

TMDs in dijet and heavy hadron pair production in SIDIS

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Outline

Kinematic region vs EIC

Factorization formula

- New dijet soft function

Evolution

- ϕ_b -angle and imaginary part
- ζ -prescription
- Scale choice and NP-model

Plots

Check our recent work:

Rafael F. del Castillo, Miguel G. Echevarría, Yiannis Makris, Ignazio Scimemi

<https://arxiv.org/abs/2008.07531v4>

<https://arxiv.org/abs/2111.03703v2>

Motivation

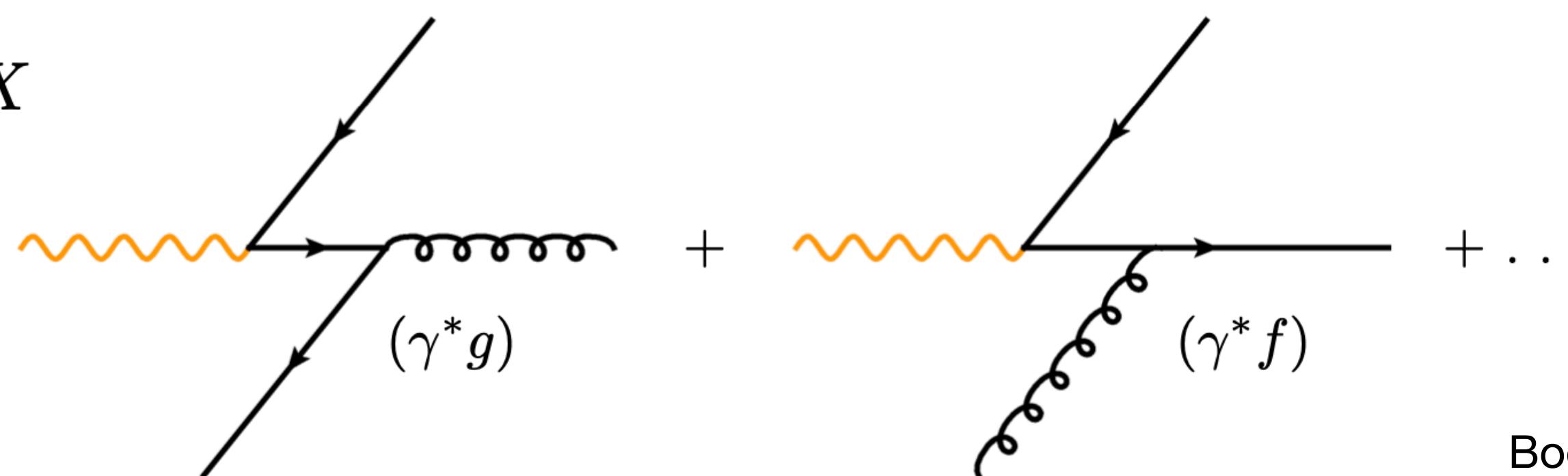
- Gluon transverse momentum dependent distributions (TMDs) are difficult to access due to the lack of clean processes where the factorization of the cross-section holds and incoming gluons constitute the dominant effect. E.g. Higgs production

Gutierrez-Reyez, Leal-Gómez, Scimemi, Vladimirov, 2019

- We consider two processes which are presently attracting increasing attention

$$\ell + h \rightarrow \ell' + J_1 + J_2 + X$$

dijet LO process:

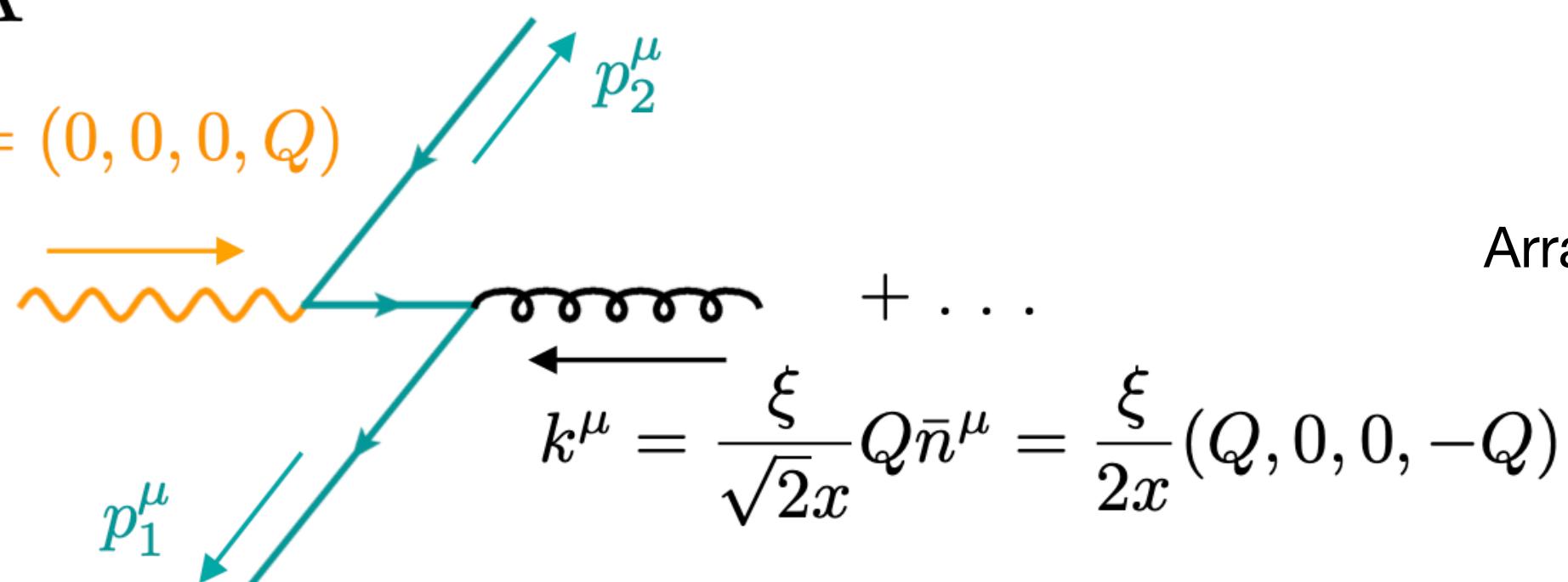


Boer, Brodsky, Mulders, Pisano, 2011

$$\ell + h \rightarrow \ell' + H + \bar{H} + X$$

$$q^\mu = \frac{Q}{\sqrt{2}}(n^\mu - \bar{n}^\mu) = (0, 0, 0, Q)$$

heavy meson pair at LO:



Arratia, Furletova, Hobbs, Olness, Nguyen et al. 2020

$$k^\mu = \frac{\xi}{\sqrt{2}x} Q \bar{n}^\mu = \frac{\xi}{2x} (Q, 0, 0, -Q)$$

Zhang, 2017

Dominguez, Xiao, Yuan, 2013

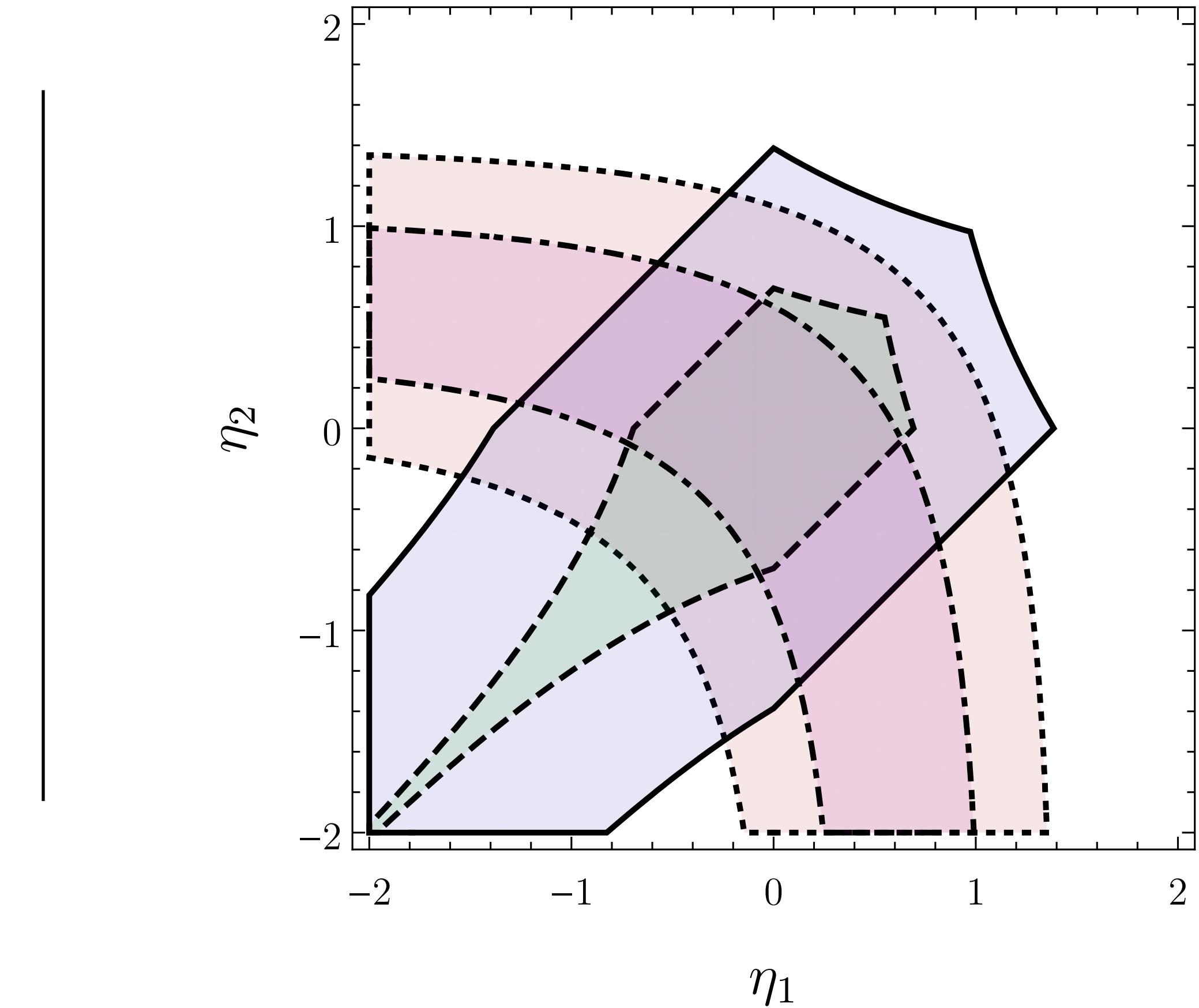
Kinematic region

Dijet production

$$\mathbf{r}_T = \mathbf{p}_{1T} + \mathbf{p}_{2T}$$

$$p_T = \frac{|\mathbf{p}_{1T}| + |\mathbf{p}_{2T}|}{2}$$

$$|\mathbf{r}_T| \ll p_T$$



$$\frac{1}{4} < \frac{\hat{s}}{|\hat{t}|}, \frac{\hat{s}}{|\hat{u}|}, \frac{|\hat{u}|}{|\hat{t}|} < 4$$

