DIS AT 4-LOOPS WITH DIFFERENTIAL EQUATIONS
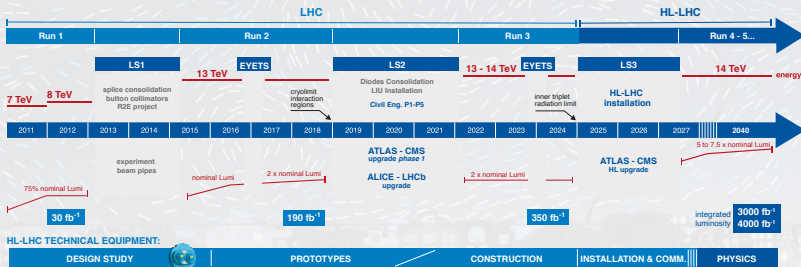
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MOTIVATION

▶ **Theoretical predictions** need to keep up with the ever-increasing precision of **experimental measurements**

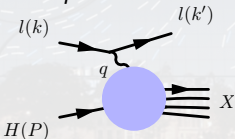▶ Need to understand the SM background in order to resolve **new physics**

▶ **Example:** Higgs inclusive: 8% → 3% expected experimental uncertainty at $3000\ fb^{-1}$. The PDF uncertainty on the theoretical prediction cannot be neglected anymore.

STRUCTURE FUNCTIONS

▶ The improvment in the high-order perturbative-QCD corrections requires matching splitting functions

▶ The computation of the corresponding 4-loop Wilson coefficients is extremely challenging from the theoretical point of view

$$\frac{\sigma_{DIS}}{dxdy} = \frac{2\pi\alpha_{em}^2}{Q^2} L^{\mu\nu} W_{\mu\nu}$$

$$W_{\mu\nu} = \left(P^\mu - \frac{(P \cdot q)q_\mu}{q^2}\right)\left(P^\nu - \frac{(P \cdot q)q_\nu}{q^2}\right) \frac{F_1(x, Q^2)}{P \cdot q}$$

$$+ \left(-g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_2(x, Q^2)$$

$$+ i\epsilon_{\mu\nu\rho\sigma} \frac{P^\rho q^\sigma}{2P \cdot q} F_3(x, Q^2)$$

▶ Where the **hadronic** and **partonic** quantities are related via the **PDF**:

$$F_a(x, Q^2) = \sum_i \left[f_i(\xi) \otimes \hat{F}_{a,i}(\xi, Q^2) \right](x)$$

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MELLIN MOMENTS

Objective:

▶ Compute the hadronic cross-section $\hat{W}_{\mu\nu}$ using the forward scatting $\hat{T}_{\mu\nu}$

How:

▶ Compute the **Mellin moments** of the structure functions:

$$F_a(x, Q^2) = \sum_i \left[f_i(\xi) \otimes \hat{F}_{a,i}(\xi, Q^2) \right](x)$$

with the **Mellin transform** defined by

$$M[f(x)](N) = \int_0^1 \mathrm{d}x \, x^{N-1} f(x)$$

▶ Compute enough moments to reconstruct $\hat{W}_{\mu\nu}$ in **x**-space

▶ The Mellin moments of **cross-section** correspond to the expansion coefficients around $\omega = \frac{1}{x} = 0$ of the **Forward Scattering Amplitude** :

$$M[\hat{W}_{\mu\nu}](N) = \frac{1}{N!} \left[\frac{\mathrm{d}^N T_{\mu\nu}}{\omega^N} \right]\bigg|_{\omega=0}$$
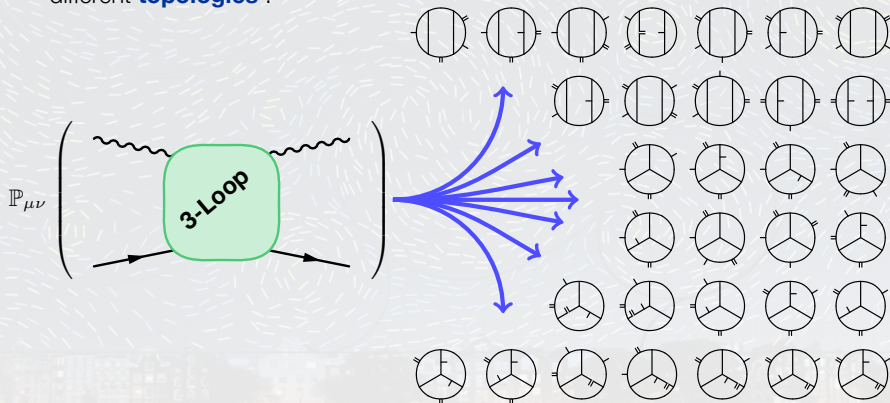
SYSTEM OF DIFFERENTIAL EQUATIONS:

