

# Moments of the longitudinal structure function of proton from lattice QCD simulations

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- Partially based on:
- Can et al., PRD102, 114505 (2020),  
[arXiv:2007.01523 \[hep-lat\]](https://arxiv.org/abs/2007.01523)
  - Can et al., PoS(LATTICE2021)324,  
[arXiv:2110.01310 \[hep-lat\]](https://arxiv.org/abs/2110.01310)

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Y. Nakamura (RIKEN, Kobe)

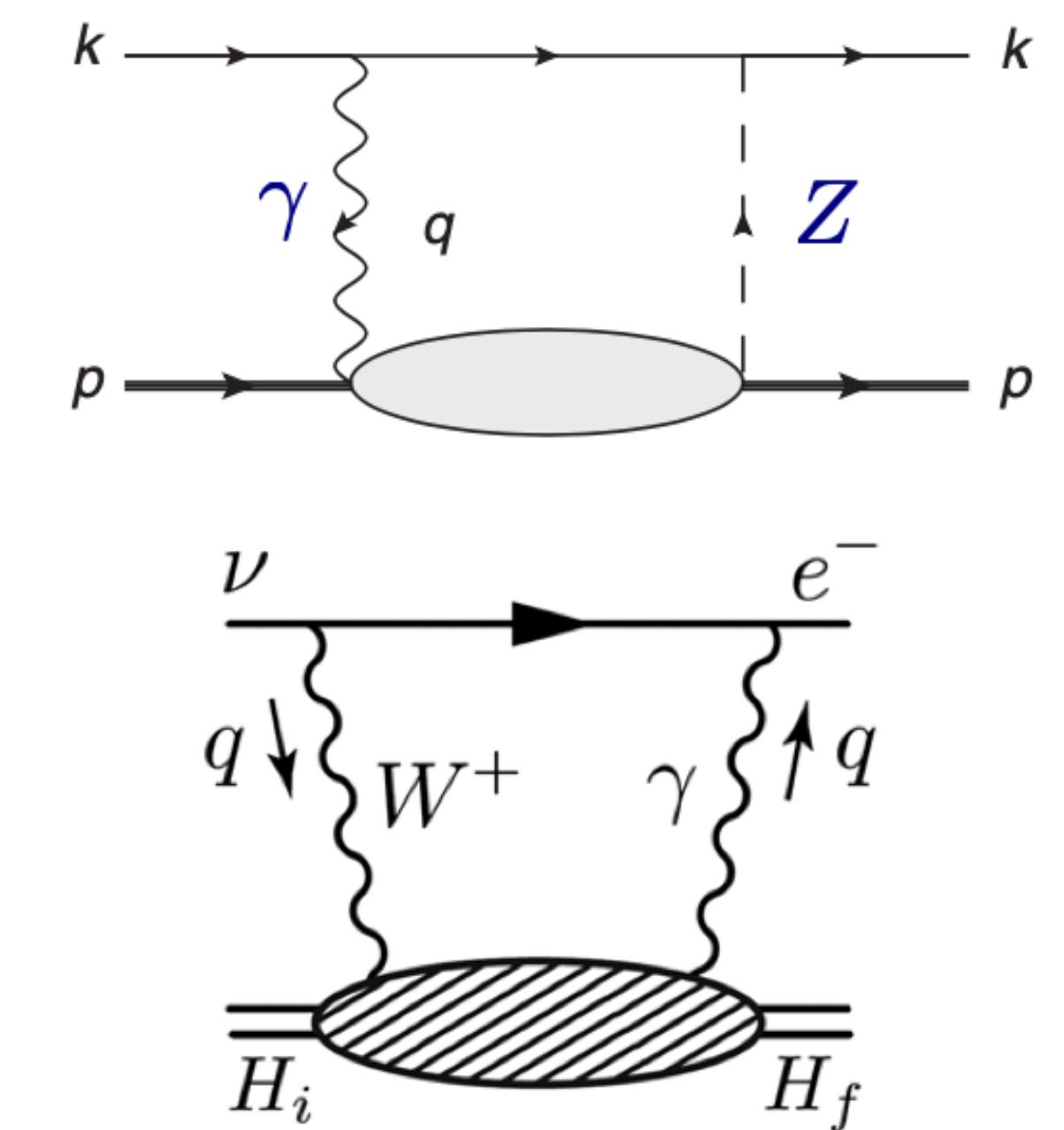
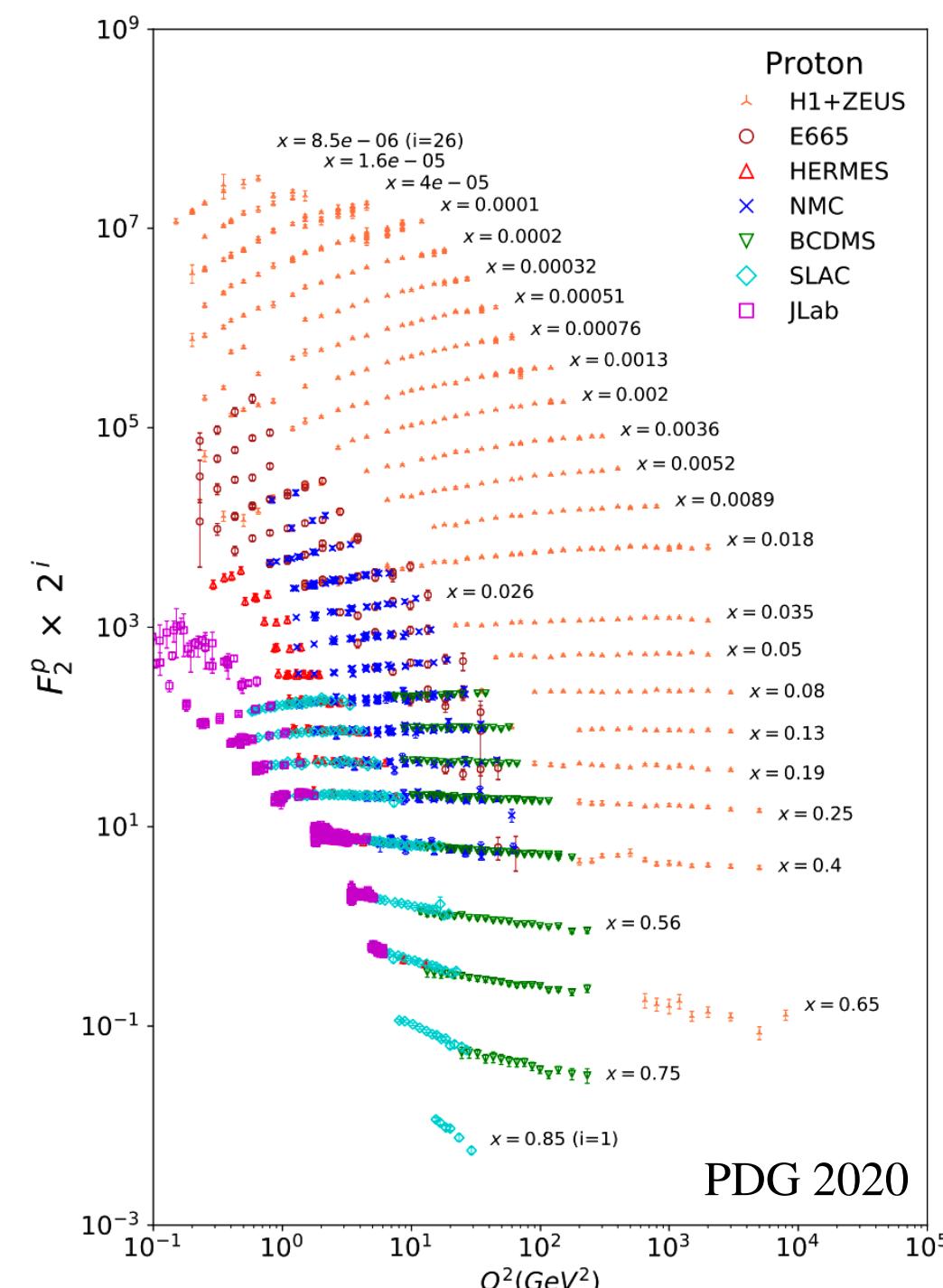
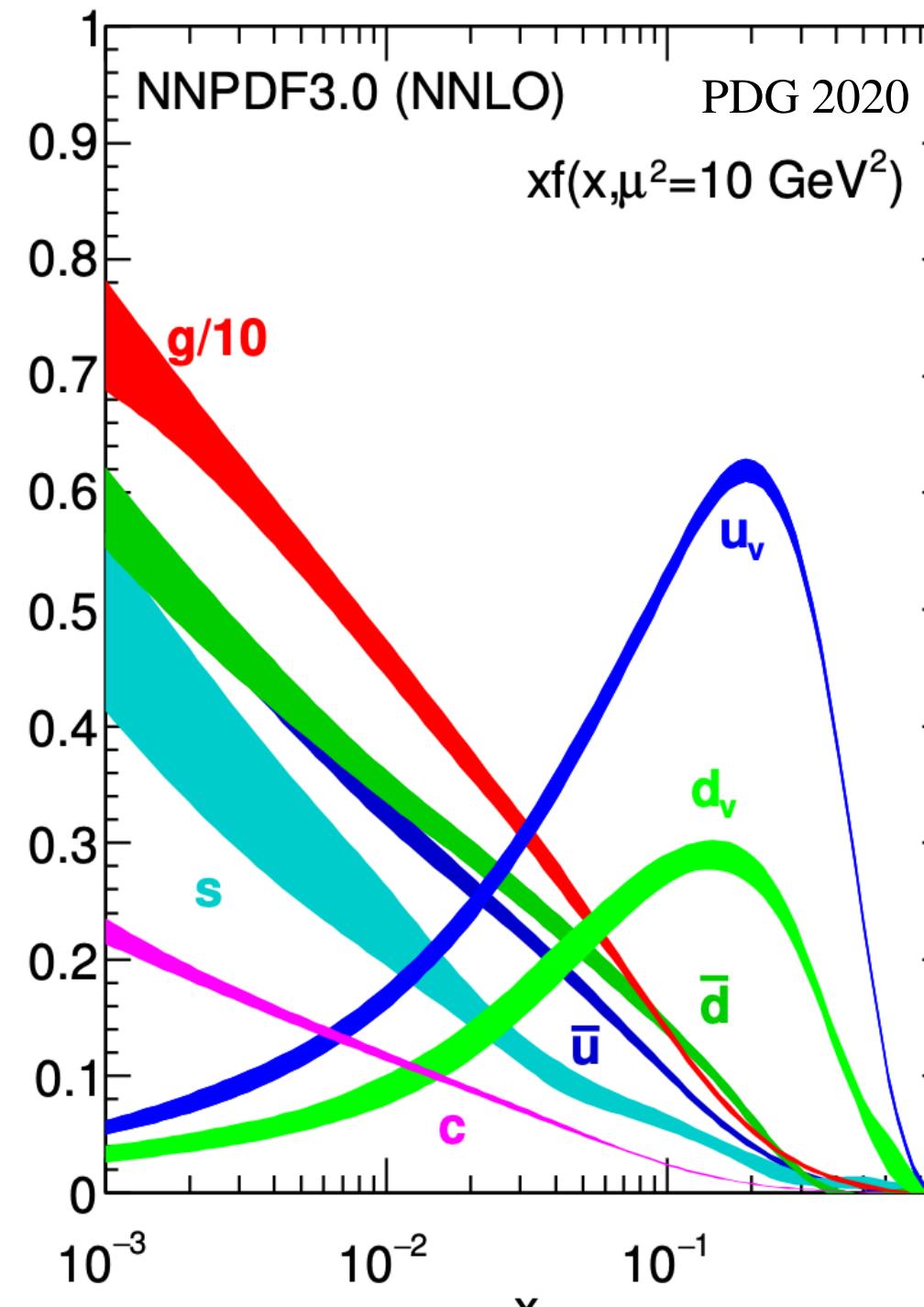
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The numerical configuration generation (using the BQCD lattice QCD program)) and data analysis (using the Chroma software library) was carried out on the IBM BlueGene/Q and HP Tesseract using DIRAC 2 resources (EPCC, Edinburgh, UK), the IBM Blue-Gene/Q (NIC, Jülich, Germany) and the Cray XC40 at HLRN (The North-German Supercomputer Alliance), the NCI National Facility in Canberra, Australia (supported by the Australian Commonwealth Government) and Phoenix (University of Adelaide). RH is supported by STFC through grant ST/P000630/1. HP is supported by DFG Grant No. PE 2792/2-1. PELR is supported in part by the STFC under contract ST/G00062X/1. GS is supported by DFG Grant No. SCHI 179/8-1. KUC, RDY and JMZ are supported by the Australian Research Council grant DP190100297.

