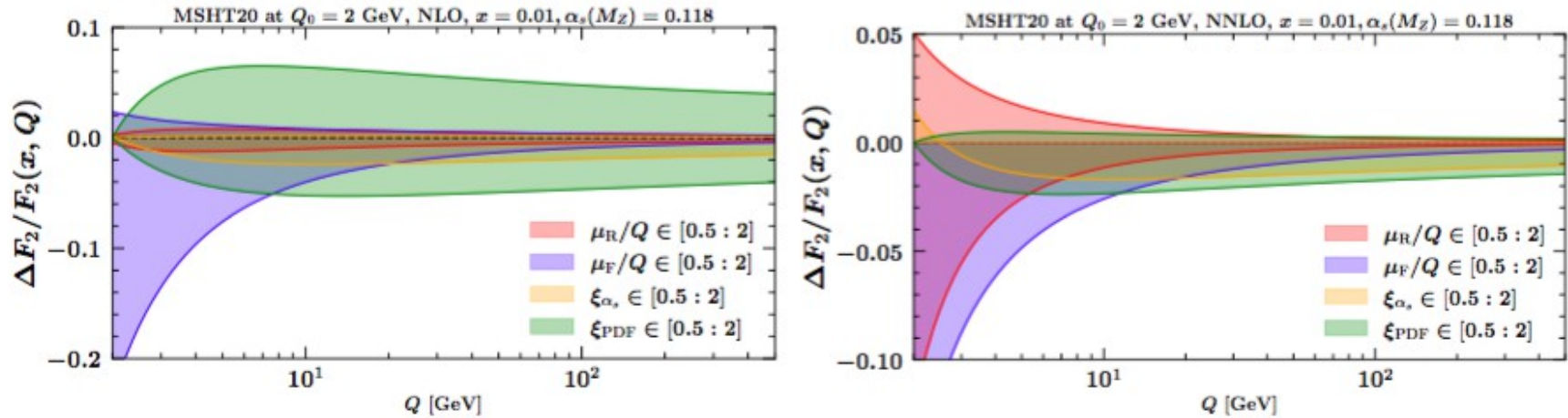

FIG. 6. Q-dependence of the relative variation $\Delta F_2/F_2$ at NLO and NNLO associated with variations of renormalisation and factorisation scales, μ_R and μ_F, and resummation scales ξ_{α_s} and ξ_{PDF}.

- The xi_PDF contribution starts from 0 and grows rapidly with Q, remaining sizeable out to large Q, while the mu_F contribution is largest at low Q and decreases with increasing Q.

- Analogously, the mu_R contribution is important at low Q and decreases with Q, while the xi_alpha is subdominant at low Q but becomes relevant at high Q.

- The xi_PDF contribution stays comparatively significant for large Q and small x, corresponding to higher-order perturbative corrections to PDF anomalous dimensions dominating the small-x region [Catani & H, Phys. Lett. B 315 (1993) 157] for sufficiently large Q.

Conclusions

- The difference between exact and perturbative solutions to RGE, in many common single-logarithmic problems such as the evolution of QCD coupling and PDFs, can be accounted for by variations of resummation scales. Such variations should be performed to assess theory uncertainties associated with RGE subleading-order terms.

- In many applications relevant for collider physics, notably the extraction of PDFs from global fits, these variations are commonly not performed. Theory uncertainties are thus not included, or underestimated.

- This points to the need for PDF extractions which include RGE theory uncertainties, e.g. by use of the resummation scale technique described here.

- This approach can be used in collider processes sensitive to collinear as well as TMD distributions. It is applicable both to PDF and to TMD extractions.