CERN-TH-2022-088

# Resummation Scales and the Assessment of Theoretical Uncertainties in Parton Distribution Functions [a]

V. Bertone[1], G. Bozzi[2,3] and F. Hautmann[4,5,6]
[1] *IRFU, CEA, Université Paris-Saclay, F-91191 Gif-sur-Yvette*
[2] *Dipartimento di Fisica, Università di Cagliari, Cittadella Universitaria, I-09042 Monserrato*
[3] *INFN, Sezione di Cagliari, Cittadella Universitaria, I-09042 Monserrato*
[4] *CERN, Theoretical Physics Department, CH 1211 Geneva*
[5] *Elementaire Deeltjes Fysica, Universiteit Antwerpen, B 2020 Antwerpen*
[6] *Theoretical Physics Department, University of Oxford, Oxford OX1 3PU*

We discuss perturbative solutions of renormalization group equations, and propose the use of resummation scale techniques in assessing theoretical uncertainties on the extraction of parton distribution functions from data.

Many applications of Quantum Chromodynamics (QCD) to precision calculations for high-energy collider physics involve the perturbative solution of renormalization group equations (RGEs). It has long been known that, at any given order of perturbation theory, the differences between RGE solutions obtained by methods which are equivalent up to formally subleading orders provide a significant source of theoretical uncertainty on predictions for physical observables.

In the literature of calculations based on QCD resummation, these uncertainties are commonly taken into account by a variety of different techniques, which appeal to the principle of introducing so-called resummation scales, of the order of the hard momentum-transfer scale of the process but otherwise arbitrary, and setting criteria to let them vary and evaluate the corresponding variation in the theoretical prediction.

In several other common applications of QCD to collider physics, on the other hand, resummation scale variations are not performed. A notable example is the determination of parton distribution functions (PDFs) from global fits to experimental data, [1,2,3,4,5,6,7] central to virtually any aspect of hadron collider physics. In this application, RGEs and their perturbative solutions are used for the evolution of the PDFs themselves and for the evolution of the running coupling. The fact that RGE theory uncertainties are not included and resummation scale variations are not performed leads to an underestimate of the uncertainties associated with the extraction of PDFs from global fits to data.

This point is illustrated in detail in our recent work [8]. There, we discuss various aspects of theory uncertainties from RGE perturbative solutions and propose resummation scale methods to characterize them. Applying this to deep-inelastic scattering (DIS) structure functions, we show that resummation scale effects are significant in kinematic regions relevant to PDF determinations. In this article we summarize some of the results from Ref. [8].

[a] Presented by F. Hautmann at the *29th Workshop on Deep-Inelastic Scattering and Related Subjects*, 2-6 May 2022.

We consider RGE of the general form

$$\frac{d\ln R}{d\ln\mu}(\mu,\alpha_s(\mu)) = \gamma(\alpha_s(\mu))\,, \tag{1}$$

where $R$ is a renormalized quantity, function of the strong coupling $\alpha_s$ and renormalization scale $\mu$, and $\gamma$ is its anomalous dimension, computable as a power series expansion in the coupling $\alpha_s$.

In Ref. [8] we examine several examples, at next-to-leading (NLO) and next-to-next-to-leading (NNLO) order, and analyze RGE solutions obtained by methods which differ by subleading-order terms.[b] To characterize the differences between such solutions, we introduce the evolution operator $G$ which connects $R$ at two scales $\mu$ and $\mu_0$, $R_\mu = G(\mu,\mu_0)R_{\mu_0}$, and trace the differences between solutions back to the fact that, due to subleading orders, one has, for a given $\mu'$, $G(\mu,\mu_0) \neq G(\mu,\mu')G(\mu',\mu_0)$. We refer to this as *perturbative hysteresis* [8].

For the application to PDF evolution, $R$ in Eq. (1) is to be thought of as the flavor multiplet of PDF Mellin moments, and $\gamma$ as the Mellin transform of DGLAP splitting functions. Besides, one has an RGE of the form (1) for the running coupling, in which $R$ is proportional to the coupling $\alpha_s$, and $\gamma$ to the QCD $\beta$ function. Hysteresis effects in both these RGEs are analyzed quantitatively in Ref. [8].