# Motivation

- Nucleon structure (leading twist)
  - Structure functions from first principles
  - Understanding the behaviour in the high- and low-x regions
- Scaling and Power corrections/ Higher twist effects
  - $Q^2$  cuts of global QCD analyses
  - Twist-4 contributions
  - Kinematic effects
- New physics searches
  - Weak charge of the proton
  - $\gamma - W/Z$  interference

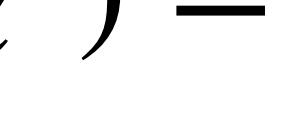


# Motivation

- Technical issues
  - Operator mixing/renormalisation issues in OPE approach in LQCD

$$\mu(Q^2) = c_2(a^2 Q^2) v_2(a) + \frac{c_4(a^2 Q^2)}{Q^2} v_4(a) + \dots$$

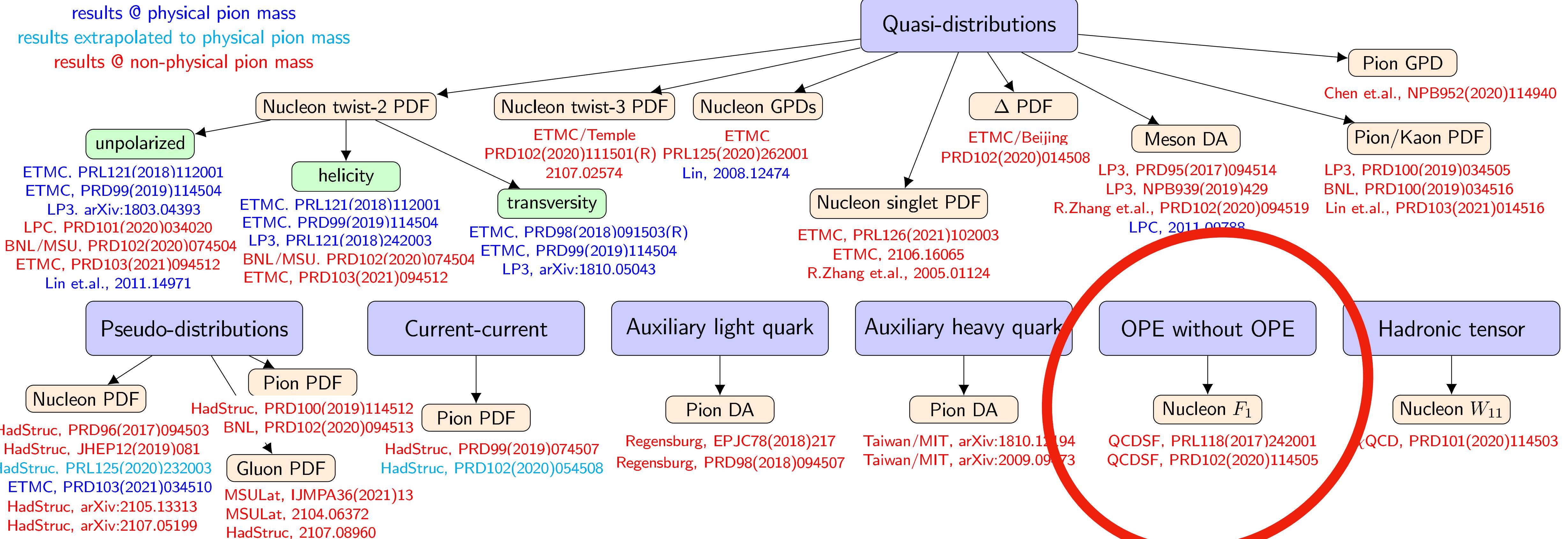
twist-2      twist-4  
mixing

1/a<sup>2</sup> divergence


- 4-point functions are costly; harder to tackle
  - Feynman-Hellmann (FH) approach needs 2-point functions only

# LQCD landscape

Krzysztof Cichy @ LATTICE'21 plenary, arXiv:2110.07440



- QCDSF/UKQCD Collaboration
- Extended to nucleon  $F_2$  and  $F_L$
- Study of higher-twist

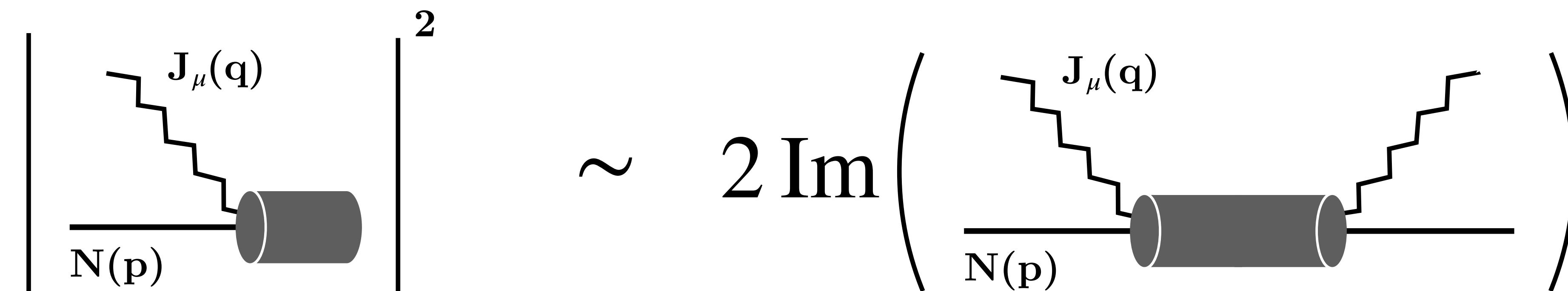
# Forward Compton Amplitude

$$T_{\mu\nu}(p, q) = i \int d^4z e^{iq \cdot z} \rho_{ss'} \langle p, s' | \mathcal{T}\{J_\mu(z) J_\nu(0)\} | p, s \rangle , \text{ spin avg. } \rho_{ss'} = \frac{1}{2} \delta_{ss'} \quad \underline{\omega} = \frac{2p \cdot q}{Q^2}$$

Same Lorentz decomposition as the Hadronic Tensor

$$= \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \mathcal{F}_1(\omega, Q^2) + \left( p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left( p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \frac{\mathcal{F}_2(\omega, Q^2)}{p \cdot q}$$

Compton Structure Functions (SF)