We process all the diagrams for the relevant process and cast them into 32 different **topologies** :
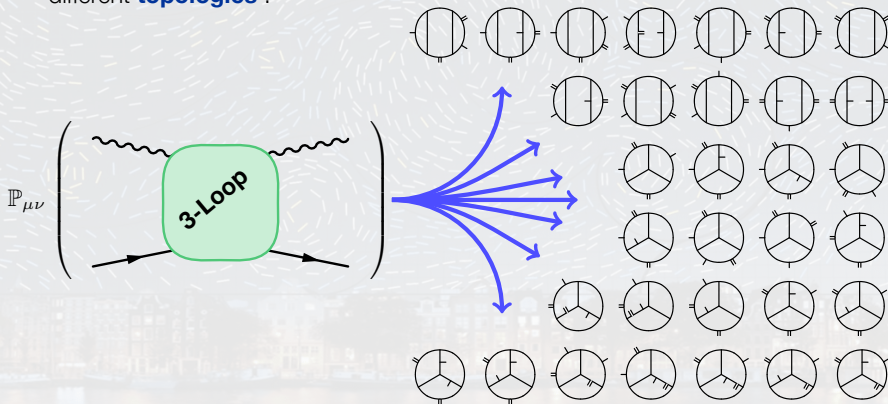
SYSTEM OF DIFFERENTIAL EQUATIONS:

We process all the diagrams for the relevant process and cast them into 32 different **topologies** :

Still left with $\approx 10^5$ different integrals to be computed!

- ▶ We can resize the problem by using **IBP** relations among these integrals.
- ▶ Many publicly available programs to perform **reductions to master integrals** (e.g FIRE, Reduze, Kira) each with its strengths and weaknesses.

SYSTEM OF DIFFERENTIAL EQUATIONS

▶ Within each **topology** we perform a reduction to **master integrals** :

$$I^{(n)}(\omega, \epsilon) = \sum_i c_i(\omega, \epsilon) \cdot M_i^{(n)}(\omega, \epsilon), \qquad \omega = \frac{1}{x}$$

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Expand in ω

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▶ The master integrals allow to construct a closed system of differential equations:

$$\frac{\partial}{\partial \omega} \vec{M}(\omega, \epsilon) = A(\omega, \epsilon) \cdot \vec{M}(\omega, \epsilon).$$

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Assuming:

- The DE matrix has at most a simple pole in ω:

$$A = \frac{A_{-1}}{w} + \sum_{k=0}^{\infty} A_k \omega^k$$

Note: Can always be done by applying a linear transformation T:

$$\vec{M} \to T \cdot \vec{M}, \qquad A \to \frac{\partial T}{\partial \omega} T^{-1} + T \cdot A \cdot T^{-1}$$

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$$\underbrace{\left((k+1)\mathbb{1} - A_{-1}\right)}_{:=B_k} \cdot \vec{m}_{k+1} = \sum_{j=0}^{k} A_j \vec{m}_{k-j}$$

$\det(B_k) \neq 0$ $\det(B_k) = 0$

Recursive Expression:

$$\vec{m}_{k+1} = B_k^{-1} \cdot \left(\sum_{j=0}^{k} A_j \vec{m}_{k-j}\right)$$

Gaussian Elimination:

Required by a **finite number** of k

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Mellin moments generation

$$\frac{\partial}{\partial \omega} \vec{M}(\omega, \epsilon) = A(\omega, \epsilon) \cdot \vec{M}(\omega, \epsilon).$$

$$A = \frac{A_{-1}}{w} + \sum_{k=0}^{\infty} A_k \omega^k \quad \Big\downarrow \quad \vec{M} = \sum_{k=0}^{\infty} \vec{m}_k \omega^k$$

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EXPANSION

► We need to fix the **boundary** condition for $p \to 0$ (FORCER)

► In our case the **transformation** matrix T consists of a simple rescaling of the master integrals

$$T = \mathrm{diag}\left(\omega^{\vec{a}}\right), \qquad \vec{a} \in \mathbb{N}_0^{\text{\# of masters}}$$

► We can perform a simultaneous expansion in the **dimensional regulator** ϵ in order to speed up the computation

► Results can be expanded to high order in **Mellin moments**

► Faster than an expansion at the integrand level (Tensor decomposition)

► The **reductions** to master integrals remain the main bottleneck of the computation

ADVENTURING INTO 4-LOOPS

▶ Starting to explore the DIS expression for a simple subgroup $[n_f^3, n_f^2]$ for $q + \gamma \rightarrow q + \gamma$:

▶ The **reductions** for all relevant topologies (12) can be performed within days.