The extraction of PDFs $f_j(x,\mu)$ from collider data uses factorization formulas relating $f_j$ to physical observables $\Sigma$, e.g. DIS structure functions, with the schematic form

$$\Sigma(x,Q) = \sum_j \int dz H_j(z,Q,\alpha_s(\mu_R),\mu_F) f_j(x/z,\mu_F), \tag{2}$$

where $H_j$ are perturbatively computable hard-scattering functions, $\mu_F$ is the factorization scale, and $\mu_R$ is the renormalization scale. Standard QCD calculations estimate theoretical uncertainties associated with unknown higher orders in Eq. (2) by setting criteria for variations of the scales $\mu_F$ and $\mu_R$.

Ref. [8] observes that theoretical uncertainties on the extraction of PDFs arise both from $\mu_F$ and $\mu_R$ variations in Eq. (2) and from unknown higher orders in the kernels $\gamma$ in Eq. (1), and proposes to take the latter into account by resummation scale techniques. To this end, two different but essentially equivalent methods are formulated. One is based on expressing the evolution operator $G$ via the analytic formalism of $g$-functions, [8] which is frequently applied in soft-gluon resummation calculations. Schematically, at the $k$-th order of logarithmic accuracy $\mathrm{N}^k\mathrm{LL}$, this is of the form

$$G^{\mathrm{N}^k\mathrm{LL}} \sim g_0^{\mathrm{N}^k\mathrm{LL}}(\lambda)\exp\left[\sum_{l=0}^{k}\alpha_s^l(\mu)g_{l+1}(\lambda)\right] \quad , \quad \lambda \sim \alpha_s(\mu)\beta_0\ln(\mu^{(\mathrm{Res})}/\mu)\,, \tag{3}$$

where the $g$'s are perturbatively computable functions of the scaling variable $\lambda$, and $\mu_{\mathrm{Res}}$ is the resummation scale. Explicit expressions for the $g$-functions up to $k=3$ are given in Ref. [12]. Another method is based on displacing the argument of the coupling appearing in the perturbative expansion of $\gamma$, so as to obtain a new effective anomalous dimension differing from the previous one by subleading terms. This method is well-suited for numerical evaluations. In either method, one ends up introducing resummation scales $\mu_{\mathrm{PDF}}^{(\mathrm{Res})}$ for the PDF evolution and $\mu_{\alpha_s}^{(\mathrm{Res})}$ for the coupling evolution.

As an application, we illustrate the case in which we take $\Sigma$ in Eq. (2) to be the DIS structure function $F_2(x,Q)$. We perform variations of renormalization and factorization scales, $\mu_R$ and $\mu_F$, and of resummation scales, which we parameterize as $\mu_{\alpha_s}^{(\mathrm{Res})} = \xi_{\alpha_s}Q$ and $\mu_{\mathrm{PDF}}^{(\mathrm{Res})} = \xi_{\mathrm{PDF}}Q$. Fig. 1

[b] The examples considered in Ref. [8] are single-logarithmic resummation problems. Similar effects in double-logarithmic Sudakov problems have been observed e.g. in Refs. [9,10,11].

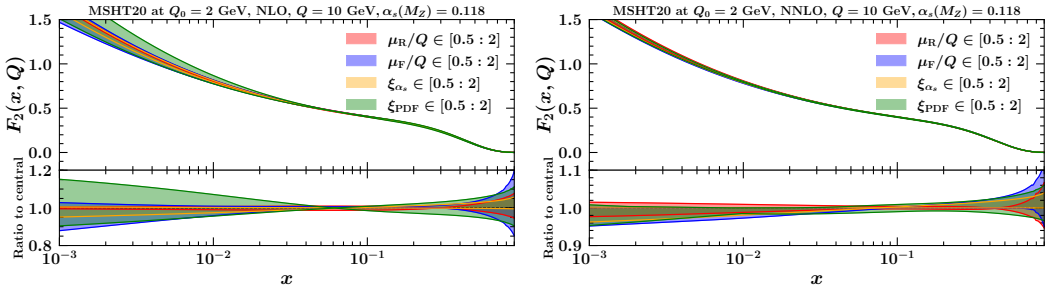


Figure 1 – The structure function $F_2$ from Ref. [8] versus $x$, at NLO and NNLO in perturbation theory, with the uncertainty bands associated with variations of renormalization and factorization scales, $\mu_R$ and $\mu_F$, and resummation scale parameters $\xi_{\alpha_s}$ and $\xi_{\rm PDF}$. We use MSHT20 PDFs at $Q_0 = 2$ GeV and $\alpha_s(M_Z) = 0.118$ as RGE inputs.