DIS Cross Section ~ Hadronic Tensor

Forward Compton Amplitude ~ Compton Tensor

# Nucleon Structure Functions

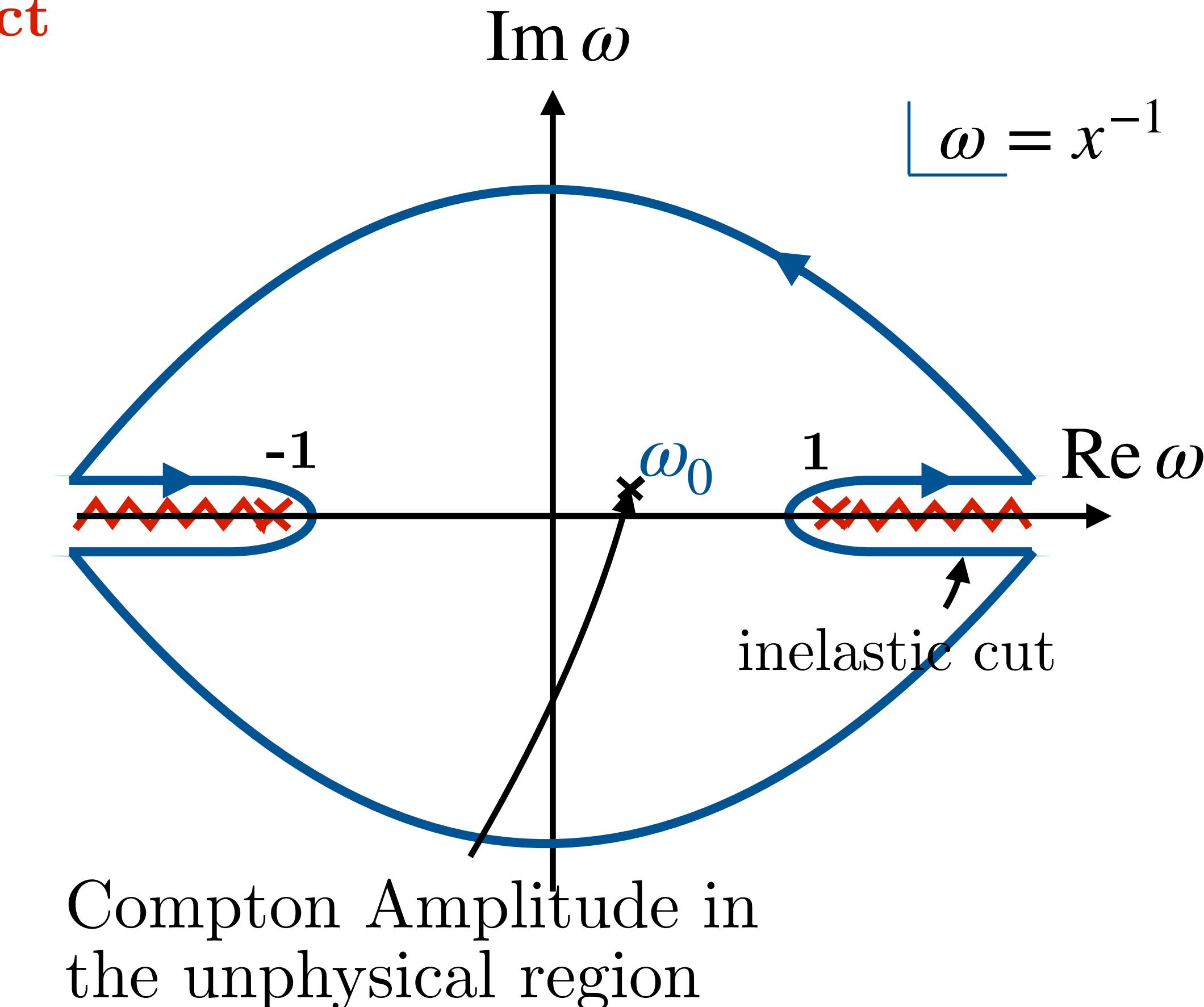
- we can write down dispersion relations and connect Compton SFs to DIS SFs:

$$\underbrace{\mathcal{F}_1(\omega, Q^2) - \mathcal{F}_1(0, Q^2)}_{\equiv \bar{\mathcal{F}}_1(\omega, Q^2)} = 2\omega^2 \int_0^1 dx \frac{2x F_1(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$

$$\mathcal{F}_2(\omega, Q^2) = 4\omega^2 \int_0^1 dx \frac{F_2(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$

$$\underbrace{\mathcal{F}_L(\omega, Q^2) + \mathcal{F}_1(0, Q^2)}_{\equiv \bar{\mathcal{F}}_L(\omega, Q^2)} = \frac{8M_N^2}{Q^2} \int_0^1 dx F_2(x, Q^2)$$

$$+ 2\omega^2 \int_0^1 dx \frac{F_L(x, Q^2)}{1 - x^2\omega^2 - i\epsilon}$$



Compton Amplitude in the unphysical region

# Nucleon Structure Functions

- **Mellin moments**

$$\underline{\omega} = \frac{2p \cdot q}{Q^2} \equiv x^{-1}$$

$$\overline{\mathcal{F}}_{1,L}(\omega, Q^2) = \sum_{n=0}^{\infty} 2\omega^{2n} M_{2n}^{(1,L)}(Q^2), \text{ where } M_{2n}^{(1)}(Q^2) = 2 \int_0^1 dx x^{2n-1} F_1(x, Q^2), \text{ and } M_0^{(1)}(Q^2) = 0$$

$$\mathcal{F}_2(\omega, Q^2) = \sum_{n=1}^{\infty} 4\omega^{2n-1} M_{2n}^{(2)}(Q^2), \text{ where } M_{2n}^{(2,L)}(Q^2) = \int_0^1 dx x^{2n-2} F_{2,L}(x, Q^2), \text{ and } M_0^{(L)}(Q^2) = \frac{4M_N^2}{Q^2} M_2^{(2)}(Q^2)$$

- $\mu = \nu = 3$  and  $p_3 = q_3 = 0 \quad \Rightarrow \quad \mathcal{F}_1(\omega, Q^2) = T_{33}(p, q)$
- $\mu = \nu = 0$  and  $p_3 = q_3 = q_0 = 0 \quad \Rightarrow \quad \mathcal{F}_2(\omega, Q^2) = [T_{00}(p, q) + T_{33}(p, q)] \frac{Q^2 \omega}{2E_N^2}$

Once we have the Compton amplitude,  $T_{\mu\nu}(p, q)$ ,  
we can extract the Mellin moments!

# FH Theorem at 1<sup>st</sup> order

in Quantum Mechanics:

$$\frac{\partial E_\lambda}{\partial \lambda} = \langle \phi_\lambda | \frac{\partial H_\lambda}{\partial \lambda} | \phi_\lambda \rangle$$

$H_\lambda$ : perturbed Hamiltonian of the system

$E_\lambda$ : energy eigenvalue of the perturbed system

$\phi_\lambda$ : eigenfunction of the perturbed system

- expectation value of the perturbed system is related to the shift in the energy eigenvalue

in Lattice QCD: energy shifts in the presence of a weak external field

$$S \rightarrow S(\lambda) = S + \lambda \int d^4x \mathcal{O}(x)$$

↑  
real parameter

e.g. local bilinear operator  $\rightarrow \bar{q}(x)\Gamma_\mu q(x)$ ,  $\Gamma_\mu \in \{1, \gamma_\mu, \gamma_5 \gamma_\mu, \dots\}$

@ 1<sup>st</sup> order

$$\frac{\partial E_\lambda}{\partial \lambda} = \frac{1}{2E_\lambda} \langle 0 | \mathcal{O} | 0 \rangle$$