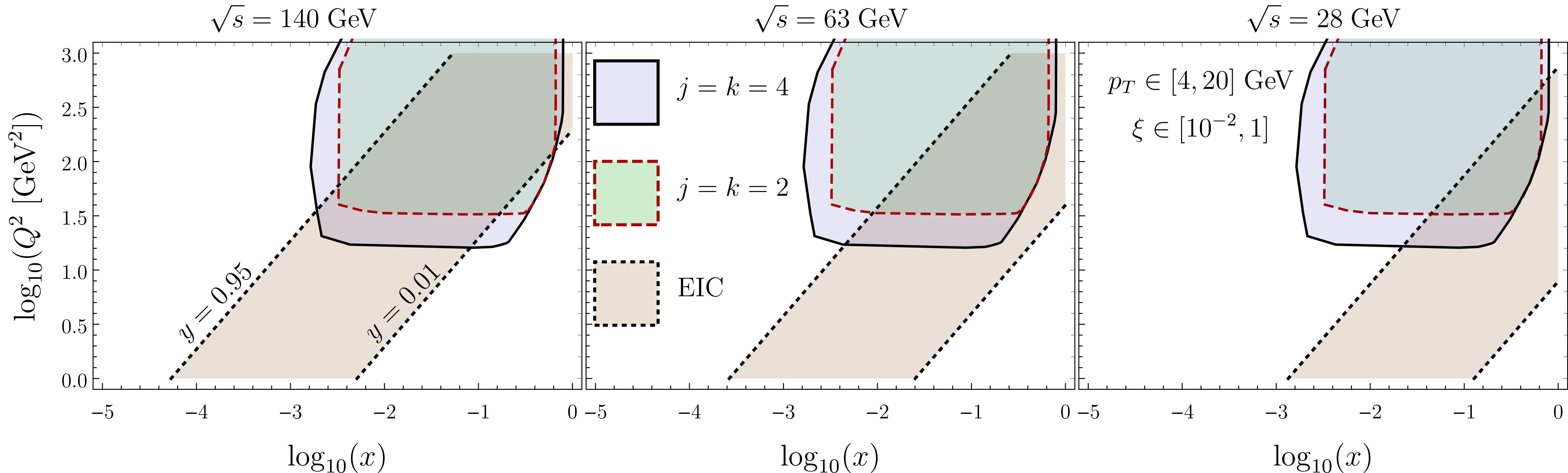
$$\frac{1}{2} < \frac{\hat{s}}{|\hat{t}|}, \frac{\hat{s}}{|\hat{u}|}, \frac{|\hat{u}|}{|\hat{t}|} < 2$$

$$\frac{1}{4} < \frac{Q^2}{4p_T^2} < 4$$

$$\frac{1}{2} < \frac{Q^2}{4p_T^2} < 2$$

Factorization holds for $|\mathbf{r}_T| \ll p_T$ and for the central rapidity region

Kinematic region vs EIC coverage



$$\frac{1}{j} < \frac{\hat{s}}{|\hat{t}|}, \frac{\hat{s}}{|\hat{u}|}, \frac{|\hat{u}|}{|\hat{t}|} < j$$

$$\frac{1}{k} < \frac{Q^2}{4p_T^2} < k$$

Overlapping increases with higher beam energies

Factorization

$$F_g^{\mu\nu}(\xi, \mathbf{b}) = f_1(\xi, \mathbf{b}) \frac{g_T^{\mu\nu}}{d-2} + h_1^\perp(\xi, \mathbf{b}) \underbrace{\left(\frac{g_T^{\mu\nu}}{d-2} + \frac{b^\mu b^\nu}{\mathbf{b}^2} \right)}$$

Dijet

$$\left\{ \begin{array}{l} \frac{d\sigma(\gamma^* g)}{dx d\eta_1 d\eta_2 dp_T d\mathbf{r}_T} = \sum_f H_{\gamma^* g \rightarrow f\bar{f}}^{\mu\nu}(\hat{s}, \hat{t}, \hat{u}, \mu) \int \frac{d^2 \mathbf{b}}{(2\pi)^2} \exp(i\mathbf{b} \cdot \mathbf{r}_T) F_{g,\mu\nu}(\xi, \mathbf{b}, \mu, \zeta_1) \\ \quad \times S_{\gamma g}(\mathbf{b}, \eta_1, \eta_2, \mu, \zeta_2) (\mathcal{C}_f(\mathbf{b}, R, \mu) J_f(p_T, R, \mu)) (\mathcal{C}_{\bar{f}}(\mathbf{b}, R, \mu) J_{\bar{f}}(p_T, R, \mu)) \\ \\ \frac{d\sigma^U(\gamma^* f)}{dx d\eta_1 d\eta_2 dp_T d\mathbf{r}_T} = \sum_f \sigma_0^{fU} H_{\gamma^* f \rightarrow g f}^U(\hat{s}, \hat{t}, \hat{u}, \mu) \int \frac{d^2 \mathbf{b}}{(2\pi)^2} \exp(i\mathbf{b} \cdot \mathbf{r}_T) F_f(\xi, \mathbf{b}, \mu, \zeta_1) \\ \quad \times S_{\gamma f}(\mathbf{b}, \zeta_2, \mu) (\mathcal{C}_g(\mathbf{b}, R, \mu) J_g(p_T, R, \mu)) (\mathcal{C}_f(\mathbf{b}, R, \mu) J_f(p_T, R, \mu)) \end{array} \right.$$

Hornig, Makris, Mehen, 2016

HHP

$$\begin{aligned} \frac{d\sigma(\gamma^* g)}{dx d\eta_H d\eta_{\bar{H}} dp_T d\mathbf{r}_T} &= H_{\gamma^* g \rightarrow Q\bar{Q}}^{\mu\nu}(\hat{s}, \hat{t}, \hat{u}, \mu) \int \frac{d\mathbf{b}}{(2\pi)^2} \exp(i\mathbf{b} \cdot \mathbf{r}_T) F_{g,\mu\nu}(\xi, \mathbf{b}, \mu, \zeta_1) \\ &\quad \times S_{\gamma g}(\mathbf{b}, \mu, \zeta_2) H_+(m_Q, \mu) \mathcal{J}_{Q \rightarrow H} \left(\mathbf{b}, \frac{m_Q}{p_T}, \mu \right) H_+(m_Q, \mu) \mathcal{J}_{\bar{Q} \rightarrow \bar{H}} \left(\mathbf{b}, \frac{m_Q}{p_T}, \mu \right) \end{aligned}$$

Fickinger, Fleming, Kim, Mereghetti, 2016

n - incoming beam direction
 v_1 - jet 1 direction
 v_2 - jet 2 direction

New dijet soft function

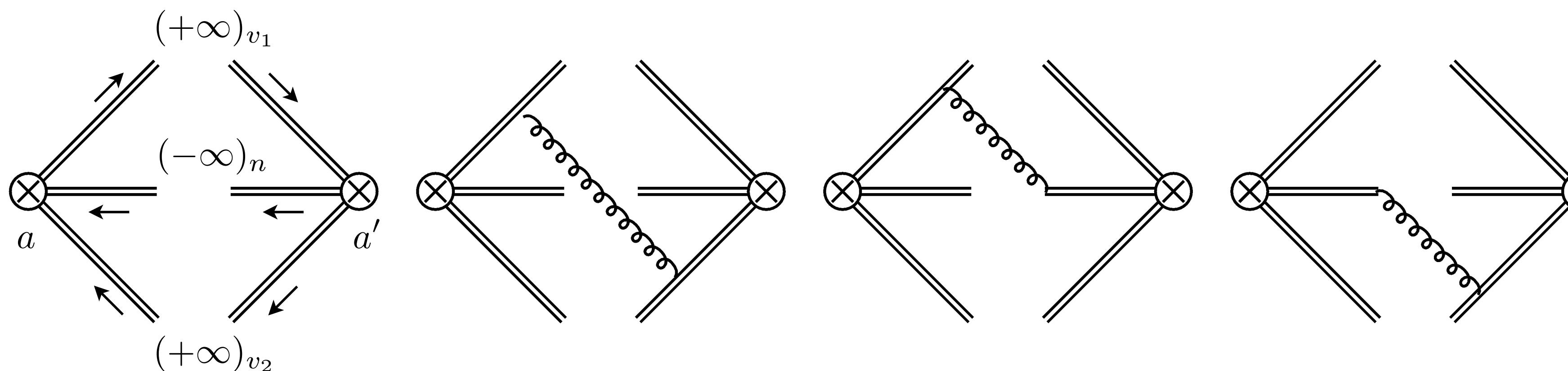
Soft function

$$\begin{aligned}\hat{S}_{\gamma g}(\mathbf{b}) &= \frac{1}{C_F C_A} \langle 0 | S_n^\dagger(\mathbf{b}, -\infty)_{ca'} \text{Tr} \left[S_{v_2}(+\infty, \mathbf{b}) T^{a'} S_{v_1}^\dagger(+\infty, \mathbf{b}) \right. \\ &\quad \times \left. S_{v_1}(+\infty, 0) T^a S_{v_2}^\dagger(+\infty, 0) \right] S_n(0, -\infty)_{ac} | 0 \rangle \\ \hat{S}_{\gamma f} &= \hat{S}_{\gamma g}(n \leftrightarrow v_2)\end{aligned}$$

