Moments of the longitudinal structure function of proton from lattice QCD simulations

K. Utku Can
CSSM, The University of Adelaide

in collaboration with QCDSF/UKQCD:
A. Hannaford-Gunn, K. Y. Somfleth, R. D. Young, J. M. Zanotti (Adelaide),
R. Horsley (Edinburgh), P. E. L. Rakow (Liverpool), H. Perlt (Leipzig), G. Schierholz (DESY), H. Stüben (Hamburg),
Y. Nakamura (RIKEN, Kobe)

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Partially based on:
Motivation

- **Nucleon structure** (leading twist)
  - Structure functions from first principles
  - Understanding the behaviour in the high- and low-x regions

- **Scaling and Power corrections/Higher twist effects**
  - $Q^2$ cuts of global QCD analyses
  - Twist-4 contributions
  - Kinematic effects

- **New physics searches**
  - Weak charge of the proton
  - $\gamma - W/Z$ interference

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![Graph showing nucleon structure](image1.png)

![Diagram illustrating PDG 2020](image2.png)
Motivation

Technical issues

- Operator mixing/renormalisation issues in OPE approach in LQCD

\[
\mu(Q^2) = c_2(a^2 Q^2) v_2(a) + \frac{c_4(a^2 Q^2)}{Q^2} v_4(a) + \cdots
\]

- 4-point functions are costly; harder to tackle
- Feynman-Hellmann (FH) approach needs 2-point functions only
LQCD landscape

Krzysztof Cichy @ LATTICE’21 plenary, arXiv:2110.07440

- QCDSF/UKQCD Collaboration
- Extended to nucleon $F_2$ and $F_L$
- Study of higher-twist
**Forward Compton Amplitude**

\[ T_{\mu \nu}(p, q) = i \int d^4z e^{iq \cdot z} \rho_{ss'} \langle p, s' | \mathcal{T} \{ J_\mu(z) J_\nu(0) \} | p, s \rangle \quad \text{spin avg.} \quad \rho_{ss'} = \frac{1}{2} \delta_{ss'} \]

\[ \omega = \frac{2p \cdot q}{Q^2} \]

\[ = \left( -g_{\mu \nu} + \frac{q_\mu q_\nu}{q^2} \right) \mathcal{F}_1(\omega, Q^2) + \left( p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left( p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \frac{\mathcal{F}_2(\omega, Q^2)}{p \cdot q} \]

Compton Structure Functions (SF)

\[ \sim 2 \text{ Im} \left( \begin{array}{c} J_\mu(q) \\ N(p) \end{array} \right) \]

**Same Lorentz decomposition as the Hadronic Tensor**

**DIS Cross Section ~ Hadronic Tensor**

**Forward Compton Amplitude ~ Compton Tensor**
we can write down dispersion relations and connect
Compton SFs to DIS SFs:

\[ \mathcal{F}_1(\omega, Q^2) - \mathcal{F}_1(0, Q^2) = 2\omega^2 \int_0^1 dx \frac{2xF_1(x, Q^2)}{1 - x^2\omega^2 - i\epsilon} \]

\[ \mathcal{F}_2(\omega, Q^2) = 4\omega^2 \int_0^1 dx \frac{F_2(x, Q^2)}{1 - x^2\omega^2 - i\epsilon} \]

\[ \mathcal{F}_L(\omega, Q^2) + \mathcal{F}_1(0, Q^2) = \frac{8M_N^2}{Q^2} \int_0^1 dx F_2(x, Q^2) + 2\omega^2 \int_0^1 dx \frac{F_L(x, Q^2)}{1 - x^2\omega^2 - i\epsilon} \]
Nucleon Structure Functions

- Mellin moments

\[ \mathcal{F}_{1,L}(\omega, Q^2) = \sum_{n=0}^{\infty} 2 \omega^{2n} M_{2n}^{(1,L)}(Q^2), \text{ where } M_{2n}^{(1)}(Q^2) = 2 \int_{0}^{1} dx x^{2n-1} F_{1}(x, Q^2), \text{ and } M_{0}^{(1)}(Q^2) = 0 \]

\[ \mathcal{F}_{2}(\omega, Q^2) = \sum_{n=1}^{\infty} 4 \omega^{2n-1} M_{2n}^{(2,L)}(Q^2), \text{ where } M_{2n}^{(2,L)}(Q^2) = \int_{0}^{1} dx x^{2n-2} F_{2,L}(x, Q^2), \text{ and } M_{0}^{(L)}(Q^2) = \frac{4 M_{N}^{2}}{Q^2} M_{2}^{(2)}(Q^2) \]