$E_\lambda \rightarrow$  spectroscopy, 2-pt function

$\langle 0 | \mathcal{O} | 0 \rangle \rightarrow$  determine 3-pt

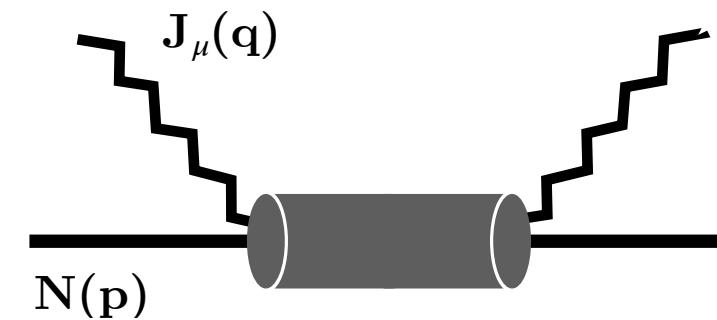
Applications:

- $\sigma$  - terms
- Form factors

# Compton Amplitude from FHT at 2<sup>nd</sup> order

- unpolarised Compton Amplitude

$$T_{\mu\mu}(p, q) = \int d^4z e^{i\mathbf{q}\cdot\mathbf{z}} \langle N(p) | \mathcal{T} \{ J_\mu(z) J_\mu(0) \} | N(p) \rangle$$



- Action modification

$$S \rightarrow S(\lambda) = S + \lambda \int d^4z (e^{i\mathbf{q}\cdot\mathbf{z}} + e^{-i\mathbf{q}\cdot\mathbf{z}}) J_\mu(z)$$

local EM current  
 $J_\mu(z) = \sum_q e_q \bar{q}(z) \gamma_\mu q(z)$

- 2<sup>nd</sup> order derivatives of the 2-pt correlator,  $G_\lambda^{(2)}(\mathbf{p}; t)$ , in the presence of the external field

$$\left. \frac{\partial^2 G_\lambda^{(2)}(\mathbf{p}; t)}{\partial \lambda^2} \right|_{\lambda=0} = \left( \frac{\partial^2 A_\lambda(\mathbf{p})}{\partial \lambda^2} - t A(\mathbf{p}) \frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial \lambda^2} \right) e^{-E_N(\mathbf{p})t}$$

from spectral decomposition

$$\left. \frac{\partial^2 G_\lambda^{(2)}(\mathbf{p}; t)}{\partial \lambda^2} \right|_{\lambda=0} = \frac{A(\mathbf{p})}{2E_N(\mathbf{p})} t e^{-E_N(\mathbf{p})t} \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(\mathbf{p}) | \mathcal{T} \{ \mathcal{J}(z) \mathcal{J}(0) \} | N(\mathbf{p}) \rangle$$

from path integral

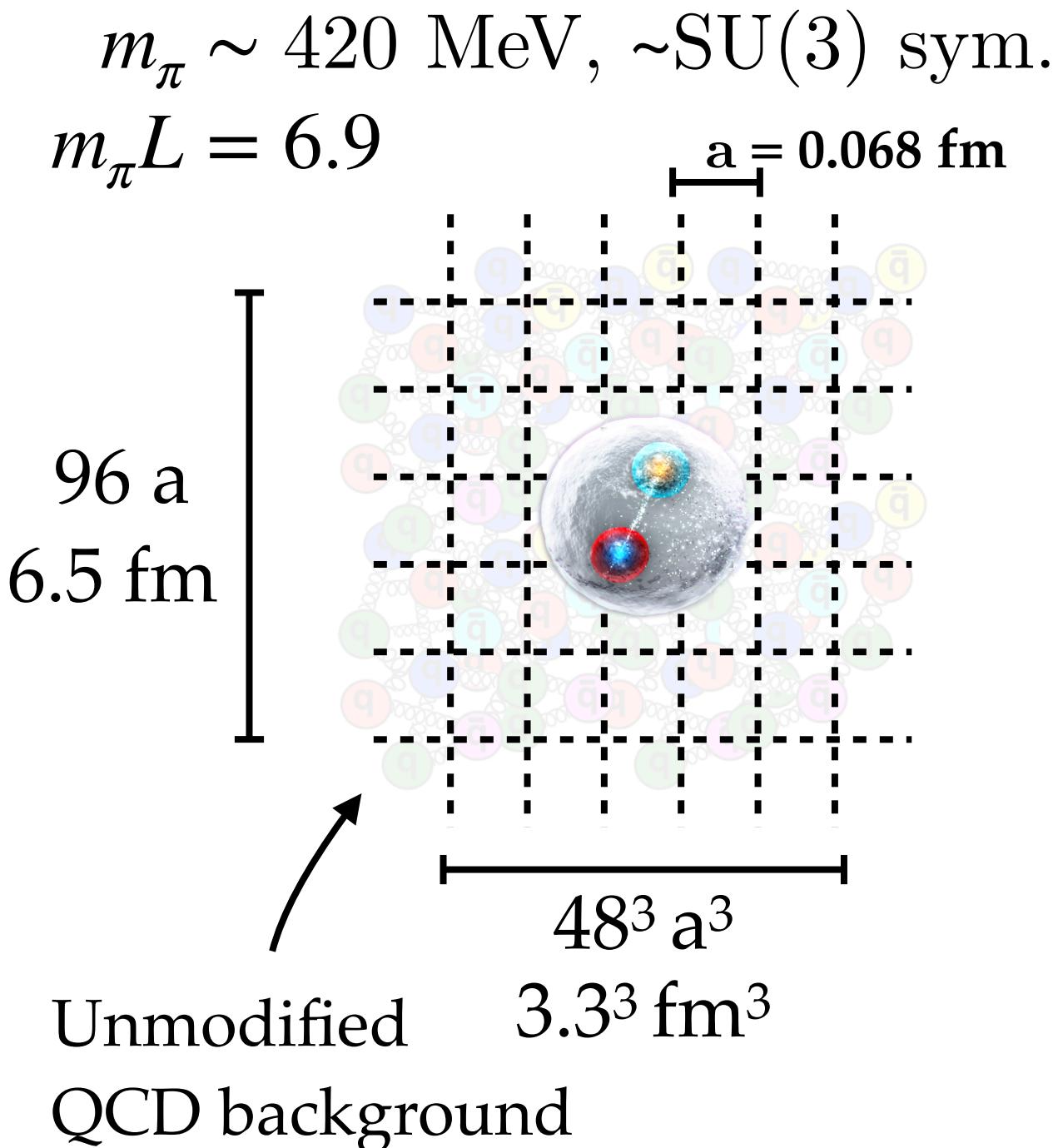
- equate the time-enhanced terms:

$$\left. \frac{\partial^2 E_{N_\lambda}(\mathbf{p})}{\partial \lambda^2} \right|_{\lambda=0} = -\frac{1}{2E_N(\mathbf{p})} \overbrace{\int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(\mathbf{p}) | \mathcal{T} \{ \mathcal{J}(z) \mathcal{J}(0) \} | N(\mathbf{p}) \rangle}^{T_{\mu\mu}(p, q)} + (q \rightarrow -q)$$