Wilson lines

$$S_v(+\infty, \xi) = P\exp \left[-ig \int_0^{+\infty} d\lambda v \cdot A(\lambda v + \xi) \right] \quad S_{\bar{v}}^\dagger(+\infty, \xi) = P\exp \left[ig \int_0^{+\infty} d\lambda \bar{v} \cdot A(\lambda \bar{v} + \xi) \right]$$

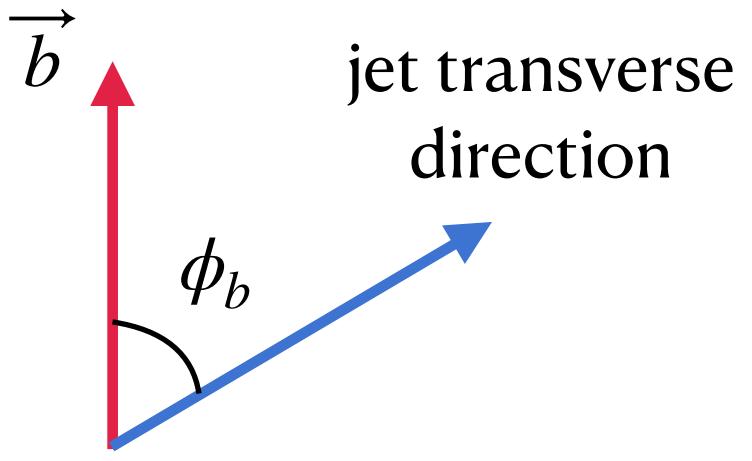
$$S_n(+\infty, \xi) = \lim_{\delta^+ \rightarrow 0} P\exp \left[-ig \int_0^{+\infty} d\lambda n \cdot A(\lambda n + \xi) e^{-\delta^+ \lambda} \right] \quad \delta \text{- regulator !!!}$$



Echevarría, Scimemi, Vladimirov, 2016

+ virtual diagrams
at one-loop order...

Evolution & imaginary part



- We find imaginary parts and ϕ_b -dependent parts in the perturbative result and ADs

$$\gamma_i(\mathbf{b}, \mu) = \gamma_{\text{cusp}} [\alpha_s] (c_i 2 \ln |\cos \phi_b| - c'_i i \pi \Theta(\phi_b)) + \text{other } \phi_b \text{ independent terms}$$

$$\sum_i c_i = \sum_i c'_i = 0 \quad \Theta(\phi_b) = \begin{cases} +1 & : -\pi/2 < \phi_b < \pi/2 \\ -1 & : \text{otherwise} \end{cases}$$

- We split the evolution kernels

$$S_{\gamma i}(\mathbf{b}, \mu_f, \zeta_{2,f}) = \exp \left[\int_{\mu_0}^{\mu_f} \left(\gamma_{S_{\gamma i}}^\phi(\phi) d \ln \mu \right) \right] \exp \left[\int_P \left(\bar{\gamma}_{S_{\gamma i}}(b, \mu, \zeta_2) d \ln \mu - \mathcal{D}_i(\mu, b) d \ln \zeta_2 \right) \right] S_{\gamma i}(\mathbf{b}, \mu_0, \zeta_{2,0})$$

$\mathcal{R}_S^\phi \rightarrow \text{Integrate over } \phi_b \qquad \mathcal{R}_S \rightarrow \zeta\text{-prescription} \qquad \text{Scimemi, Vladimirov, 2018}$
 $\text{Scimemi, Vladimirov, 2020}$

$$\mathcal{C}_i(\mathbf{b}, R, \mu_f) = \exp \left[\int_{\mu_i}^{\mu_f} \gamma_{\mathcal{C}_i}^\phi(\phi) d \ln \mu \right] \exp \left[\int_{\mu_i}^{\mu_f} \bar{\gamma}_{\mathcal{C}_i}(b, R, \mu) d \ln \mu \right] \mathcal{C}_i(\mathbf{b}, R, \mu_i)$$

$\mathcal{R}_{\mathcal{C}}^\phi \rightarrow \text{Integrate over } \phi_b \qquad \mathcal{R}_{\mathcal{C}} \rightarrow \text{Single scale evolution} \qquad \text{Hornig, Makris, Mehen, 2016}$

- ϕ_b angle is integrated out with the Fourier transform and imaginary parts cancel

Evolution & imaginary part

- After this manipulation b -space cross-section is proportional to:

$$d\sigma(\mathbf{b}) \sim |\cos \phi_b|^{2A} (\cos(\mathcal{B}\pi) - i\Theta(\phi_b) \sin(\mathcal{B}\pi)) \mathcal{R}(\{\mu_k\} \rightarrow \mu) \left[1 + \sum_{k \in \{H,F,J,S,C\}} a_s(\mu_k) f_k^{[1]}(b, \cos \phi_b) \right]$$

ϕ -independent and real kernel Perturbative result

$$\mathcal{A}(\{\mu_i\}) = \sum_{i \in \{S,C\}} c_i \int_{\mu_i}^{\mu} \gamma_{\text{cusp}} [\alpha_s] d \ln \mu'$$

$$\mathcal{B}(\{\mu_i\}) = \sum_{i \in \{S,C\}} c'_i \int_{\mu_i}^{\mu} \gamma_{\text{cusp}} [\alpha_s] d \ln \mu'$$

$$\sum_i c_i = \sum_i c'_i = 0$$

- All ϕ_b -integrals can be written in terms of a master integral

Master integral: $I_n(\mathcal{A}) \equiv \int_{-\pi}^{+\pi} d\phi_b |\cos \phi_b|^{2A} \ln^n |\cos \phi_b|$

- We need $2A > -1$ in order for the ϕ_b -integral to be well-defined \Rightarrow restriction over initial scales
- This restriction do not let us completely resum logs in collinear-soft and heavy meson jet function

Evolution & imaginary part

- We need $2\mathcal{A} > -1$ in order for the ϕ_b -integral to be well-defined \Rightarrow restriction over initial scales

Master integral: $I_n(\mathcal{A}) \equiv \int_{-\pi}^{+\pi} d\phi_b |\cos \phi_b|^{2\mathcal{A}} \ln^n |\cos \phi_b|$

$$I_0(\mathcal{A}) = \frac{2\sqrt{\pi} \Gamma(1/2 + \mathcal{A})}{\Gamma(1 + \mathcal{A})}, \quad \text{Not well-defined if } 2\mathcal{A} < -1$$

$$I_1(\mathcal{A}) = \frac{\sqrt{\pi} \Gamma(1/2 + \mathcal{A})}{\Gamma(1 + \mathcal{A})} (H_{\mathcal{A}-1/2} - H_{\mathcal{A}})$$

$$I_2(\mathcal{A}) = \frac{\sqrt{\pi} \Gamma(1/2 + \mathcal{A})}{2\Gamma(1 + \mathcal{A})} \left[(H_{\mathcal{A}-1/2} - H_{\mathcal{A}})^2 + \psi^{(1)}\left(\frac{1}{2} + \mathcal{A}\right) - \psi^{(1)}(1 + \mathcal{A}) \right]$$

$$\mathcal{A}(\{\mu_i\}) = \sum_{i \in \{S, C\}} c_i \int_{\mu_i}^{\mu} \gamma_{\text{cusp}} [\alpha_s] d \ln \mu'$$

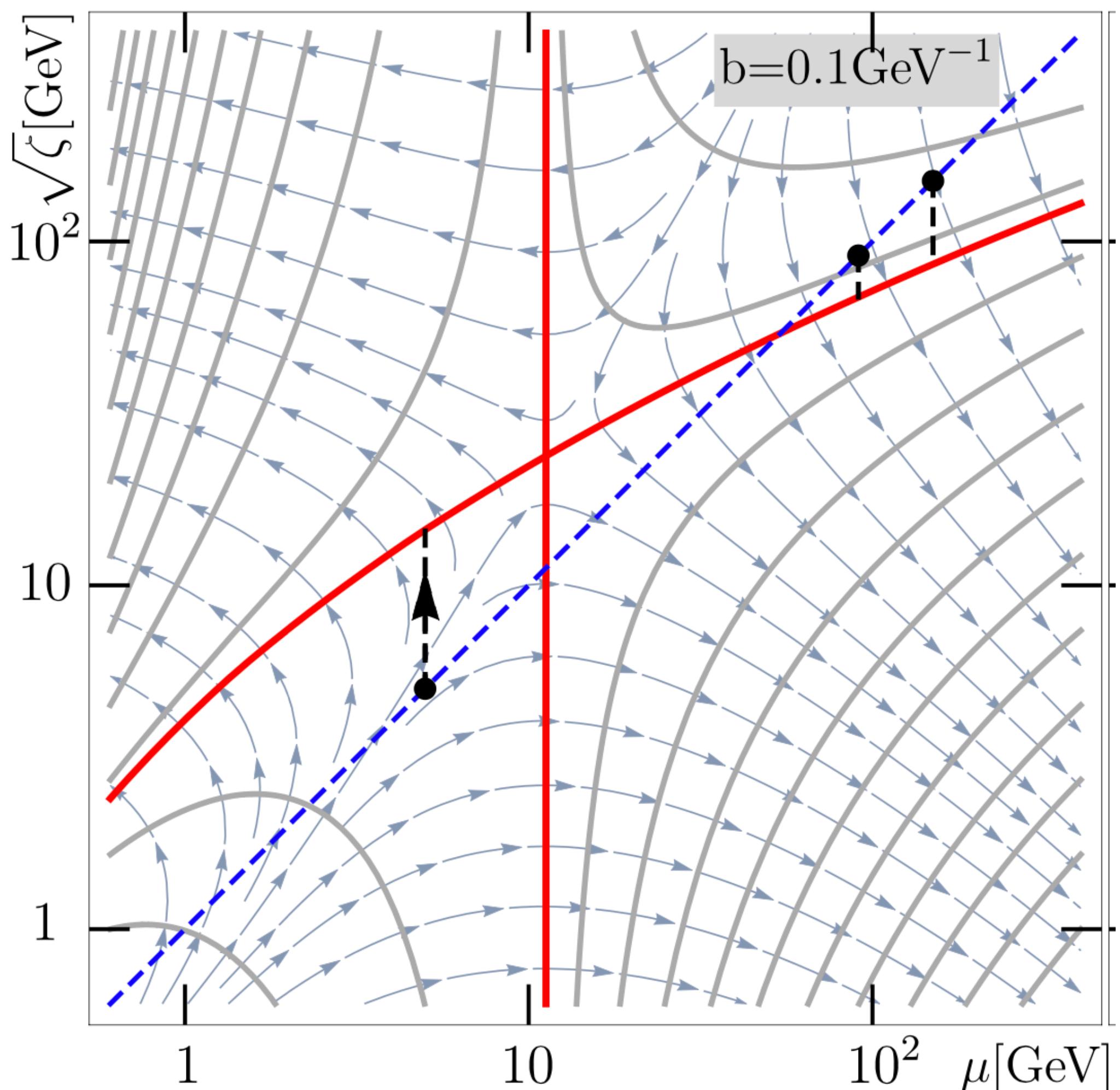
For linearly polarized gluons we have an extra $\cos 2\phi_b$:

