

The 3-loop Anomalous Dimensions from Off-shell Operator Matrix Elements

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TTP KARLSRUHE

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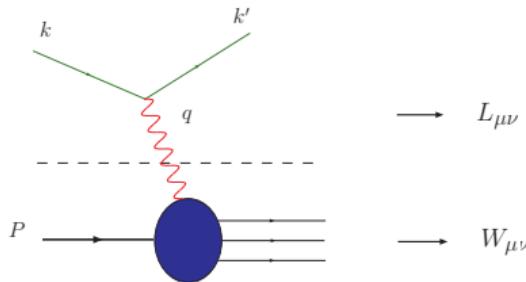
[based on: Nucl.Phys.B948 (2019) 114753, Nucl.Phys.B971 (2021) 115542, JHEP 01 (2022) 193, 2202.03216 (Nucl.Phys.B in print)]



Outline

- 1 Introduction
- 2 Status
- 3 Calculation
- 4 Results
- 5 Conclusions and Outlook

Theory of Deep Inelastic Scattering



- Kinematic invariants:

$$Q^2 = -q^2, \quad x = \frac{Q^2}{2P \cdot q}$$

- The cross section factorizes into leptonic and hadronic tensor:

$$\frac{d^2\sigma}{dQ^2 dx} \sim L_{\mu\nu} W^{\mu\nu}$$

- The hadronic tensor can be expressed through structure functions:

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, | [J_\mu^{\text{em}}(\xi), J_\nu^{\text{em}}(\xi)] | P \rangle \\ &= \frac{1}{2x} \left(g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) \\ &\quad + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho S^\sigma}{q \cdot P} g_1(x, Q^2) + i\epsilon_{\mu\nu\rho\sigma} \frac{q^\rho (q \cdot PS^\sigma - q \cdot SP^\sigma)}{(q \cdot P)^2} g_2(x, Q^2) \end{aligned}$$

- F_L , F_2 , g_1 and g_2 contain contributions from both, charm and bottom quarks.

The Evolution Equations

- The anomalous dimensions govern the scale evolution of the non-perturbative parton distribution functions:

$$\frac{d}{d \ln(\mu^2)} \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix} = \sum_{k=0}^{\infty} a_s^{k+1} \begin{pmatrix} \gamma_{qq}^{(k)} & \gamma_{qg}^{(k)} \\ \gamma_{gq}^{(k)} & \gamma_{gg}^{(k)} \end{pmatrix} \cdot \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix}$$

$$\frac{d}{d \ln(\mu^2)} q^{\text{NS},\pm,(s)}(N) = \sum_{k=0}^{\infty} a_s^{k+1} \gamma_{qq}^{\text{NS},\pm,(s)} \cdot q^{\text{NS},\pm,(s)}(N)$$

with $\gamma_{ij} = \sum_{k=0}^{\infty} \gamma_{ij}^{(k)}$ and $\Sigma(N) = \sum_{k=1}^{n_f} q_k(N) + \bar{q}_k(N)$

- 3 non-singlet anomalous dimensions: $\gamma_{qq}^{\text{NS},\pm}, \gamma_{qq}^{\text{NS},(s)}$
- 4 singlet anomalous dimensions: $\gamma_{qq}^{\text{PS}}, \gamma_{qg}, \gamma_{gq}, \gamma_{gg}$
- All are present in the unpolarized and polarized case: $\gamma_{ij} \rightarrow \Delta \gamma_{ij}$
- 2 additional anomalous dimensions for transversity: $\gamma_{qq}^{\text{tr},\pm}$

The Status of the QCD Anomalous Dimensions

Unpolarized Anomalous Dimensions

■ 1 Loop

- Gross, Wilczek 1974; Georgi, Politzer 1974

■ 2 Loop

- Floratos, Ross, Sachrajda 1977, 1978
- Gonzalez-Arroyo, Lopez, Yndurain 1979, Gonzalez-Arroyo and Lopez 1980
- Curci, Furmanski and Petronzio 1980, Furmanski and Petronzio 1980
- Floratos, Kounnas and Lacaze 1981
- Hamberg and van Neerven 1992 [all former flaws clarified.]
- Ellis and Vogelsang 1996; Matiounine, Smith and van Neerven 1998; Moch and Vermaseren 1999

