Proton spin, topology and confinement: lessons from $QCD_2$.

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based on arXiv:2204.10827 with Adrien Florio and Dmitri E. Kharzeev
Outline

• $\text{QCD}_2$ and the Sine-Gordon model

• Constructing a baryon state in $\text{QCD}_2$

• Chirality distributions inside topological baryons and non-topological "mesons" in $\text{QCD}_2$

• Outlook
Motivation

Operator product expansion:

\[ J^\mu J^\nu \sim 2\epsilon^{\mu\nu\lambda\rho} \frac{p_\lambda}{Q^2} \left[ C^{NS} \left( j_{\rho 5}^3 + \frac{1}{\sqrt{3}} j_{\rho 5}^8 \right) + \frac{2\sqrt{2}}{\sqrt{3}} C^S j_{\rho 5}^0 \right], \]

\[ j_{\mu 5}^a = \bar{q} \gamma_\mu \gamma_5 T^a q \] are the QCD axial currents.

In parton model

\[ \int_0^1 dx B g_1(x_B) = \frac{1}{2} \left[ \frac{4}{9} \Delta u + \frac{1}{9} \Delta d + \frac{1}{9} \Delta s \right]. \]

So we are interested in

\[ 2 M s \mu g_A^{(0)} = \langle p, s | j_{\mu 5}^0 | p, s \rangle. \]
QCD$_2$ in the 't Hooft limit

\[ \mathcal{L}_{QCD_2} = -\frac{1}{4} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + i\bar{q}\gamma^\mu(\partial_\mu - igA_\mu)q - m\bar{q}q \]

't Hooft limit: \( N \to \infty, \; g \to 0, \; N \cdot g^2 \equiv \lambda = \text{const.} \).

Similarities to QCD$_4$:

- Mass gap generation
- Chiral symmetry breaking

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Leading order in \( \frac{1}{N} \) after bosonization is the Sine-Gordon model [2]:

\[ \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - m'^2 \cos \left( \frac{\phi}{f} \right), \]

where \( f = \sqrt{\frac{N}{4\pi}}, \ m' = \sqrt{\frac{N}{\sqrt{\pi}}}mg. \)

Topological soliton in the Sine-Gordon model

A static soliton "kink" solution of the classical equation of motion in the Sine-Gordon model:

\[
\phi_c(x) = \sqrt{\frac{4N}{\pi}} \arctan e^{\sqrt{\frac{4\pi}{N}} m'x}.
\]

The mass of the kink is (from the classical Hamiltonian):

\[
M_{kink} = 4\sqrt{\frac{N}{\pi}} m'.
\]
Baryon as a kink

Bosonization of the quark vector current

\[ \bar{\psi} \gamma^\mu \psi = j^\mu = \sqrt{\frac{N}{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi \]

gives the charge of a solution in the Sine-Gordon model

\[ Q = \sqrt{\frac{N}{\pi}} \left[ \phi(x \to \infty) - \phi(x \to -\infty) \right] . \]
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Charge of the topological kink is encoded in topology:

\[ Q_{kink} = N. \]
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We want to study chirality inside a baryon:

\[ \langle kink | j^5_\mu | kink \rangle . \]

Axial current in 1+1:

\[ j^5_\mu = \epsilon^{\mu\nu} j_\nu = \sqrt{\frac{N}{\pi}} \partial_\mu \phi. \]
Constructing the quantum soliton state

The defining property of a soliton state [1]:

\[ \langle \text{kink} | \hat{\phi}(x) | \text{kink} \rangle = \phi_c(x). \]

Classical field profile and quantum field operator can be decomposed into Fourier components:

\[ \phi_c(x) = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega(k)}} (\alpha_k e^{ikx} + \alpha_k^* e^{-ikx}), \]

\[ \hat{\phi}(x) = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega(k)}} (a_{k}^{\text{sol}} e^{ikx} + a_{k}^{\text{sol}\dagger} e^{-ikx}) \]