$$I_n(\mathcal{A}) \rightarrow -I_n(\mathcal{A} + 1) + \frac{1}{2} I_n(\mathcal{A})$$

Same for angular modulation and Sivers asymmetry...

Evolution, ζ -prescription fixed μ evolution

Figure: Alexey Vladimirov & Ignazio Scimemi



Scimemi, Vladimirov, 2018
Scimemi, Vladimirov, 2020

Evolution kernel is given by

$$S(\mathbf{b}; \mu_f, \zeta_{2,f}) = \exp \left[\int_P (\gamma_S(\mu, \zeta_2) d \ln \mu - \mathcal{D}_S(\mu, b) d \ln \zeta_2) \right] S(\mathbf{b}; \mu_0, \zeta_{2,0})$$

$$\left. \begin{aligned} \frac{d}{d \ln \mu} S(\mathbf{b}; \mu, \zeta) &= \gamma_S(b; \mu, \zeta) S(\mathbf{b}; \mu, \zeta) \\ \frac{d}{d \ln \zeta} S(\mathbf{b}; \mu, \zeta) &= -\mathcal{D}_S(b, \mu) S(\mathbf{b}; \mu, \zeta) \end{aligned} \right\} \longrightarrow \boxed{\nabla F = E F}$$

$$E = (\gamma_S(b, \mu, \zeta), -\mathcal{D}_S(b, \mu))$$

Equipotential (null-evolution) line is given by $\gamma_S = 2\mathcal{D}_S \frac{d \ln \zeta_\mu}{d \ln \mu^2}$

gluon channel solution $\zeta_{2,\mu}^{\gamma^* g}(\mathbf{b}, \mu) = \left(\frac{\mu}{\mu_0} \right)^{\frac{2C_F}{C_A}} \zeta_{2,0} e^{v_S(\mathbf{b}, \mu)}$ perturbative

$$R_S((\mu_0, \zeta_{2,0}) \rightarrow (\mu_f, \zeta_f)) = \left(\frac{\zeta_f}{\zeta_{2,\mu}(\mathbf{b}, \mu_f)} \right)^{-\mathcal{D}_S(\mathbf{b}, \mu_f)}$$

Scale choices and NP-model

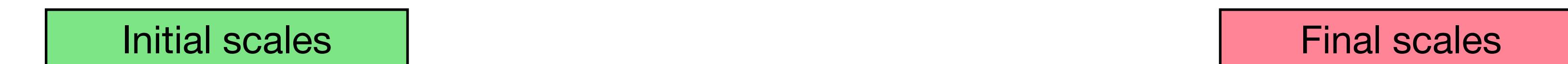
- For the new b -dependent function we consider a gaussian model for NP contribution

$$S_{\gamma i}(b; p_T, 1) = \mathcal{R}_S(\{\mu_0, \zeta_0\} \rightarrow \{p_T, 1\}) S_{\gamma i}^{pert}(b; \mu_0, \zeta_0) f_S^{\text{NP}}(b)$$

$$\mathcal{C}(b, R; p_T) = \mathcal{R}_{\mathcal{C}}(b, R; p_T, \mu_{\mathcal{C}}) \mathcal{C}^{pert}(b, R; \mu_{\mathcal{C}}) f_{\mathcal{C}}^{\text{NP}}(b, R)$$

$$\mathcal{J}(b, m_Q/p_T; p_T) = \mathcal{R}_{\mathcal{J}}(b, m_Q/p_T; p_T, \mu_{\mathcal{J}}) \mathcal{J}^{pert}(b, m_Q/p_T; \mu_{\mathcal{J}}) f_{\mathcal{J}}^{\text{NP}}(b; m_Q)$$

$$f_i^{\text{NP}}(b) = \exp \left(-\frac{b^2}{(B_{\text{NP}}^i)^2} \right)$$



$$\mu_{\mathcal{C}} = 2e^{-\gamma_E} \left(\frac{1}{b} + \frac{1}{b_{\max}} \right)$$

$$\mu_J = p_T R$$

$$\mu_f = p_T$$

$$\mu_{\mathcal{J}} = \frac{1}{2} e^{-\gamma_E} \left(\frac{1}{b} + \frac{1}{b_{\max}} \right)$$

$$\mu_+ = m_Q$$

$$\zeta_{2,0} = 1$$

$$\mu_S = \frac{2e^{-\gamma_E}}{b^*}, \quad b^* = \frac{b}{\sqrt{1 + b^2/b_{\max}^2}} \quad \zeta_{2,0}^{\gamma g} = \left(\frac{4p_T^2}{\hat{s}} \right)^{\frac{2C_F}{C_A}}$$

	\mathcal{C}	\mathcal{J}	S		\mathcal{C}	\mathcal{J}
$B_{\text{NP}}^i \text{ (GeV}^{-1}\text{)}$	2.5	2.5	2.5	$b_{\max} \text{ (GeV}^{-1}\text{)}$	0.5	0.3

Plots for phenomenological analysis

<https://teorica.fis.ucm.es/artemide/>
[https://github.com/vladimirovalexey/artemide-public.”](https://github.com/vladimirovalexey/artemide-public)

- We use **arTeMiDe** to obtain the plots
- TMDPDF and TMDFF structure and evolution is included arTeMiDe
- SF double-scale evolution and jet functions included as new modules

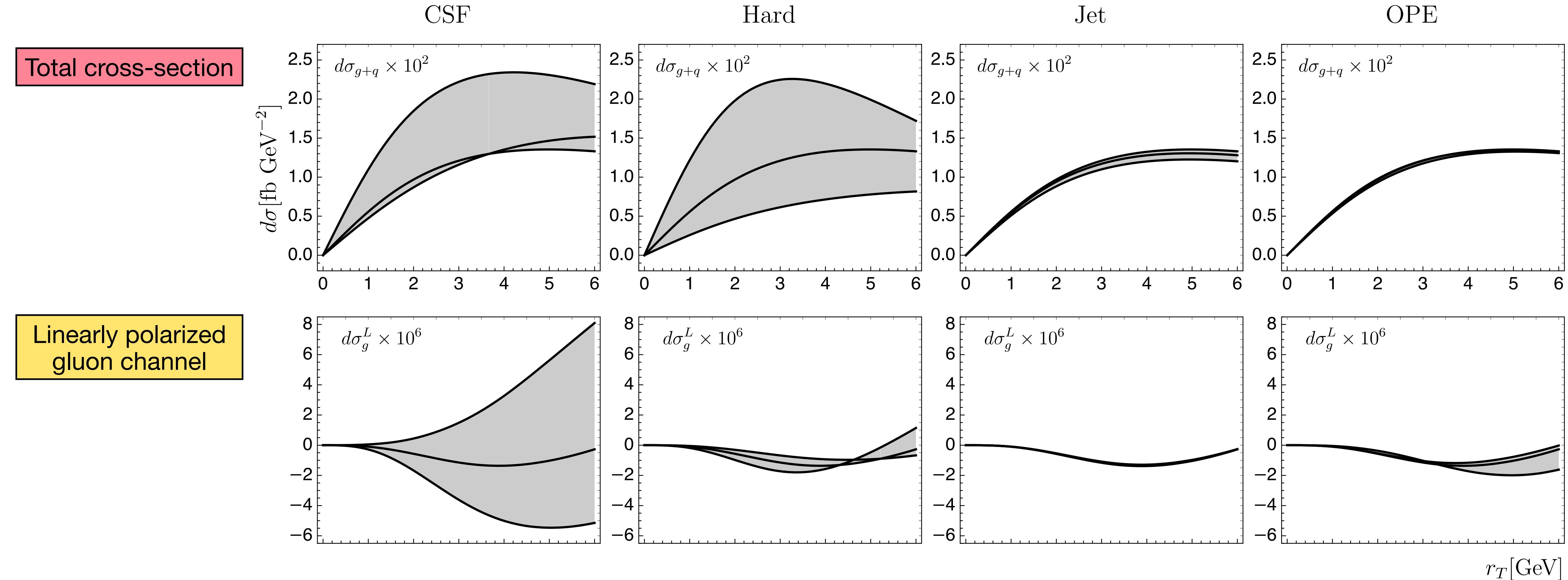
$p_T = 20 \text{ GeV}$ ($p_T \sim Q$)