■ 3 Loop

- Moch, Vermaseren, Vogt 2004
- Ablinger et al. 2010-2017 ($\sim T_F$ from massive OMEs)
- Anastasiou et al. 2015, Duhr, Dulat and Mistlberger, 2020 (implicit)
- Blümlein, Marquard, Schönwald and Schneider 2021 (non-singlet)

■ 4 and 5 Loop Moments

- Baikov and Chetyrkin 2006, Baikov, Chetyrkin and Kühn 2015
- Velizhanin 2012, 2014
- Ruijl et al. 2016, Davies et al., Moch et al., 2017, 2021
- Herzog et al. 2019

The Status of the QCD Anomalous Dimensions

Polarized Anomalous Dimensions

■ 1 Loop

- Sasaki 1975
- Ahmed and Ross 1975

■ 2 Loop

- Mertig and van Neerven 1995
- Vogelsang 1995
- Matiounine, Smith and van Neerven 1998

■ 3 Loop

- Moch, Vermaseren, Vogt 2014, 2015
- Behring et al. 2019 ($\sim T_F$ from massive OMEs)
- Blümlein, Marquard, Schönwald and Schneider 2021

The Status of the QCD Anomalous Dimensions

Transversity Anomalous Dimensions

■ 1 Loop

- Artru and Mekhfi 1990
- Ioffe and Khojamirian 1995
- Blümlein 2001 [and many other papers]

■ 2 Loop

- Hayashigaki, Kanazawa and Koike 1995
- Kumano and Miyama 1997
- Vogelsang 1998
- Blümlein, Klein, Tödtli 2009 (including 3-loop TF moments)

■ 3 Loop

- Velizhanin 2012 (diophantine guess)
- Blümlein, Marquard, Schönwald and Schneider 2021

Computational Methods

■ Forward Compton Amplitude

- Used by: [Moch, Vermaseren, Vogt 2004-2014](#)
- Is a gauge invariant framework, but needs 4-point functions.
- Necessitates scalar or tensor currents to obtain the gluonic anomalous dimensions.

■ Massive On-Shell Operator Matrix Elements

- Used by: [Ablinger et al. 2014-2019](#)
- Allows the calculation of all contributions $\sim T_F$.
- Is a gauge invariant framework.

■ Massless Off-Shell Operator Matrix Elements (this talk)

- Allows to calculate all anomalous dimensions.
- Is a gauge dependent framework.
- In the unpolarized singlet case mixing with new OMEs occurs, the polarized and non-singlet cases are unaffected.
- The previous 2-loop studies by [Matiounine, van Neerven and Smith 1998](#) contain some errors.

Technical Aspects of the Calculation

Unpolarized

- Twist-2 local operators:

$$O_{q,r;\mu_1,\dots,\mu_N}^{\text{NS}} = i^{N-1} \mathbf{S} \left[\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{traces},$$

$$O_{q;\mu_1,\dots,\mu_N}^{\text{S}} = i^{N-1} \mathbf{S} \left[\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi \right] - \text{traces},$$

$$O_{g;\mu_1,\dots,\mu_N}^{\text{S}} = 2i^{N-2} \mathbf{SSp} \left[F_{\mu_1 \alpha}^a D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N}^{\alpha,a} \right] - \text{traces}$$

- OMEs are defined as expectation values of the operators between external particles:

$$A_{ij}^{(I)} = \langle j(p) | O_i^I | j(p) \rangle, \quad \text{with } i,j = q,g$$

- There are different Lorentz structures contributing to the OMEs, e.g.

$$A_{qq}^{\text{NS}} = \left[\Delta A_{qq}^{\text{NS,phys}} + \not{p} \frac{\Delta.p}{p^2} A_{qq}^{\text{NS,eom}} \right] (\Delta.p)^{N-1}, \Delta A_{gg,\mu\nu} = \epsilon_{\mu\nu\alpha\beta} \Delta^\alpha p^\beta \Delta A_{gg}^{\text{phys}} (\Delta.p)^{N-2}.$$

- We can extract the splitting functions from the physical parts of the OMEs.