Coherent states: \( a_{k}^{\text{sol}} |\alpha_k\rangle = \alpha_k |\alpha_k\rangle. \) We define the kink state as

\[ |\text{kink}\rangle = \bigotimes_k |\alpha_k\rangle, \]

\[ \langle \text{kink} | a_{k}^{\text{sol}} | \text{kink} \rangle = \alpha_k \]

Topology and energy parts in solitonic constituents

\[ \phi_c(x) = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega(k)}} \left( \alpha_k e^{ikx} + \alpha_k^* e^{-ikx} \right), \]

Fourier coefficients can be represented as a product of topological and non-topological contributions:

\[ \alpha_k = t_k c_k, \quad t_k = \frac{i\sqrt{\omega(k)}}{k}, \quad c_k = -\sqrt{\frac{N\pi}{2}} \frac{1}{\cosh \sqrt{\frac{N}{4\pi} \frac{\pi k}{2m'}}} \]

This decomposition amounts to representing the kink as a convolution of topology and "energy":

\[ \sqrt{\frac{4N}{\pi}} \arctan e^{\sqrt{\frac{4\pi}{N} m'x}} = \sqrt{\frac{N}{\pi}} (\text{sign} \ast \text{sech}) \left( \sqrt{\frac{N}{4\pi} m'x} \right) + \frac{\pi}{2} \]
A baryon state on the light-cone

Boost into the Infinite Momentum Frame:

\[
\sqrt{\frac{4N}{\pi}} \arctan e^{\sqrt{\frac{4\pi}{N}} m'\gamma(x+\beta t)} \xrightarrow{\beta \to 1} \sqrt{\frac{4N}{\pi}} \arctan e^{\sqrt{\frac{4\pi}{N}} m'\gamma\sqrt{2x^+}}.
\]

The Bjorken $x$ is $x_B = \frac{k_+}{p_+}$, $p_+ = \sqrt{2\gamma M_p}$.

The occupation number of constituents

\[
N_{k_+} = |\alpha_{k_+}|^2 = \frac{\pi N}{\sqrt{2x_Bp_+}} \frac{1}{\cosh^2 (N x_B)}
\]

has a logarithmic divergence at low $x_B$ due to topology.
Chirality distribution inside a baryon

Axial current in bosonic language:

$$\bar{\psi} \gamma^\mu \gamma^5 \psi = j_5^\mu = \partial^\mu \phi.$$ 

In the momentum space in terms of $x_B$:

$$\langle kink | j_{5+} | kink \rangle = N \frac{1}{\cosh [N x_B]}.$$
Non-topological contribution

\[ \sqrt{\frac{N}{\pi}} \left( \text{sign} \ast \text{sech} \right) \left( \frac{\pi p_+ x^+}{2N} \right) \rightarrow \sqrt{\frac{N}{\pi}} \text{sech} \left( \frac{\pi p_+ x^+}{2N} \right) \]

Eliminate topology by crossing out the sign function. This way we get a non-topological ("meson") state.

\[ \langle \text{non-top.} | j_{5+} | \text{non-top.} \rangle \]

\[ = \frac{2N^2 x_B i}{\pi \cosh(Nx_B)}. \]

Chirality vanishes at \( x_B \to 0 \).
A close comparison of objects with and without topology

- Topology leads to an enhancement of chirality at small $x_B$
- For large $N$ the $x_B \sim 1$ region is suppressed
Summary and conclusion

- QCD$_2$ in the 't Hooft limit reduces to the exactly soluble Sine-Gordon model.
- An exact non-perturbative baryon state can be constructed, so the chirality distribution inside a baryon is accessible.
- Topological structure leads to a strong chirality enhancement at small $x_B$. This enhancement disappears if topology is "switched off". This is an exact result in QCD$_2$.

This suggests a sharp difference in small-$x_B$ behavior of polarized structure functions of baryons and mesons in QCD$_4$. 