- \( \mu = \nu = 3 \) and \( p_3 = q_3 = 0 \) \( \implies \) \( \mathcal{F}_{1}(\omega, Q^2) = T_{33}(p, q) \)

- \( \mu = \nu = 0 \) and \( p_3 = q_3 = q_0 = 0 \) \( \implies \) \( \mathcal{F}_{2}(\omega, Q^2) = [T_{00}(p, q) + T_{33}(p, q)] \frac{Q^2 \omega}{2E_{N}^2} \)

Once we have the Compton amplitude, \( T_{\mu\nu}(p, q) \), we can extract the Mellin moments!
**FH Theorem at 1st order**

*in Quantum Mechanics:*

$$\frac{\partial E_\lambda}{\partial \lambda} = \langle \phi_\lambda | \frac{\partial H_\lambda}{\partial \lambda} | \phi_\lambda \rangle$$

- $H_\lambda$: perturbed Hamiltonian of the system
- $E_\lambda$: energy eigenvalue of the perturbed system
- $\phi_\lambda$: eigenfunction of the perturbed system

- expectation value of the perturbed system is related to the shift in the energy eigenvalue

*in Lattice QCD: energy shifts in the presence of a weak external field*

$$S \rightarrow S(\lambda) = S + \lambda \int d^4x \, \mathcal{O}(x)$$

e.g. local bilinear operator

$$\bar{q}(x) \Gamma_\mu q(x)$$

- $\Gamma_\mu \in \{1, \gamma_\mu, \gamma_5 \gamma_\mu, \ldots\}$

@ 1st order

$$\frac{\partial E_\lambda}{\partial \lambda} = \frac{1}{2E_\lambda} \langle 0 | \mathcal{O} | 0 \rangle$$

- $E_\lambda \rightarrow$ spectroscopy, 2-pt function
- $\langle 0 | \mathcal{O} | 0 \rangle \rightarrow$ determine 3-pt

**Applications:**
- $\sigma$ - terms
- Form factors
Compton Amplitude from FHT at 2\textsuperscript{nd} order

- unpolarised Compton Amplitude
  \[ T_{\mu\mu}(p,q) = \int d^4z e^{iq\cdot z} \langle N(p) | \mathcal{T} \{ J_\mu(z) J_\mu(0) \} | N(p) \rangle \]

- Action modification
  \[ S \rightarrow S(\lambda) = S + \lambda \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) J_\mu(z) \]

2\textsuperscript{nd} order derivatives of the 2-pt correlator, \( G_\lambda^{(2)}(p;t) \), in the presence of the external field

\[
\left. \frac{\partial^2 G_\lambda^{(2)}(p;t)}{\partial \lambda^2} \right|_{\lambda=0} = \left( \frac{\partial^2 A_\lambda(p)}{\partial \lambda^2} - tA(p) \frac{\partial^2 E_{N\lambda}(p)}{\partial \lambda^2} \right) e^{-E_N(p)t} \quad \text{from spectral decomposition} \\
\left. \frac{\partial^2 G_\lambda^{(2)}(p;t)}{\partial \lambda^2} \right|_{\lambda=0} = \frac{A(p)}{2E_N(p)} t e^{-E_N(p)t} \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(p) | \mathcal{T} \{ J(z) J(0) \} | N(p) \rangle \quad \text{from path integral} \\
\]

- equate the time-enhanced terms:
  \[ \left. \frac{\partial^2 E_{N\lambda}(p)}{\partial \lambda^2} \right|_{\lambda=0} = -\frac{1}{2E_N(p)} \int d^4z (e^{iq\cdot z} + e^{-iq\cdot z}) \langle N(p) | \mathcal{T} \{ J(z) J(0) \} | N(p) \rangle + (q \rightarrow -q) \]