Technical Aspects of the Calculation

Polarized

- Twist-2 local operators:

$$O_{q,r;\mu_1,\dots,\mu_N}^{\text{NS}} = i^{N-1} S \left[\bar{\psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{traces},$$

$$O_{q;\mu_1,\dots,\mu_N}^S = i^{N-1} S \left[\bar{\psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi \right] - \text{traces},$$

$$O_{g;\mu_1,\dots,\mu_N}^S = 2i^{N-2} S S p \left[\frac{1}{2} \epsilon_{\mu_1 \alpha \beta \gamma} F^{\beta \gamma, a} D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N}^{\alpha, a} \right] - \text{traces}$$

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- We can extract the splitting functions from the physical parts of the OMEs.

Technical Aspects of the Calculation

Transversity

- Twist-2 local operators:

$$O_{q,r;\mu,\mu_1,\dots,\mu_N}^{\text{NS}} = i^{N-1} S \left[\bar{\psi} \sigma_{\mu\mu_1} D_{\mu_2} \dots D_{\mu_N} \frac{\lambda_r}{2} \psi \right] - \text{traces}$$

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Technical Aspects of the Calculation

Transversity

- Twist-2 local operators:

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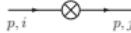
- There are different Lorentz structures contributing to the OMEs, e.g.

$$A_{qq}^{\text{NS}} = \left[\Delta A_{qq}^{\text{NS,phys}} + p \frac{\Delta \cdot p}{p^2} A_{qq}^{\text{NS,eom}} \right] (\Delta \cdot p)^{N-1}, \Delta A_{gg,\mu\nu} = \epsilon_{\mu\nu\alpha\beta} \Delta^\alpha p^\beta \Delta A_{gg}^{\text{phys}} (\Delta \cdot p)^{N-2}.$$

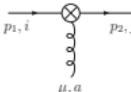
- We can extract the splitting functions from the physical parts of the OMEs.

Technical Aspects of the Calculation

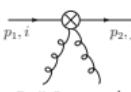
The OMEs are calculated using the QCD Feynman rules together with operator insertions:



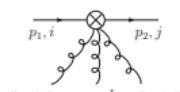
$$\delta^{ij} \Delta \gamma_{\pm} (\Delta \cdot p)^{N-1}, \quad N \geq 1$$



$$g t_{ji}^{\mu} \Delta^{\mu} \Delta \gamma_{\pm} \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2}, \quad N \geq 2$$



$$g^2 \Delta^{\mu} \Delta^{\nu} \Delta \gamma_{\pm} \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2} \\ [(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1}], \quad N \geq 3$$



$$g^3 \Delta_{\mu} \Delta_{\nu} \Delta_{\rho} \Delta \gamma_{\pm} \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta \cdot p_2)^j (\Delta \cdot p_1)^{N-m-2} \\ [(t^a t^b t^c)_{ji} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} + (t^a t^c t^b)_{ji} (\Delta p_4 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} + (t^b t^a t^c)_{ji} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_5 + \Delta p_1)^{m-l-1} + (t^b t^c t^a)_{ji} (\Delta p_3 + \Delta p_5 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1} + (t^c t^a t^b)_{ji} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_4 + \Delta p_1)^{m-l-1} + (t^c t^b t^a)_{ji} (\Delta p_3 + \Delta p_4 + \Delta p_1)^{l-j-1} (\Delta p_3 + \Delta p_1)^{m-l-1}], \quad N \geq 4$$

$$\gamma_+ = 1, \quad \gamma_- = \gamma_5.$$

$$\begin{aligned} & \text{Feynman diagram with 2-gluon vertex and 2-gluon loop insertion} \\ & \frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2} \\ & \left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right], \quad N \geq 2 \end{aligned}$$