▶ **Expansion** in ω is possible to **high orders** within a day:

$$\mathcal{O}\left(\omega^{1500}\right)$$

▶ Allows for the reconstruction of the **structure functions** in **x**-space at **all orders**

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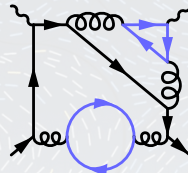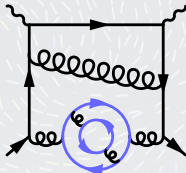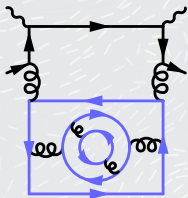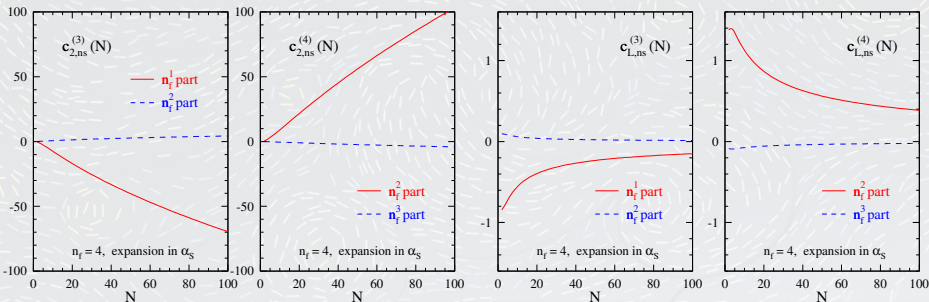

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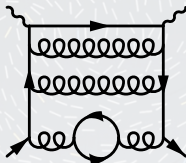▶ **Expansion** in ω is possible to **high orders** within a day:

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ADVENTURING INTO 4-LOOPS

The **new** goal is to push the same technique to new horizons:

▶ Consider the diagrams contributing to $C_F^3 n_f$

▶ Effectively 3-loop topologies with **bubble insertions**!

▶ The added degrees of freedom start to become a real problem for the reduction to **master integrals:**

  • Moving from 11 to 12 propagators

  • Higher powers in the numerator

TACKLING THE PROBLEMS

General Reduction programs :

▶ Reliable on a wide range of problems

▶ Thoroughly checked through years of usage and feedbacks

▶ Paralelization of the reduction problem

▶ Multiple ways to solve the problem and implementation of general optimization

▶ Already too slow for the integrals we are dealing with

Problem we are facing :

▶ Relatively contained number of integrals to be reduced (compared with the d.o.f)

▶ Very few integrals have the highest complexity (numerators/denominators powers)

TACKLING THE PROBLEMS

Preliminary results: We have implemented our own reduction procedure to try to improve the main bottleneck of our approach.

Why:

► With the current available reduction programs it would be impossible to obtain the necessary DEs in a reasonable ammount of time

First Idea:

► Obtain a fast partial reduction (not master integrals) to be able to give simpler problems to the public reduction programs

Surpising result:

► We obtained a quick and complete reduction for a topology which was running for a long time without success using available tools.

SUMMARY

► Use IBP identities for a **reduction to master integrals** and build a system of differential equations

► **Transform** the system to allow for an efficient recursive expression for the extraction of the series coefficients

► Tested the method by computing high numbers of Mellin moments for the DIS **Wilson coefficients** $\hat{F}_1$, $\hat{F}_2$ and $\hat{F}_3$ at **3-loops**

► Generated 1500 Mellin moments for the n_f^2 contribution to DIS at **4-loops** to obtain for the first time the corresponding **Wilson coefficients**

Future:

► Apply the same method for extracting Mellin moments at **4-loops** for the $C_F^3 n_f$ to extract the **splitting functions**

► Validate and use our **reduction** to Master integrals to allow for the construction of the Differential System.

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Thank you!

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