$\sqrt{s} = 140 \text{ GeV}$

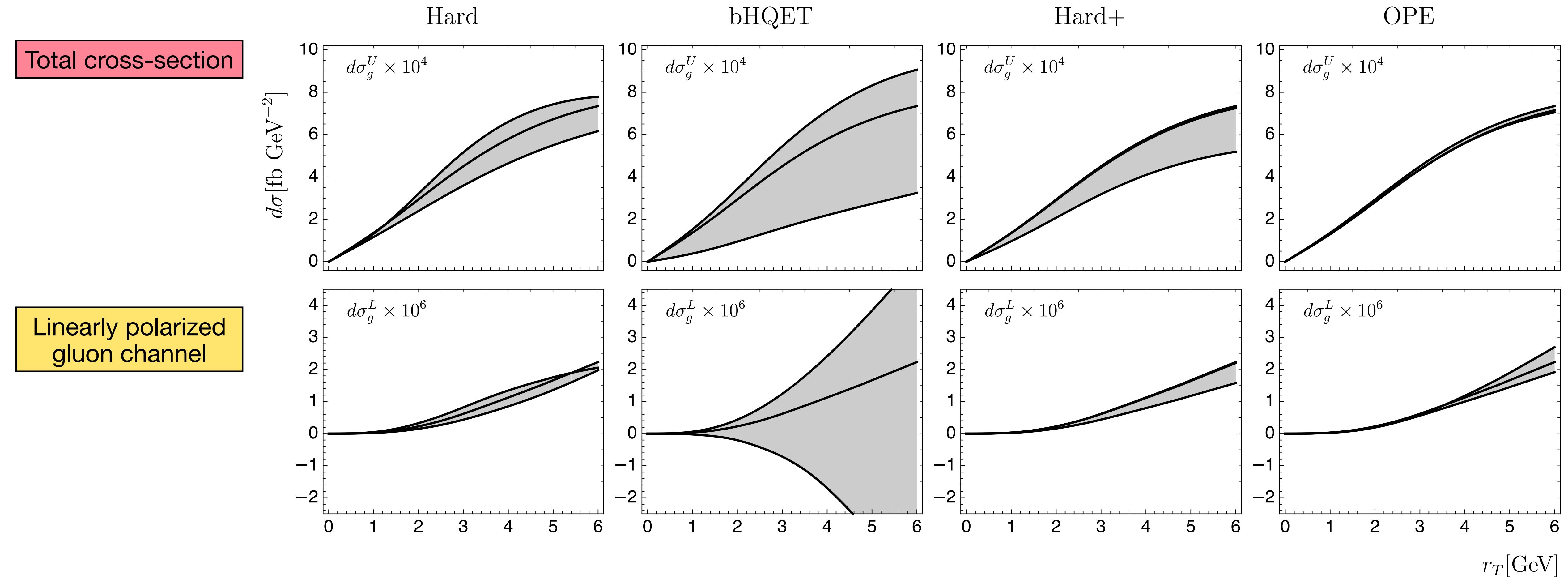
Integrated over x

Central rapidity region

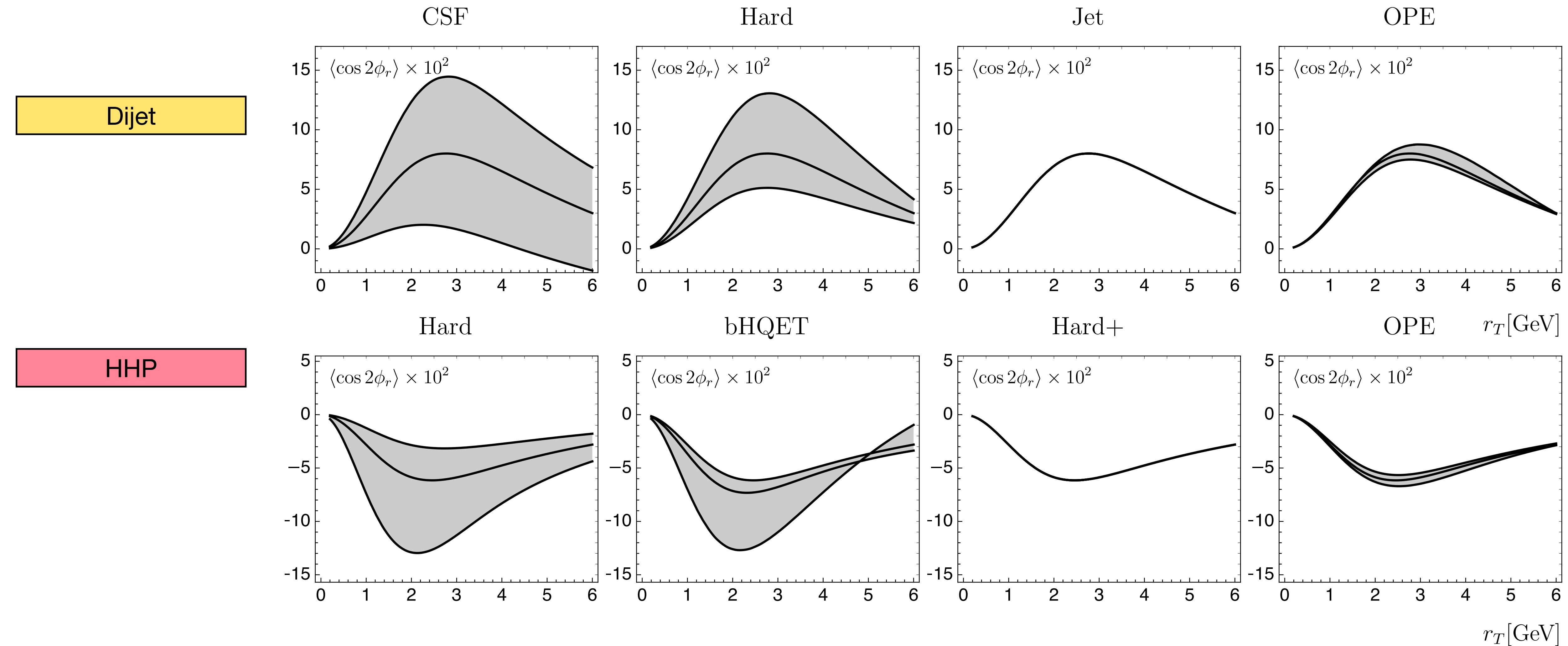
Dijet production



Heavy hadron pair production



$\langle \cos 2\phi_r \rangle$ - asymmetry



Conclusion

- We have established factorization for dijet and heavy hadron pair production
- Can be potentially observed in the future EIC
- We have been able to compute the new TMD Soft Function up to NLO and its anomalous dimension up to three-loops
- Rapidity structure of this new SF allows us to use the ζ -prescription
- The presence of the new SF makes the gluon TMDPDF extraction non-trivial
- Analysis of the numerical result for the cross-section shows the effect of linearly polarized gluon TMDs can be neglected compared to unpolarized gluon TMDs
- Future work: Gluon Sivers function, di-hadron production,...

Thank you for listening!

Backup

Evolution & imaginary part

Constant terms

$$I_{\text{const.}}(\mathcal{A}, \mathcal{B}) \equiv \int_{-\pi}^{+\pi} d\phi_b |\cos \phi_b|^{2\mathcal{A}} \left(\cos(\mathcal{B}\pi) - i\Theta(\phi_b) \sin(\mathcal{B}\pi) \right) = I_0(\mathcal{A}) \cos(\mathcal{B}\pi)$$

Single logarithmic terms

$$I_{\log}(\mathcal{A}, \mathcal{B}) \equiv \int_{-\pi}^{+\pi} d\phi_b |\cos \phi_b|^{2\mathcal{A}} \left(\cos(\mathcal{B}\pi) - i\Theta(\phi_b) \sin(\mathcal{B}\pi) \right) \ln(-i \cos \phi_b)$$

From the perturbative result

We rewrite $\ln(-i \cos \phi_b) = \ln |\cos \phi_b| - \frac{i\pi}{2} \Theta(\phi_b)$

$$I_{\log}(\mathcal{A}, \mathcal{B}) = I_1(\mathcal{A}) \cos(\mathcal{B}\pi) - \frac{\pi}{2} I_0(\mathcal{A}) \sin(\mathcal{B}\pi)$$

Imaginary part cancels in this way for every case

Zero-bin subtraction

Due to the rapidity divergencies structure of the two-direction soft function (zero-bin)
can be split as

$$S(\mathbf{b}, \mu, \sqrt{\delta^+ \delta^-}) = S^{\frac{1}{2}}(\mathbf{b}, \mu, \delta^+ \nu) S^{\frac{1}{2}}(\mathbf{b}, \mu, \delta^- / \nu)$$

ν arbitrary positive number

$$\left(S_i^{\text{bare}}(\mathbf{b}, \delta)\right)^{\frac{1}{2}} = 1 + a_s C_i \left\{ -\frac{2}{\epsilon^2} + \frac{4}{\epsilon} \ln\left(\frac{\sqrt{2}\delta}{\mu}\right) + \ln(B\mu^2 e^{2\gamma_E}) \left[4 \ln\left(\frac{\sqrt{2}\delta}{\mu}\right) + \ln(B\mu^2 e^{2\gamma_E}) \right] + \frac{\pi^2}{6} \right\}$$

In this way we defined the rapidity divergence-free objects

Universal TMDPDF

Rapidity divergence-free dijet soft function

$$F_i(\xi, \mathbf{b}, \mu, \zeta_1) = \frac{B_i^{\text{un.}}(\xi, \mathbf{b}, \mu, k^- / \delta^-)}{S^{\frac{1}{2}}(\mathbf{b}, \mu, \delta^- / \nu)} \Bigg|_{\sqrt{2}k^- / \nu \rightarrow \sqrt{\zeta_1}} \quad S_{\gamma i}(\mathbf{b}, \mu, \zeta_2) = \frac{\hat{S}_{\gamma i}(\mathbf{b}, \mu, \sqrt{A_n} \delta^+)}{S^{\frac{1}{2}}(\mathbf{b}, \mu, \delta^+ \nu)} \Bigg|_{\nu / \sqrt{2A_n} \rightarrow \sqrt{\zeta_2}}$$