$$\begin{aligned} & \text{Feynman diagram with 3-gluon vertex and 2-gluon loop insertion} \\ & -ig \frac{1+(-1)^N}{2} f^{abc} \left(\begin{aligned} & [(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu)] (\Delta \cdot p_1)^{N-2} \\ & + \Delta_\lambda [\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu] \\ & \times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j} \\ & + \left\{ \begin{array}{l} p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_1 \\ \mu \rightarrow \nu \rightarrow \lambda \rightarrow \mu \end{array} \right\} + \left\{ \begin{array}{l} p_1 \rightarrow p_3 \rightarrow p_2 \rightarrow p_1 \\ \mu \rightarrow \lambda \rightarrow \nu \rightarrow \mu \end{array} \right\} \end{aligned} \right), \quad N \geq 2 \end{aligned}$$

$$\begin{aligned} & \text{Feynman diagram with 4-gluon vertex and 3-gluon loop insertion} \\ & g^2 \frac{1+(-1)^N}{2} \left(\begin{aligned} & f^{abc} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) \\ & + f^{fac} f^{bdc} O_{\mu\nu\lambda\sigma}(p_1, p_3, p_2, p_4) + f^{fad} f^{bec} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \\ & O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \left\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2} \right. \\ & \left. + [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i} \right. \\ & \left. - [p_{1,\mu} \Delta_\sigma - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i} \right. \\ & \left. + [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma] \right. \\ & \left. \times \sum_{i=0}^{N-4} \sum_{j=0}^i (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \right\} \\ & - \left\{ \begin{array}{l} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{array} \right\} - \left\{ \begin{array}{l} p_2 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{array} \right\} + \left\{ \begin{array}{l} p_1 \leftrightarrow p_2, p_2 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \lambda \leftrightarrow \sigma \end{array} \right\}, \quad N \geq 2 \end{aligned} \right)$$

The polarized Feynman rules are different and also the 5-gluon operator is needed.

Technical Aspects of the Calculation

In the polarized case we use the Larin scheme:

$$\gamma_5 = \frac{i}{24} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma ,$$

$$\epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\lambda\tau\gamma} = -\text{Det} [g_\omega^\beta] , \quad \beta = \alpha, \lambda\tau, \gamma; \omega = \mu, \nu, \rho, \sigma.$$

Transition to the so-called M-scheme is known up to $\mathcal{O}(\alpha_s^3)$ [Matiounine, Neerven, Smith '98].

Renormalization:

- In the unpolarized case this requires the mixing with unphysical (alien) operators.
⇒ Recent progress for fixed moments [Falcioni, Herzog '22].
- Even after renormalization the OMEs remain gauge dependent, but the extracted anomalous dimensions are gauge independent.

Technical Aspects of the Calculation

The program chain:

- Diagram generation: QGRAF [Nogueira '91]
- Lorentz and Dirac algebra: TFORM [Ruijl, Ueda, Vermaseren, Tentyukov '00-'17]
- Color algebra: Color [Ritbergen, Schellekens, Vermaseren '99]
- IBP reduction and differential equations: Crusher [Marquard, Seidel]
- Initial values: custom code using the master integrals from [Germann et al '10, Lee, Smirnov, Smirnov '10]
- Method to compute large number of moments: SolveCoupledSystems [Blümlein, Schneider '17]
- Determination of recurrences from large number of moments: Guess [Kauers '09-'15]
- Solution of the recurrences: Sigma [Schneider '07-]
- Reduction and inverse Mellin transformation: HarmonicSums [Ablinger '10-]

Technical Aspects of the Calculation

We consider the unpolarized 3-loop singlet case (all other cases are smaller):