shows results at NLO (left panel) and NNLO (right panel). Resummation scale uncertainties are observed to be generally non-negligible with respect to the renormalization and factorization scale uncertainties in the kinematic region considered. In the left hand side panel, we see that the effect of the resummation scale parameter $\xi_{\rm PDF}$ dominates the low-$x$ region while the effect of the factorization scale $\mu_F$ dominates at the largest $x$. The right hand side panel illustrates that the size of the uncertainty bands is reduced at NNLO. The behavior above reflects higher-order corrections to $F_2$ at small $x$ being dominated by flavor-singlet quark anomalous dimensions. [13,14]

The results shown in Fig. 1 are for $Q = 10$ GeV. The importance of resummation scale effects, relative to that of renormalization and factorization scale effects, increases with $Q$ (and is eventually becoming relevant for increasing $x$). To illustrate the behavior of the uncertainty bands with varying $Q$, in Fig. 2 we plot the $Q$-dependence of the relative variation $\Delta F_2/F_2$. The $\xi_{\rm PDF}$ contribution starts from zero and grows rapidly with $Q$, remaining sizeable out to large $Q$ while the $\mu_F$ contribution is largest at low $Q$ and decreases with increasing $Q$. Analogously, the $\mu_R$ contribution is important at low $Q$ and decreases with $Q$, while the $\xi_{\alpha_s}$ is subdominant at low $Q$ but becomes relevant at high $Q$.

The above numerical results indicate that RGE uncertainties are relevant in kinematical regions which influence current and forthcoming PDF extractions. They will also be relevant in regions explored at future lepton-hadron colliders [15,16]. As the $\xi_{\rm PDF}$ contribution remains significant at large $Q$, RGE uncertainties are important for high-scale PDF probes such as very energetic jets and top quarks.

To conclude, we note that the method we have described can be used in processes sensitive to collinear as well as transverse momentum dependent (TMD) [17] distributions. It will be applicable in extractions both of PDF and of TMD.

**Acknowledgment.** We warmly thank N. Armesto and the organizers of the 2022 Deep-Inelastic Scattering Workshop for putting up a very interesting conference and for allowing us to present these results from remote.

## References

1. R. D. Ball, J. Butterworth, A. M. Cooper-Sarkar, A. Courtoy, T. Cridge, A. De Roeck, J. Feltesse, S. Forte, F. Giuli and C. Gwenlan, *et al.* arXiv:2203.05506 [hep-ph].
2. R. D. Ball, S. Carrazza, J. Cruz-Martinez, L. Del Debbio, S. Forte, T. Giani, S. Iranipour, Z. Kassabov, J. I. Latorre and E. R. Nocera, *et al.* Eur. Phys. J. C **82** (2022) 428 [arXiv:2109.02653 [hep-ph]].
3. R. Abdul Khalek *et al.* [NNPDF], Eur. Phys. J. C **79** (2019) 931 [arXiv:1906.10698 [hep-ph]].