Compton amplitude is related to the second-order energy shift

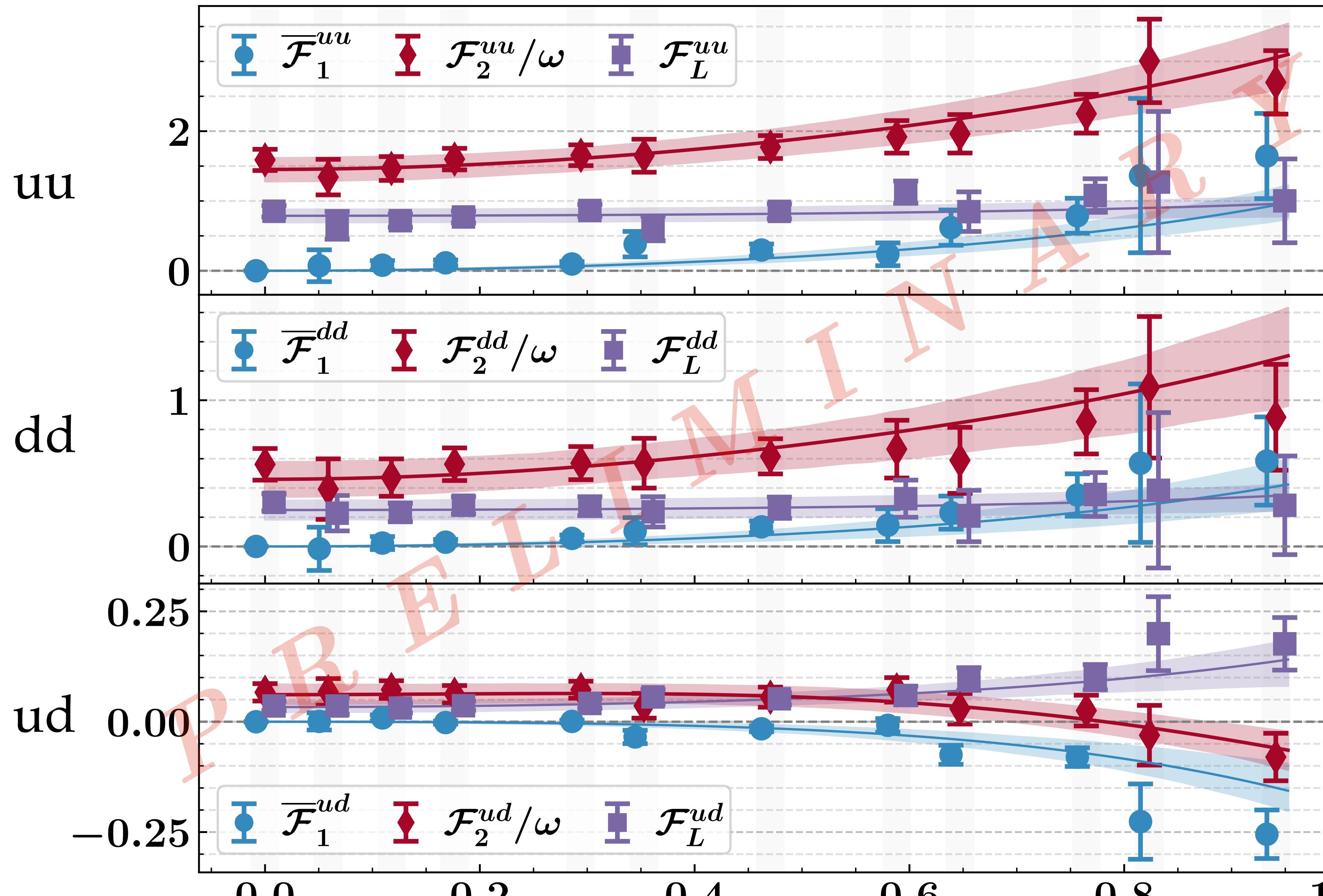
# Lattice Details

QCDSF/UKQCD,  $48^3 \times 96$ , 2+1 flavor (u/d+s)  
 $\beta = 5.65$ , NP-improved Clover action  
 Phys. Rev. D 79, 094507 (2009), arXiv:0901.3302 [hep-lat]



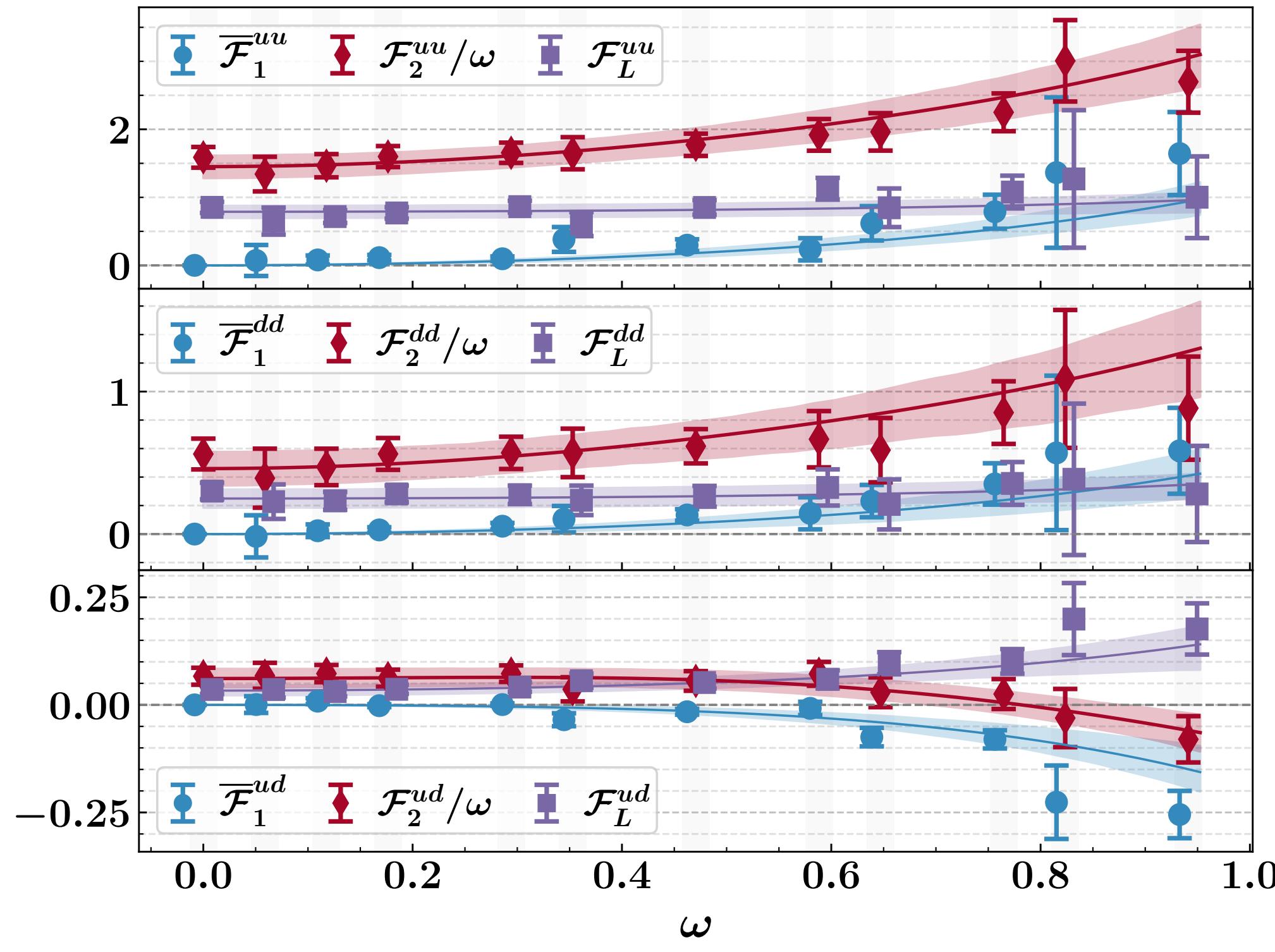
- FH implementation at the valence quark level
- Valence u/d quark props with modified action,  $S(\lambda)$
- Local EM current insertion,  $J_\mu(x) = Z_V \bar{q}(x) \gamma_\mu q(x)$
- 4 Distinct field strengths,  $\lambda = [\pm 0.0125, \pm 0.025]$
- 5 different current momenta in the range,  $1.5 \lesssim Q^2 \lesssim 7$  GeV $^2$
- $\mathcal{O}(10^3)$  measurements for each pair of  $Q^2$  and  $\lambda$
- Access to a range of  $\omega$  values for several  $(p, q)$  pairs
  - An inversion for each  $q$  and  $\lambda$ , varying  $p$  is relatively cheap
- Connected 2-pt correlators calculated only, no disconnected