- Number of diagrams: 3324
- Number of moments needed to establish recurrences:
 - $\gamma_{qq}^{(2),PS}$: 588
 - $\gamma_{qg}^{(2)}$: 1248
 - $\gamma_{gg}^{(2)}$: 1274
 - $\gamma_{gg}^{(2)}$: 1860
- Largest difference equation: d=373, o=16
- Overall computation time: ~ 3 weeks
- The 3-loop anomalous dimensions can be represented by 10 basis harmonic sums:

$$\{S_1, S_{2,1}, S_{-2,1}, S_{-3,1}, S_{-4,1}, S_{2,1,1}, S_{-2,1,1}, S_{2,1,-2}, S_{-31,1}, S_{-2,1,1,1}\}$$

(Some) Results: $\gamma_{qq}^{\text{NS},(s)}$

- $\gamma_{qq}^{\text{NS},(s)}$ comes from the interference of vector and axial-vector couplings.
- It stems from considering the **odd** moments of the unpolarized OME (only pure singlet diagrams contribute).
- [Moch, Vermaseren, Vogt '04; Blümlein, Marquard, Schönwald, Schneider '21]

$$\begin{aligned}\gamma_{qq}^{\text{NS},(s)} = & 4N_f \frac{d_{abc} d^{abc}}{N_c} \left[\frac{2P_{60}}{(N-1)N^5(N+1)^5(N+2)} - \frac{P_{61}}{(N-1)N^4(N+1)^4(N+2)} S_1 \right. \\ & - \left(\frac{2P_{62}}{(N-1)N^3(N+1)^3(N+2)} + \frac{4(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} S_1 \right) S_{-2} \\ & \left. - \frac{N^2+N+2}{N^2(N+1)^2} [S_3 - 2S_{-3} + 4S_{-2,1}] \right]\end{aligned}$$

(Some) Results: $\gamma_{qq}^{\text{NS},+}$

$$\begin{aligned}
 \gamma_{\text{NS}}^{(2),+} = & \frac{1}{2} [1 + (-1)^N] \\
 & \times \left[\text{C}_F^2 \left\{ \text{C}_A \left[\frac{72P_3}{N^2(1+N)^2} \zeta_1 + \frac{32P_{15}}{9N^2(1+N)^2} S_{-2,1} - \frac{16P_{17}}{9N^2(1+N)^2} S_1 + \frac{P_{33}}{18N^4(1+N)^4} \right. \right. \right. \\
 & + \left. \left. \left. \left(-\frac{16P_9}{9N^4(1+N)^4} - \frac{488S_2}{9} + \frac{64(-24+31N+31N^2)}{3N(1+N)} S_3 + 320S_4 - 1024S_{3,1} \right. \right. \right. \\
 & + \left. \left. \left. \left. + \frac{64(-8+31N+31N^2)}{3N(1+N)} S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-3,1,1} \right) S_1 + \left(256S_3 \right. \right. \right. \\
 & + 1792S_{-2,1} \right) S_1^2 + \left(\frac{4P_{19}}{9N^2(1+N)^2} - 832S_3 - 5248S_{-2,1} \right) S_2 + \frac{352S_2^2}{3} \\
 & + \frac{16(-30+15N+15N^2)}{3N(1+N)} S_4 + \left(-\frac{16P_{22}}{9N^2(1+N)^2} + \left(-\frac{64P_5}{9N^2(1+N)^2} - 256S_7 \right) S_1 \right. \\
 & + \left. \left. \left. + 32(12+31N+31N^2)S_6 + 64S_4 + 5376S_{2,1} - 3848S_{-2,1} + 576S_8 \right) S_{-2,2} \right. \\
 & + \left(\frac{32(8+3N+3N^2)}{N(1+N)} + 512S_7 \right) S_2^2 + \left(\frac{32(108+31N+31N^2)}{3N(1+N)} S_1 - \frac{16P_{16}}{9N^2(1+N)^2} \right. \\
 & - 1152S_1^2 + 2624S_2 + 960S_{-2} \right) S_{-3,1} + \left(\frac{16(138+35N+35N^2)}{3N(1+N)} - 1472S_1 \right) S_{-4} \\
 & + 2304S_{-3,2} + 768S_{2,3} + 2688S_{-2,3} - \frac{64(-24+29N+29N^2)}{3N(1+N)} S_{3,1} - 768S_{4,1} \\
 & + \frac{32(-174+31N+31N^2)}{3N(1+N)} S_{-2,2} - \frac{1920}{N(1+N)} S_{-3,1} + 1728S_{-4,1} \\
 & - 5376S_{2,1,-2} + 5365S_{1,1,-2} - \frac{128(-84+31N+31N^2)}{3N(1+N)} S_{-2,1,1} - 1536S_{-2,1,-2} \\
 & - 5376S_{-2,1,1} - 5376S_{-3,1,1} + 10752S_{-2,1,1,1} \Bigg] + \text{Tr}_{\text{NP}} \left[\frac{16P_5}{9N^2(1+N)^2} S_0 + \frac{4P_{14}}{9N^4(1+N)^4} \right. \\
 & + \left(-\frac{8P_{13}}{9N^2(1+N)^2} + \frac{1280}{9} S_2 - \frac{512}{3} S_3 - \frac{512}{3} S_{-2,1} + 128S_5 \right) S_1 - \frac{128}{3} S_2^2 \\
 & + \frac{64(12+29N+29N^2)}{9N(1+N)} S_3 - \frac{512}{3} S_4 + \left(-\frac{128(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{2560}{9} S_1 - \frac{256}{3} S_2 \right) \\
 & \times S_{-2,2} + \left(\frac{128(3+10N+10N^2)}{9N(1+N)} - \frac{256}{3} S_1 \right) S_{-3,2} - \frac{256}{3} S_{-3,4} - \frac{256(-3+10N+10N^2)}{9N(1+N)} S_{-2,1,1} \\
 & + \frac{256}{3} S_{1,1} - \frac{256}{3} S_{-2,2} + \frac{1024}{3} S_{-2,1,1} - \frac{32(2+3N+3N^2)}{N(1+N)} \zeta_1 \Bigg] \\
 & + \text{C}_F \left\{ \text{T}_F^2 N_F^2 \left[\frac{8P_{26}}{27N^3(1+N)^3} - \frac{128}{27} S_1 - \frac{640}{27} S_2 + \frac{128}{9} S_3 \right] + \text{C}_A^2 \left[\frac{-24P_5}{N^2(1+N)^2} \zeta_1 \right. \right. \\
 \end{aligned}$$