ζ scale associated with the δ -regulator and zero-bin split

In the Breit frame

$$\zeta_1 \zeta_2 = p_T^2$$

Kinematic definitions

We define three light-like vectors for the incoming beam and the outgoing jets

$$n^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, 1), \quad n^2 = \bar{n}^2 = 0, \quad \bar{n} \cdot n = 1$$

$$v_J^2 = \bar{v}_J^2 = 0, \quad v_J \cdot \bar{v}_J = 1, \quad \text{with } J = 1, 2$$

The standard Lorentz invariants

$$Q^2 = -q^2, \quad x = \frac{Q^2}{2P \cdot q}, \quad \xi = \frac{k^+}{P^+}$$

The transverse momentum imbalance

$$\mathbf{r}_T = \mathbf{p}_{1T} + \mathbf{p}_{2T}, \quad p_T = \frac{|\mathbf{p}_{1T}| + |\mathbf{p}_{2T}|}{2}$$

Kinematic definitions

We can rewrite them in terms of the Born level kinematics

$$Q = 2p_T \cosh(\eta_-) \exp(\eta_+), \quad \xi = 2x \cosh(\eta_+) \exp(-\eta_+)$$

$$\eta_{\pm} = \frac{\eta_1 \pm \eta_2}{2}$$

The parsonic Mandelstam variables are given by

$$\hat{s} = (q + k)^2 = +4p_T^2 \cosh^2(\eta_-)$$

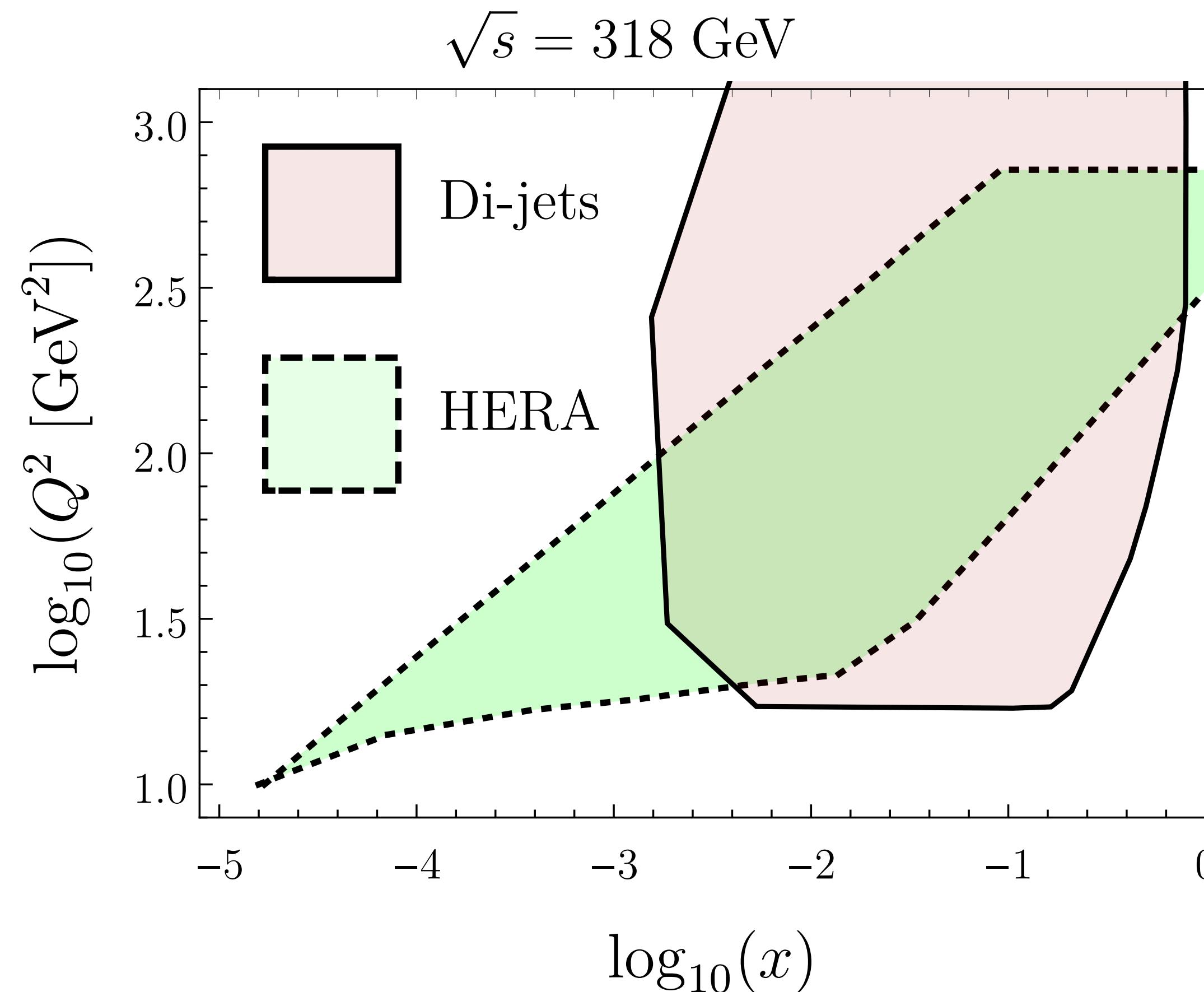
$$\hat{t} = (q - p_2)^2 = -4p_T^2 \cosh(\eta_-) \cosh(\eta_+) \exp(\eta_1)$$

$$\hat{u} = (q - p_1)^2 = -4p_T^2 \cosh(\eta_-) \cosh(\eta_+) \exp(\eta_2)$$

$$\hat{s} + \hat{t} + \hat{u} = -Q^2$$

Kinematic region vs HERA coverage

Dijet production



Cross-section factorization

Dijet production

$$\frac{d\sigma}{dx d\eta_1 d\eta_2 dp_T dr_T}$$

We measure over

- x Bjorken variable
- η_i jet pseudorapidity
- p_T transverse momentum
- \mathbf{r}_T transverse momentum imbalance

$(\gamma^* g)$

$$\frac{d\sigma(\gamma^* g)}{dx d\eta_1 d\eta_2 dp_T dr_T} = \sum_f H_{\gamma^* g \rightarrow f\bar{f}}^{\mu\nu}(\hat{s}, \hat{t}, \hat{u}, \mu) \int \frac{d^2 \mathbf{b}}{(2\pi)^2} \exp(i\mathbf{b} \cdot \mathbf{r}_T) F_{g,\mu\nu}(\xi, \mathbf{b}, \mu, \zeta_1) \\ \times S_{\gamma g}(\mathbf{b}, \eta_1, \eta_2, \mu, \zeta_2) \left(\mathcal{C}_f(\mathbf{b}, R, \mu) J_f(p_T, R, \mu) \right) \left(\mathcal{C}_{\bar{f}}(\mathbf{b}, R, \mu) J_{\bar{f}}(p_T, R, \mu) \right)$$

$(\gamma^* f)$

$$\frac{d\sigma^U(\gamma^* f)}{dx d\eta_1 d\eta_2 dp_T dr_T} = \sigma_0^{fU} \sum_f H_{\gamma^* f \rightarrow g\bar{f}}^U(\hat{s}, \hat{t}, \hat{u}, \mu) \int \frac{d^2 \mathbf{b}}{(2\pi)^2} \exp(i\mathbf{b} \cdot \mathbf{r}_T) F_f(\xi, \mathbf{b}, \mu, \zeta_1) \\ \times S_{\gamma f}(\mathbf{b}, \zeta_2, \mu) \left(\mathcal{C}_g(\mathbf{b}, R, \mu) J_g(p_T, R, \mu) \right) \left(\mathcal{C}_{\bar{f}}(\mathbf{b}, R, \mu) J_{\bar{f}}(p_T, R, \mu) \right)$$

Unpolarized & linearly polarized cross-section

Dijet production

$$F_g^{\mu\nu}(\xi, \mathbf{b}) = \boxed{f_1(\xi, \mathbf{b})} \frac{g_T^{\mu\nu}}{d-2} + \boxed{h_1^\perp(\xi, \mathbf{b})} \left(\frac{g_T^{\mu\nu}}{d-2} + \frac{b^\mu b^\nu}{\mathbf{b}^2} \right)$$

$$H_{\gamma^* g \rightarrow f\bar{f}}^{\mu\nu} = \sigma_0^{gU} H_{\gamma^* g \rightarrow f\bar{f}}^U \frac{g_T^{\mu\nu}}{d-2} + \sigma_0^{gL} H_{\gamma^* g \rightarrow f\bar{f}}^L \left(-\frac{g_T^{\mu\nu}}{d-2} + \frac{v_{1T}^\mu v_{2T}^\nu + v_{2T}^\mu v_{1T}^\nu}{2 v_{1T} \cdot v_{2T}} \right)$$

Unpolarized cross-section

$$\begin{aligned} \frac{d\sigma^U(\gamma^* g)}{dx d\eta_1 d\eta_2 dp_T d\mathbf{r}_T} &= \sigma_0^{gU} \sum_f H_{\gamma^* g \rightarrow f\bar{f}}^U(\hat{s}, \hat{t}, \hat{u}, \mu) \int \frac{d^2 \mathbf{b}}{(2\pi)^2} \exp(i\mathbf{b} \cdot \mathbf{r}_T) f_1(\xi, \mathbf{b}, \mu, \zeta_1) \\ &\quad \times S_{\gamma g}(\mathbf{b}, \zeta_2, \mu) \left(\mathcal{C}_f(\mathbf{b}, R, \mu) J_f(p_T, R, \mu) \right) \left(\mathcal{C}_{\bar{f}}(\mathbf{b}, R, \mu) J_{\bar{f}}(p_T, R, \mu) \right) \end{aligned}$$

Linearly polarized
cross-section

$$\begin{aligned} \frac{d\sigma^L(\gamma^* g)}{dx d\eta_1 d\eta_2 dp_T d\mathbf{r}_T} &= \sigma_0^{gL} \sum_f H_{\gamma^* g \rightarrow f\bar{f}}^L(\hat{s}, \hat{t}, \hat{u}, \mu) \int \frac{d^2 \mathbf{b}}{(2\pi)^2} \exp(i\mathbf{b} \cdot \mathbf{r}_T) h_1^\perp(\xi, \mathbf{b}, \mu, \zeta_1) \\ &\quad \times \frac{s_{\mathbf{b}}^2 - c_{\mathbf{b}}^2}{2} S_{\gamma g}(\mathbf{b}, \zeta_2, \mu) \left(\mathcal{C}_f(\mathbf{b}, R, \mu) J_f(p_T, R, \mu) \right) \left(\mathcal{C}_{\bar{f}}(\mathbf{b}, R, \mu) J_{\bar{f}}(p_T, R, \mu) \right) \end{aligned}$$