# Compton Structure Functions



$48^3 \times 96$ , 2+1 flavour  
 $a = 0.068 \text{ fm}$   
 $m_\pi \sim 420 \text{ MeV}$   
 $Q^2 = 4.9 \text{ GeV}^2$

# Moments | Fit details



- Bayesian approach by MCMC method

Sample the moments from Uniform priors  
*individually for u- and d-quark*

$$M_2(Q^2) \sim \mathcal{U}(0, 1)$$

$$M_{2n}(Q^2) \sim \mathcal{U}(0, M_{2n-2}(Q^2))$$

$$\bar{\mathcal{F}}_1^{qq}(\omega, Q^2) = \sum_{n=0}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2)$$

$$\frac{\mathcal{F}_2^{qq}(\omega, Q^2)}{\omega} = \frac{\tau}{1 + \tau\omega^2} \sum_{n=0}^{\infty} 4\omega^{2n} \left[ M_{2n}^{(1)} + M_{2n}^{(L)} \right](Q^2), \text{ where } \tau = \frac{Q^2}{4M_N^2}$$

- Enforce monotonic decreasing of moments for  $u$  and  $d$  only, not necessarily true for  $u - d$

$$M_2^{(1)}(Q^2) \geq M_4^{(1)}(Q^2) \geq \dots \geq M_{2n}^{(1)}(Q^2) \geq \dots \geq 0$$

We truncate at  $n = 6$

No dependence to truncation order for  $3 \leq n \leq 10$

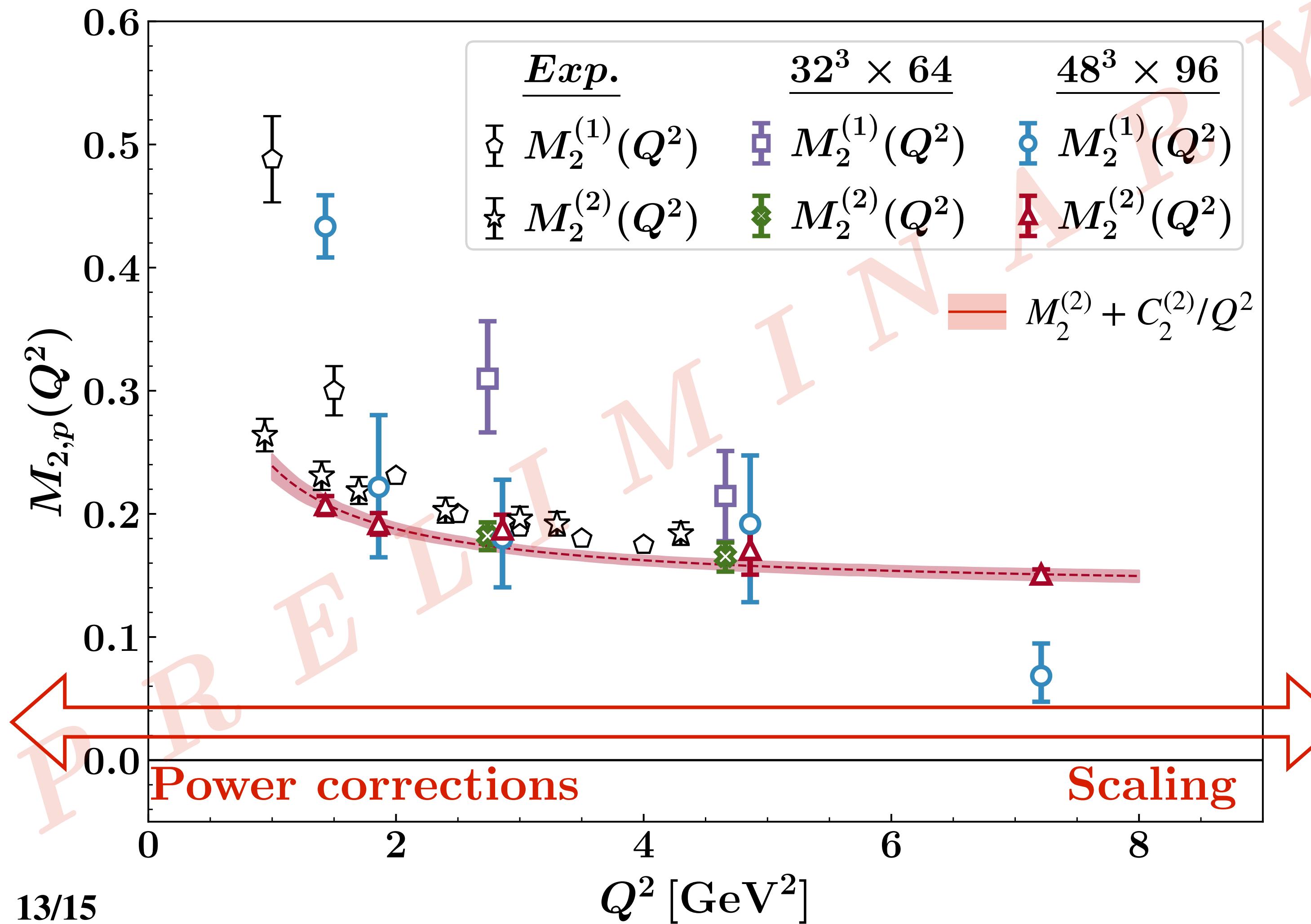
Normal Likelihood function,  $\exp(-\chi^2/2)$