$$\begin{aligned}
 & \left. \left. \left. \left. + \frac{32P_{11}}{9N^2(1+N)^2} S_{-2,1} + \frac{8P_{18}}{9N^3(1+N)^2} S_3 + \frac{P_{32}}{54N^4(1+N)^3} + \left(\frac{4P_{35}}{3N^3(1+N)^4} \right. \right. \right. \right. \\
 & - \frac{16(-8+11N+11N^2)}{N(1+N)} S_5 - 256S_6 + 512S_{3,1} - \frac{64(-24+11N+11N^2)}{3N(1+N)} S_{-2,1} \\
 & - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \Bigg) S_1 + \left(-128S_3 - 512S_{-2,1} \right) S_1^2 + \left(-\frac{8344}{27} \right. \\
 & + 384S_5 + 1536S_{-2,1} \Bigg) S_2 - \frac{16(-24+55N+55N^2)}{3N(1+N)} S_4 + 64S_6 + \left(\frac{32P_{10}}{9N^2(1+N)^2} S_1 \right. \\
 & + \frac{16P_{27}}{9N^3(1+N)^3} - \frac{352}{3} S_2 - 64S_3 - 1536S_{2,1} + 128S_{-2,1} - 192S_5 \Bigg) S_{-2,2} \\
 & + \left(\frac{48(2+N+N^2)}{N(1+N)} - 192S_1 \right) S_1^2 + \left(\frac{16P_{12}}{9N^2(1+N)^2} - \frac{32(24+11N+11N^2)}{3N(1+N)} S_1 \right. \\
 & + 256S_1^2 - 768S_2 - 320S_{-2} \Bigg) S_{-3,2} + \left(-\frac{16(30+13N+13N^2)}{3N(1+N)} + 320S_7 \right) S_{-4} \\
 & - 704S_{-5,2} - 384S_{3,2} - 768S_{2,2} - \frac{64(-12+11N+11N^2)}{3N(1+N)} S_{3,1} + 384S_{4,1} \\
 & - 32(-24+11N+11N^2) S_{-4,2} + 1088S_{-2,3} + \frac{512}{N(1+N)} S_{-3,1} - 448S_{-4,1} \\
 & + 1536S_{2,1,-2} - 768S_{3,1,-2} + \frac{128(-24+11N+11N^2)}{3N(1+N)} S_{-2,1,1} + 512S_{-2,1,-2} + 1536S_{-2,2,1} \\
 & + S_{-3,1,1} - 3072S_{-2,1,1,1} \Bigg] + \text{C}_A \text{Tr}_{\text{NP}} \left[\frac{8P_{30}}{27N^3(1+N)^3} + \left(-\frac{16P_{30}}{27N^3(1+N)^3} + 64S_3 \right. \right. \\
 & + \frac{256}{3} S_{-2,1} - 128S_5 \Bigg) S_1 + \frac{5344}{27} S_2 - \frac{32(3+14N+14N^2)}{3N(1+N)} S_3 + \frac{320}{3} S_4 + \left(-\frac{1280}{9} S_1 \right. \\
 & + \frac{64(-3+10N+16N^2)}{9N^2(1+N)^2} + \frac{128}{3} S_2 \Bigg) S_{-2} + \left(-\frac{64(3+10N+10N^2)}{9N(1+N)} + \frac{128}{3} S_1 \right) S_{-3} \\
 & + \frac{128}{3} S_{-4,2} - \frac{256}{3} S_{6,1} + \frac{128(-3+10N+10N^2)}{9N(1+N)} S_{-2,3} + \frac{128}{3} S_{-2,2} - \frac{512}{3} S_{-2,1,1} \\
 & + \frac{32(2+3N+3N^2)}{N(1+N)} \zeta_3 \Bigg] + \text{C}_P \left\{ \frac{-48P_6}{N^2(1+N)^2} \zeta_1 + \frac{8P_4}{N^2(1+N)^2} S_1 + \frac{P_{36}}{N^3(1+N)^3} \right. \\
 & + \left(\frac{8P_{26}}{N^4(1+N)^4} - \frac{128(1+2N)}{N^2(1+N)^2} S_0 + 128S_2^2 - 384S_4 + 128S_6 + 512S_{3,1} - 3328S_{-2,2} \right. \\
 & - \frac{384(-4+N+N^2)}{N(1+N)} S_{-2,1} - 3564S_{-3,1} + 6144S_{-2,1,1} \Bigg) S_1 + \left(-\frac{64(1+3N+3N^2)}{N^3(1+N)^3} \right. \\
 & \left. \left. \left. \left. - 1536S_{-2,1} \right) S_1^2 + \left(\frac{4P_{25}}{N^3(1+N)^3} + 512S_3 + 4352S_{-2,1} \right) S_2 - \frac{32(2+3N+3N^2)}{N(1+N)} S_2^2 \right\} \right. \\
 \end{aligned}$$