Compton amplitude is related to the second-order energy shift
Lattice Details

- FH implementation at the valence quark level
- Valence u/d quark props with modified action, $S(\lambda)$
- Local EM current insertion, $J_\mu(x) = Z V \bar{q}(x) \gamma_\mu q(x)$
- 4 Distinct field strengths, $\lambda = [\pm 0.0125, \pm 0.025]$
- 5 different current momenta in the range, $1.5 \lesssim Q^2 \lesssim 7 \text{GeV}^2$
- $\mathcal{O}(10^3)$ measurements for each pair of $Q^2$ and $\lambda$
- Access to a range of $\omega$ values for several $(p, q)$ pairs
- An inversion for each $q$ and $\lambda$, varying $p$ is relatively cheap
- Connected 2-pt correlators calculated only, no disconnected

QCDSF/UKQCD, 48$^3 \times 96$, 2+1 flavor (u/d+s)

$\beta = 5.65$, NP-improved Clover action


$m_\pi \sim 420 \text{ MeV}, \sim \text{SU(3) sym.}$

$m_\pi L = 6.9$

$a = 0.068 \text{ fm}$

$96 \times a$

$6.5 \text{ fm}$

$48^3 a^3$

$3.3^3 \text{ fm}^3$

Unmodified QCD background

Moments of the Nucleon Structure Functions

Moments of the Nucleon Structure Functions


Compton Structure Functions

\(48^3 \times 96, \text{2+1 flavour}\)

\(a = 0.068 \text{ fm}\)

\(m_\pi \sim 420 \text{ MeV}\)

\(Q^2 = 4.9 \text{ GeV}^2\)

\(\omega = x^{-1} = 2p \cdot q/Q^2\)
Moments of the Nucleon Structure Functions

Moments | Fit details

\[ \mathcal{F}_1^{qq}(\omega, Q^2) = \sum_{n=0}^{\infty} 2\omega^{2n}M_{2n}^{(1)}(Q^2) \]

\[ \mathcal{F}_2^{qq}(\omega, Q^2) = \frac{\tau}{1 + \tau \omega^2} \sum_{n=0}^{\infty} 4\omega^{2n} \left[ M_{2n}^{(1)} + M_{2n}^{(L)} \right](Q^2), \text{ where} \]

\[ \tau = \frac{Q^2}{4M_N^2} \]

- Enforce monotonic decreasing of moments for \( u \) and \( d \) only, not necessarily true for \( u - d \)

\[ M_{2n}(Q^2) \geq M_{4n}(Q^2) \geq \cdots \geq M_{2n}(Q^2) \geq \cdots \geq 0 \]

We truncate at \( n = 6 \)

No dependence to truncation order for \( 3 \leq n \leq 10 \)

- Bayesian approach by MCMC method
  
  Sample the moments from Uniform priors

  \( M_2(Q^2) \sim \mathcal{U}(0, 1) \)
  \( M_{2n}(Q^2) \sim \mathcal{U}(0, M_{2n-2}(Q^2)) \)

  Normal Likelihood function, \( \exp(-\chi^2/2) \)

  \[ \chi^2 = \sum_{i} \frac{(\mathcal{F}_i - \mathcal{F}_{\text{obs}}(\omega_i))^2}{\sigma_i^2} \]

  errors via bootstrap analysis

Scaling and Power Corrections

Moments | Scaling and Power Corrections

- Unique ability to study the $Q^2$ dependence of the moments!

- Global PDF-fit cuts $\sim 10 \text{ GeV}^2$

- Need $Q^2 > 10 \text{ GeV}^2$ data to reliably extract partonic moments

- Power corrections below $\sim 3 \text{ GeV}^2$?

- Modelling via

$$M_2^{(2)}(Q^2) = M_2^{(2)} + C_2^{(2)}/Q^2$$

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Scaling and Power Corrections

Moments | $F_L$

- **Unique ability to study the moments of $F_L$!**

Possible for the first time in a lattice QCD simulation!

- **Direct:** Fit to data points
  - Determines upper bounds

- **Indirect:** Use the moments of $F_2$:
  - $M_2^{(L), QCD}(Q^2) = \frac{4}{9 \pi} \alpha_s(Q^2) M_2^{(2)}(Q^2)$
  - Better precision, good agreement with exp. behaviour

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Summary

- A versatile approach! \( F_1, F_2, \) and \( F_L \)
- Systematic investigation of power corrections, higher-twist effects and scaling is within reach
- Overcomes the operator mixing/renormalisation issues
- Can be extended to:
  - mixed currents, interference terms
  - spin-dependent structure functions
  arXiv:2110.11532