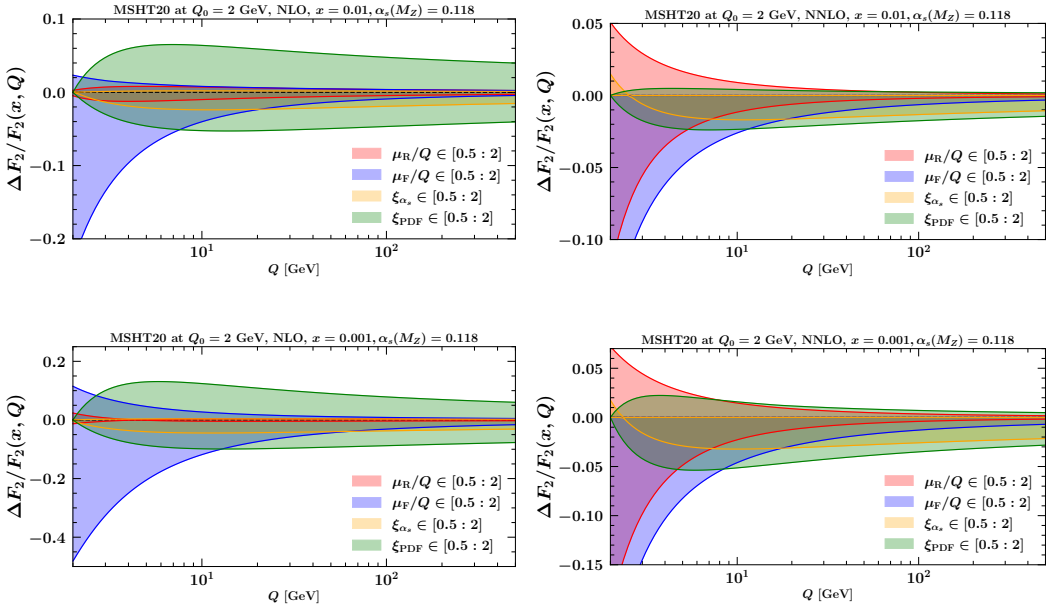


Figure 2 – $Q$-dependence of the relative variation $\Delta F_2/F_2$ for $x = 10^{-2}$ (top) and $x = 10^{-3}$ (bottom), at NLO (left) and NNLO (right), associated with variations of renormalization and factorization scales, $\mu_R$ and $\mu_F$, and resummation scale parameters $\xi_{\alpha_s}$ and $\xi_{\rm PDF}$.


4. S. Bailey, T. Cridge, L. A. Harland-Lang, A. D. Martin and R. S. Thorne, Eur. Phys. J. C **81** (2021) 341 [arXiv:2012.04684 [hep-ph]].
5. T. J. Hou, J. Gao, T. J. Hobbs, K. Xie, S. Dulat, M. Guzzi, J. Huston, P. Nadolsky, J. Pumplin and C. Schmidt, *et al.* Phys. Rev. D **103** (2021) 014013 [arXiv:1912.10053 [hep-ph]].
6. S. Alekhin, J. Bluemlein, S. Moch and R. Placakyte, Phys. Rev. D **96** (2017) 014011 [arXiv:1701.05838 [hep-ph]].
7. H. Abramowicz *et al.* [H1 and ZEUS], Eur. Phys. J. C **75** (2015) 580 [arXiv:1506.06042 [hep-ex]].
8. V. Bertone, G. Bozzi and F. Hautmann, Phys. Rev. D **105** (2022) 096003 [arXiv:2202.03380 [hep-ph]].
9. M. A. Ebert, JHEP **02** (2022) 136 [arXiv:2110.11360 [hep-ph]].
10. G. Billis, F. J. Tackmann and J. Talbert, JHEP **03** (2020) 182 [arXiv:1907.02971 [hep-ph]].
11. F. Hautmann, L. Keersmaekers, A. Lelek and A.M. Van Kampen, Nucl. Phys. B **949** (2019) 114795 [arXiv:1908.08524 [hep-ph]].
12. V. Bertone *et al.*, in preparation.
13. S. Catani and F. Hautmann, Phys. Lett. B **315** (1993) 157; Nucl. Phys. B **427** (1994) 475; arXiv:hep-ph/9408251.
14. R. K. Ellis, F. Hautmann and B. R. Webber, Phys. Lett. B **348** (1995) 582; arXiv:hep-ph/9506303.
15. P. Agostini *et al.* [LHeC and FCC-he Study Group], J. Phys. G **48** (2021) 110501 [arXiv:2007.14491 [hep-ex]].
16. Y. Hatta *et al.*, arXiv:2002.12333 [hep-ph].
17. R. Angeles-Martinez *et al.*, Acta Phys. Polon. B **46** (2015) 2501 [arXiv:1507.05267 [hep-ph]].