$$\begin{aligned}
 & \frac{32(2+15N+15N^2)}{N(1+N)} S_4 + \left(\frac{32P_{23}}{N^3(1+N)^3} + \left(-\frac{128(5+7N+3N^2)}{N^2(1+N)^2} + 512S_2 \right) S_1 \right. \\
 & - \frac{64(4+3N+3N^2)}{N(1+N)} S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} - 384S_5 \Bigg) S_{-2} + \left(\frac{128}{N(1+N)} \right. \\
 & - 256S_1 S_2^2 + \left(\frac{32(5+5N+9N^2)}{N^2(1+N)^2} - \frac{64(20+3N+3N^2)}{N(1+N)} S_1 + 1280S_1^2 - 2176S_2 \\
 & - 460S_{-2} S_{-3} + \left(-\frac{32(26+3N+3N^2)}{N(1+N)} + 1664S_5 \right) S_{-4} - 1792S_{-5,2} - 384S_{2,3} \\
 & - 2304S_{2,1,-2} + \frac{128(-2+3N+3N^2)}{N(1+N)} S_{3,1} + 384S_{4,1} - \frac{64(4-N+3N^2)}{N^2(1+N)^2} S_{-2,1} \\
 & - \frac{64(-26+3N+3N^2)}{N(1+N)} S_{-2,2} + 2944S_{-2,3} + \frac{1792}{N(1+N)} S_{-3,1} - 1664S_{-4,1} \\
 & + 4608S_{2,1,2} - 768S_{1,1,1} + \frac{768(-4+N+N^2)}{N(1+N)} S_{-2,1,1,1} + 1024S_{-3,1,1} \\
 & + 4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1} \Bigg\},
 \end{aligned}$$