$$\chi^2 = \sum_i \frac{(\bar{\mathcal{F}}_i - \bar{\mathcal{F}}^{obs}(\omega_i))^2}{\sigma_i^2}$$

errors via bootstrap analysis

# Moments | Scaling and Power Corrections

- Unique ability to study the  $Q^2$  dependence of the moments!



- Global PDF-fit cuts  $\sim 10$  GeV $^2$
- Need  $Q^2 > 10$  GeV $^2$  data to reliably extract partonic moments

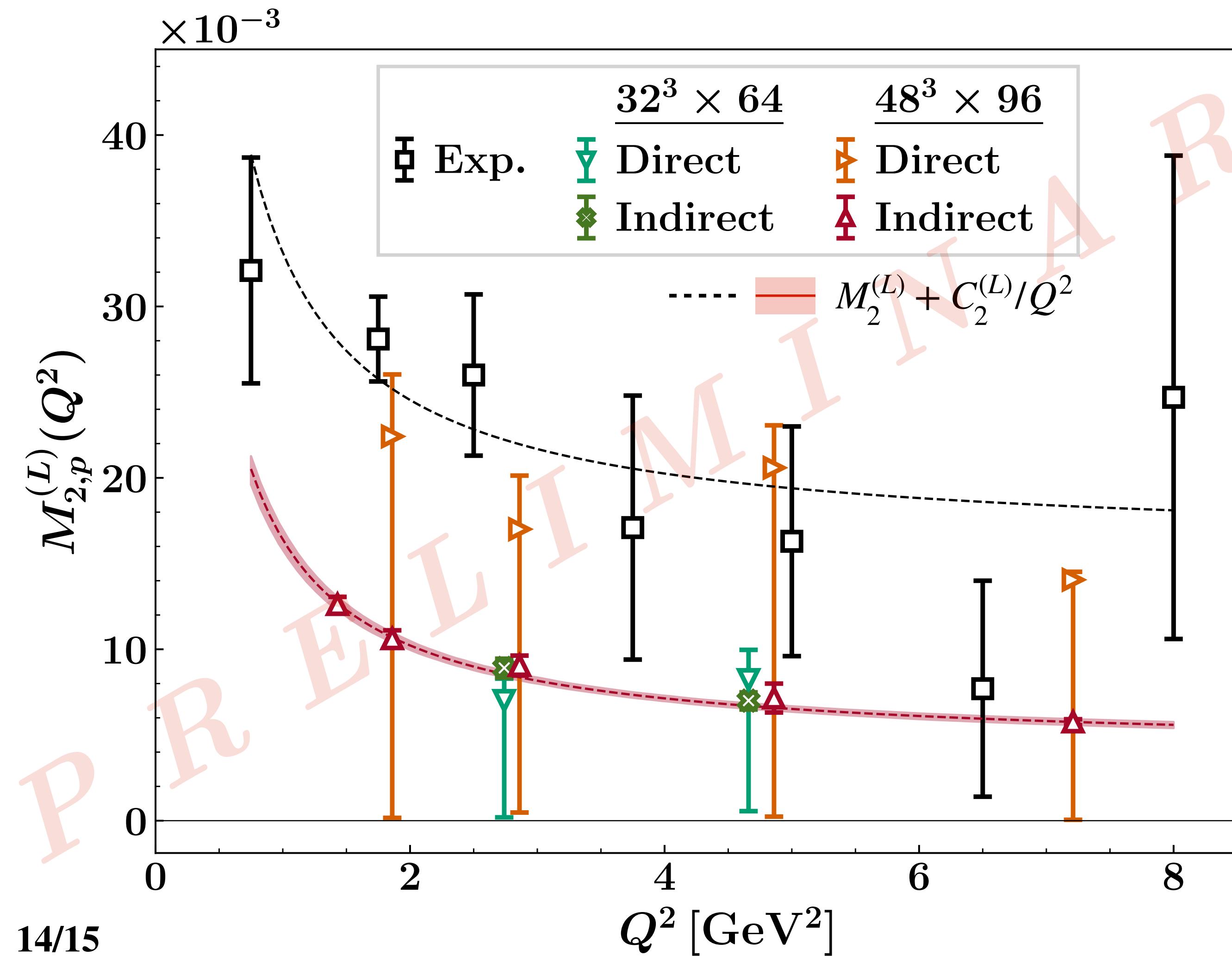
- Power corrections below  $\sim 3$  GeV $^2$  ?
- Modelling via
- $M_2^{(2)}(Q^2) = M_2^{(2)} + C_2^{(2)}/Q^2$

Exp  $M_2^{(1)}$ : W. Melnitchouk, R. Ent, and C. Keppel, Phys. Rept. 406, 127 (2005), arXiv:hep-ph/0501217.

Exp  $M_2^{(2)}$ : C. S. Armstrong, R. Ent, C. E. Keppel, S. Liuti, G. Niculescu, and I. Niculescu, Phys. Rev. D 63, 094008 (2001), arXiv:hep-ph/0104055.

# Moments | $F_L$

- Unique ability to study the moments of  $F_L$ !



Possible for the first time  
in a lattice QCD simulation!

- Direct: Fit to data points
- Determines upper bounds
- Indirect: Use the moments of  $F_2$ :
  - $M_2^{(L),QCD}(Q^2) = \frac{4}{9\pi} \alpha_s(Q^2) M_2^{(2)}(Q^2)$
  - Better precision, good agreement with exp. behaviour

Exp Nachtmann  $M_2^{(L)}$ : P. Monaghan, A. Accardi, M. E. Christy, C. E. Keppel, W. Melnitchouk, and L. Zhu, [Phys. Rev. Lett. 110, 152002 \(2013\)](#), arXiv:1209.4542 [nucl-ex].

# Summary

- A versatile approach!  $F_1, F_2$ , and  $F_L$
- Systematic investigation of power corrections, higher-twist effects and scaling is within reach
- Overcomes the operator mixing/renormalisation issues
- Can be extended to:
  - mixed currents, interference terms
  - spin-dependent structure functions
- GPDs: A. Hannaford-Gunn et al. Phys. Rev. D **105**, 014502