Zero-bin subtraction

- We need to subtract the zero-bin from the TMD beam function
- The zero-bin corresponds to the two-direction back-to-back soft function (the one used in Drell-Yan or SIDIS). Here, we use the subtraction as done in Echevarría, Idilbi, Scimemi, 2013.
- We can reorganize the zero-bin to obtain rapidity divergence-free function as expressed in the cross-section factorization
- This leads to the universal TMDPDF and a rapidity divergence-free new TMD soft function and the introduction of the scale ζ

$$\begin{aligned} B_i^{\text{un.}}(\xi, \mathbf{b}, \mu, k^-/\delta^-) &\longrightarrow F_i(\xi, \mathbf{b}, \mu, \zeta_1) \\ \hat{S}_{\gamma i}(\mathbf{b}, \mu, \sqrt{A_n} \delta^+) &\longrightarrow S_{\gamma i}(\mathbf{b}, \mu, \zeta_2) \end{aligned}$$

Zero-bin subtraction

ζ scale

The scale can be removed from the final result by introducing the constraint

$$\zeta_1 \zeta_2 = \frac{(k^-)^2}{A_n} = \frac{\hat{u} \hat{t}}{\hat{s}}$$

In the Breit-frame this leads to

$$\zeta_1 \zeta_2 = p_T^2$$

Notice that

ζ_1 has square mass dimension

ζ_2 is dimensionless



$\zeta_1 = p_T^2$ natural way of
 $\zeta_2 = 1$ choosing the scale

Procedure totally analogous to the one used in Drell-Yan or SIDIS

This allows to use ζ -prescription for TMDPDF and SF evolution

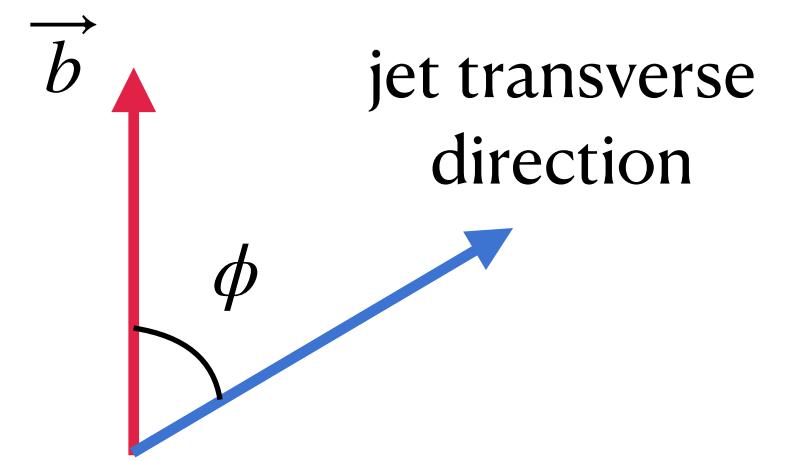
Zero-bin subtraction

Subtracted soft function, finite result

($\gamma^* g$) - channel

$$S_{\gamma g}(\mathbf{b}, \mu, \zeta_2) = 1 + a_s \left\{ C_F \left[\frac{\pi^2}{3} + 2 \ln^2 \left(\frac{B\mu^2 e^{2\gamma_E}}{-A_{\mathbf{b}}} \right) + 4 \text{Li}_2(1 + A_{\mathbf{b}}) \right] \right.$$
$$\left. + C_A \left[-2 \ln(B\mu^2 e^{2\gamma_E}) \boxed{\ln \zeta_2} - \ln^2(-A_{\mathbf{b}}) - \frac{\pi^2}{3} - 2 \text{Li}_2(1 + A_{\mathbf{b}}) \right] \right\} + \mathcal{O}(a_s^2)$$

with... $A_{\mathbf{b}} = \frac{(v_1 \cdot v_2)}{2(v_1 \cdot \hat{b})(v_2 \cdot \hat{b})} = -\frac{\hat{s}}{4 p_T^2 c_b^2}$



Consistency check

Dijet-production

$$\frac{d}{d \ln \mu} G(\mu) = \gamma_G(\mu) G(\mu)$$

$(\gamma^* g)$ -channel

$$\gamma_{H_{\gamma g}} + \gamma_{S_{\gamma g}} + \gamma_{F_g} + 2\gamma_{J_f} + \gamma_{C_1} + \gamma_{C_2} + \gamma_\alpha = 0$$

$(\gamma^* f)$ -channel

$$\gamma_{H_{\gamma f}} + \gamma_{S_{\gamma f}} + \gamma_{F_f} + \gamma_{J_f} + \gamma_{J_g} + \gamma_{C_f} + \gamma_{C_g} + \gamma_\alpha = 0$$

The sum of all anomalous dimensions should cancel for each channel

$$\gamma_{S_{\gamma g}}^{[1]} = 4 \left\{ -C_A \textcolor{red}{\boxed{\ln \zeta_2}} + 2C_F \left[\ln(B\mu^2 e^{2\gamma_E}) - \ln \hat{s} + \ln p_T^2 + \textcolor{green}{\boxed{\ln(4c_b^2)}} \right] \right\},$$

ζ -logs

ϕ -logs

$$\gamma_{S_{\gamma f}}^{[1]} = 4 \left\{ (C_F + C_A) \left[\ln(B\mu^2 e^{2\gamma_E}) - \ln \hat{s} + \ln p_T^2 + \textcolor{green}{\boxed{\ln(4c_b^2)}} \right] + (C_F - C_A) \left[\ln \left(\frac{\hat{t}}{\hat{u}} \right) - \kappa(v_f) \right] - C_F \textcolor{red}{\boxed{\ln \zeta_2}} \right\}$$

$$\gamma_{F_i}^{[1]} = 4C_i \left[-\textcolor{red}{\boxed{\ln \left(\frac{\zeta_1}{\mu^2} \right)}} + \gamma_i \right],$$

$$\gamma_{C_g}^{[1]} = 4C_A \left[-\ln(B\mu^2 e^{2\gamma_E}) + \ln R^2 - \textcolor{green}{\boxed{\ln(4c_b^2)}} + \kappa(v_g) \right]$$

$$\kappa(v_f) = -\kappa(v_{\bar{f}}) = -\kappa(v_g) = i\pi \text{sign}(c_b)$$

$$\gamma_{C_i}^{[1]} = 4C_F \left[-\ln(B\mu^2 e^{2\gamma_E}) + \ln R^2 - \textcolor{green}{\boxed{\ln(4c_b^2)}} + \kappa(v_i) \right]$$

They cancel !!!

Heavy-meson pair production

$$\ell + h \rightarrow \ell' + H + \bar{H} + X$$

$$q^\mu = \frac{Q}{\sqrt{2}}(n^\mu - \bar{n}^\mu) = (0, 0, 0, Q)$$

heavy meson pair at LO:

$$+ \dots$$
$$k^\mu = \frac{\xi}{\sqrt{2}x} Q \bar{n}^\mu = \frac{\xi}{2x} (Q, 0, 0, -Q)$$

- Experimentally more challenging
- Observation of charmed mesons could be possible

Arratia, Furletova, Hobbs, Olness, Nguyen et al. 2020

Li, Liu, Vitev, 2020

Chudakov, Higinbotham, Hyde, Furletov, Furletova, Nguyen, 2016

Cross-section factorization

Heavy meson pair production

$$\frac{d\sigma(\gamma^* g)}{dx d\eta_H d\eta_{\bar{H}} dp_T d\mathbf{r}_T}$$

We measure over

- x Bjorken variable
- $\eta_H, \eta_{\bar{H}}$ heavy meson pseudorapidity
- p_T transverse momentum
- \mathbf{r}_T transverse momentum imbalance

$$\frac{d\sigma(\gamma^* g)}{dx d\eta_H d\eta_{\bar{H}} dp_T d\mathbf{r}_T} = H_{\gamma^* g \rightarrow Q\bar{Q}}^{\mu\nu}(\hat{s}, \hat{t}, \hat{u}, \mu) \int \frac{d\mathbf{b}}{(2\pi)^2} \exp(i\mathbf{b} \cdot \mathbf{r}_T) F_{g,\mu\nu}(\xi, \mathbf{b}, \mu, \zeta_1)$$

$$\times S_{\gamma g}(\mathbf{b}, \mu, \zeta_2) J_{Q \rightarrow H}(\mathbf{b}, p_T, m_Q, \mu) J_{\bar{Q} \rightarrow \bar{H}}(\mathbf{b}, p_T, m_Q, \mu)$$

Fickinger, Fleming, Kim, Mereghetti, 2016

- Region sensitive to TMD
Factorization for highly boosted heavy mesons
We have a new scale

$$|\mathbf{r}_T| \ll p_T^{H, \bar{H}} \\ p_T^H \gg m_H \\ m_Q$$

Connection to the fragmentation shape function

Heavy meson pair production

We can see that the shape function is related to the heavy meson jet function

$$\mathcal{J}_{Q \rightarrow H}(\mathbf{b}) = \frac{m_H}{\sqrt{2} p_H^-} \tilde{S}_{Q \rightarrow H}\left(\tau \rightarrow \frac{\mathbf{v} \cdot \mathbf{b}}{\sqrt{2}}\right)$$

- We can check our NLO calculation (finite terms)
- We can get the jet function AD up to two loops

Fickinger, Fleming, Kim, Mereghetti, 2016

This sum is known up to three-loops...

$$\gamma_{S_{\gamma g}} = - (\gamma_{H_{\gamma g}} + \gamma_{F_g} + \gamma_\alpha + \gamma_{\mathcal{J}}(\mathbf{v}_1) + \gamma_{\mathcal{J}}(\mathbf{v}_2) + 2\gamma_+)$$