Comparison with the Literature

- Our results of the 3-loop non-singlet anomalous dimensions agree with [Moch, Vermaseren, Vogt 2004, 2015].
- In the transversity case we agree with the formerly guessed expressions in [Velizhanin, 2012].
- In the polarized case we agree with the 3-loop anomalous dimensions in [Moch, Vermaseren, Vogt 2014].
- We agree with the earlier results at $\mathcal{O}(T_F)$ from massive OMEs.
- At 2-loop we recalculated the singlet anomalous dimensions using unpolarized OMEs together with all unphysical OMEs and alien operators.

Conclusions and Outlook

Conclusions:

- We have calculated all flavor non-singlet and the polarized singlet 3-loop anomalous dimensions by using the method of massless off-shell operator matrix elements.
- The unpolarized and polarized transversity anomalous dimensions have been calculated for the first time ab initio.
- We agree with the results of earlier calculations based on either the forward Compton amplitude or massive on-shell OMEs.
- The 2-loop unpolarized case is completely understood, correcting some errors in the literature.

Outlook:

- The calculation of the 3-loop unpolarized singlet anomalous dimensions is forthcoming.
- The method appears to be useful to calculate 4-loop quantities.

Backup

The Larin and M-scheme

The structure function has to be scheme independent:

$$g_1 = C_{g_1,L} \Delta f_L = (C_{g_1,L} Z_5^{-1}) (Z_5 \Delta f_L) = C_{g_1,M} \Delta f_M,$$

For the transition to the M-scheme we have:

$$Z_5 = 1 + \sum_{n=1}^{\infty} a_s^n \begin{pmatrix} z_{qq}^{(n),\text{NS}} + z_{qq}^{(n),\text{PS}} & 0 \\ 0 & 0 \end{pmatrix},$$

$$z_{qq}^{(1),\text{NS}} = -\frac{8C_F}{N(N+1)}$$

$$z_{qq}^{(2),\text{NS}} = -C_F T_F N_F \frac{16(3+N-5N^2)}{9N^2(N+1)^2} + \dots$$

$$z_{qq}^{(1),\text{PS}} = 0$$

$$z_{qq}^{(2),\text{PS}} = 8C_F T_F N_F \frac{(N+2)(1+N-N^2)}{N^3(N+1)^3}$$

Mathematical Structures

Harmonic Sums: [Vermaseren '98; Blümlein, Kurth '98]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z}/\{0\}, \quad N \in \mathbb{N}/\{0\}.$$

The Mellin inversion of the splitting functions $P_{ij}(z)$ defines the anomalous dimensions:

$$\gamma_{ij} = - \int_0^1 dz z^{N-1} P_{ij}(z)$$

Harmonic Polylogarithms: [Remiddi, Vermaseren '99]

$$H_{b,\vec{a}}(z) = \int_0^z dx f_b(x) H_{\vec{a}}(x), \quad H_{\emptyset} = 1, \quad H_{0,\dots,0} = \frac{1}{n!} \ln^n(x), \quad b, a_i \in \{-1, 0, 1\},$$

with the alphabet:

$$\mathcal{A}_H = \left\{ f_0(x) = \frac{1}{x}, f_{-1}(x) = \frac{1}{1+x}, f_1(x) = \frac{1}{1-x} \right\}.$$