Soft function

anomalous dimension

$$\gamma_{S_{\gamma g}} = \gamma_{\text{cusp}} \left[2C_F \ln \left(\frac{B\mu^2 e^{2\gamma_E}}{-A_b} \right) - C_A \ln \zeta_2 \right] + \delta\gamma_{S_{\gamma g}}$$

$$\delta\gamma_{S_{\gamma g}}^{[1]} = 0$$

$$\delta\gamma_{S_{\gamma g}}^{[2]} = C_F \left[C_A \left(\frac{1616}{27} - \frac{22}{9}\pi^2 - 56\zeta_3 \right) + n_f T_F \left(-\frac{448}{27} + \frac{8}{9}\pi^2 \right) \right]$$

$$\delta\gamma_{S_{\gamma g}}^{[3]} = \dots$$

Connection to the fragmentation shape function

Heavy meson pair production

Shape function is defined as Fickinger, Fleming, Kim, Mereghetti, 2016

$$S_{Q \rightarrow H}(\omega) = \frac{1}{2N_c} \sum_X \langle 0 | \delta(\omega - i\sqrt{2} \bar{v} \cdot \partial) W_v^\dagger h_{v\beta_+} | H_\beta X \rangle \langle H_\beta X | \bar{h}_{v,\beta_+} W_v \frac{\not{v}}{\sqrt{2}} | 0 \rangle$$

and its Fourier transformation...

$$\begin{aligned} \tilde{S}_{Q \rightarrow H}(\tau) &= \int d\omega \exp(i\omega\tau) S_{Q \rightarrow H}(\omega) \\ &= \frac{1}{2m_H N_C} \sum_X \langle 0 | \exp(-\sqrt{2}\tau \bar{v} \cdot \partial) W_v^\dagger h_{v\beta_+} | XH \rangle \langle XH | \bar{h}_{Qv} W_v \frac{\not{v}}{\sqrt{2}} | 0 \rangle \end{aligned}$$

We can see that it is related to the bHQET jet function

$$\mathcal{J}_{Q \rightarrow H}(\mathbf{b}) = \frac{m_H}{\sqrt{2} p_H^-} \tilde{S}_{Q \rightarrow H} \left(\tau \rightarrow \frac{\mathbf{v} \cdot \mathbf{b}}{\sqrt{2}} \right)$$

- We can check our NLO calculation (finite terms)
- We can get the jet function AD up to two loops

Refactorization of heavy-quark fragmentation

Heavy meson pair production

We use heavy-quark jet function to describe the fragmentation of heavy mesons from heavy quarks. In the limit $r_T \ll p_T$ there are two scales that need to be resummed

$$\mu_+ = m_Q, \quad \text{and} \quad \mu_{\mathcal{J}} = m_Q \frac{r_T}{p_T}$$

To do this we use **bHQET (boosted heavy quark effective theory)** to factorize the jet function into a hard matching coefficient and a TMD matrix element

$$J_{Q \rightarrow H}(\mathbf{b}, p_T, m_Q, \mu) = H_+(m_Q, \mu) \mathcal{J}_{Q \rightarrow H}\left(\mathbf{b}, \frac{m_Q}{p_T}, \mu\right)$$

Appears for the first time

Known up to two-loops

Refactorization of heavy-quark fragmentation

Heavy meson pair production

Up to one loop order we find

$$H_+(m_Q, \mu) = 1 + \frac{\alpha_s}{4\pi} C_F \left\{ \ln \left(\frac{\mu^2}{m_Q^2} \right) + \ln^2 \left(\frac{\mu^2}{m_Q^2} \right) + 8 + \frac{\pi^2}{6} \right\}$$

$$\gamma_+ = \frac{\alpha_s C_F}{\pi} \left\{ \frac{1}{2} - \ln \left(\frac{m_Q^2}{\mu^2} \right) \right\}$$

$$\mathcal{J}_{Q \rightarrow Q}^{\text{bare}} \left(\mathbf{b}, \frac{m_Q}{p_T} \right) = 1 + \frac{\alpha_s C_F}{\pi} \left\{ -\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} \left[1 - 2 \ln \mathcal{R} \right] + \ln \mathcal{R} - \ln^2 \mathcal{R} - \frac{5\pi^2}{24} \right\}$$

$$\gamma_{\mathcal{J}} = \frac{\alpha_s C_F}{\pi} \{ 1 - 2 \ln \mathcal{R} \}$$

with... $\mathcal{R} = -\frac{i p_T \mu e^{\gamma_E} (\mathbf{v} \cdot \mathbf{b})}{m_Q |\mathbf{v}|}$

We have separated scales and can now be resummed

Anomalous dimension are consistent $\gamma_{\mathcal{J}} + \gamma_+ = \gamma_J + \gamma_{\mathcal{C}_f}$



Consistent !!!

$$A_{\boldsymbol{b}} = \frac{(v_1 \cdot v_2)}{2(v_1 \cdot \hat{\boldsymbol{b}})(v_2 \cdot \hat{\boldsymbol{b}})} = -\frac{\hat{s}}{4 p_T^2 c_b^2}$$

$$A_n = \frac{(v_1 \cdot v_2)}{2(v_1 \cdot n)(v_2 \cdot n)}$$

New soft function

Bare soft
function

$$\begin{aligned} \hat{S}_{\gamma g}^{\text{bare}}(\boldsymbol{b}) &= \hat{S}_{\gamma g}^{\text{finite}}(\boldsymbol{b}) + a_s \left\{ C_A \left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \left(2 \ln \left(\frac{\sqrt{2} \delta^+}{\mu} \right) + \ln(2A_n) \right) \right] \right. \\ &\quad \left. + 2C_F \left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \ln \left(\frac{B \mu^2 e^{2\gamma_E}}{-A_{\boldsymbol{b}}} \right) \right] \right\} \end{aligned}$$

$$\begin{aligned} \hat{S}_{\gamma f}^{\text{bare}}(\boldsymbol{b}) &= \hat{S}_{\gamma f}^{\text{finite}}(\boldsymbol{b}) + a_s \left\{ C_A \left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \left(\ln \frac{(n \cdot v_1)(\boldsymbol{v}_2 \cdot \boldsymbol{b})}{(n \cdot v_2)(\boldsymbol{v}_1 \cdot \boldsymbol{b})} + \ln \left(\frac{B \mu^2 e^{2\gamma_E}}{-A_{\boldsymbol{b}}} \right) \right) \right] \right. \\ &\quad \left. + \frac{4}{\epsilon} C_F \ln \left(-\frac{i \boldsymbol{v}_1 \cdot \boldsymbol{b} \delta^+ e^{\gamma_E}}{n \cdot v_1} \right) \right\} \end{aligned}$$

Finite soft
function

$$\begin{aligned} \hat{S}_{\gamma g}^{\text{finite}}(\boldsymbol{b}) &= 1 + a_s \left\{ C_A \left[\ln(B \mu^2 e^{2\gamma_E}) \left(\ln(B \mu^2 e^{2\gamma_E}) + 4 \ln \left(\frac{\sqrt{2} \delta^+}{\mu} \right) + 2 \ln(2A_n) \right) - \ln^2(-A_{\boldsymbol{b}}) \right. \right. \\ &\quad \left. \left. - \frac{\pi^2}{6} - 2 \text{Li}_2(1 + A_{\boldsymbol{b}}) \right] + C_F \left[\frac{\pi^2}{3} + 2 \ln^2 \left(\frac{B \mu^2 e^{2\gamma_E}}{-A_{\boldsymbol{b}}} \right) + 4 \text{Li}_2(1 + A_{\boldsymbol{b}}) \right] \right\}, \end{aligned}$$

$$\begin{aligned} \hat{S}_{\gamma f}^{\text{finite}}(\boldsymbol{b}) &= 1 + a_s \left\{ C_A \left[\frac{\pi^2}{6} + \ln^2 \left(\frac{B \mu^2 e^{2\gamma_E}}{-A_{\boldsymbol{b}}} \right) + 2 \text{Li}_2(1 + A_{\boldsymbol{b}}) + 2 \ln(B \mu^2 e^{2\gamma_E}) \ln \frac{(n \cdot v_1)(\boldsymbol{v}_2 \cdot \boldsymbol{b})}{(n \cdot v_2)(\boldsymbol{v}_1 \cdot \boldsymbol{b})} \right. \right. \\ &\quad \left. \left. + 4 C_F \ln(B \mu^2 e^{2\gamma_E}) \ln \left(-\frac{i \boldsymbol{v}_1 \cdot \boldsymbol{b} \delta^+ e^{\gamma_E}}{n \cdot v_1} \right) \right\} \right\} \end{aligned}$$

$$A_{\mathbf{b}} = \frac{(v_1 \cdot v_2)}{2(v_1 \cdot \hat{b})(v_2 \cdot \hat{b})} = -\frac{\hat{s}}{4 p_T^2 c_b^2}$$

$$A_n = \frac{(v_1 \cdot v_2)}{2(v_1 \cdot n)(v_2 \cdot n)}$$

Zero-bin subtraction

Subtracted soft function

Finite soft
function

$$\begin{aligned} S_{\gamma g}(\mathbf{b}, \mu, \zeta_2) &= 1 + a_s \left\{ C_F \left[\frac{\pi^2}{3} + 2 \ln^2 \left(\frac{B\mu^2 e^{2\gamma_E}}{-A_{\mathbf{b}}} \right) + 4 \text{Li}_2(1 + A_{\mathbf{b}}) \right] \right. \\ &\quad \left. + C_A \left[-2 \ln(B\mu^2 e^{2\gamma_E}) \ln \zeta_2 - \ln^2(-A_{\mathbf{b}}) - \frac{\pi^2}{3} - 2 \text{Li}_2(1 + A_{\mathbf{b}}) \right] \right\} + \mathcal{O}(a_s^2) \end{aligned}$$

$$\begin{aligned} S_{\gamma f}(\mathbf{b}, \mu, \zeta_2) &= 1 + a_s \left\{ C_A \left[\frac{\pi^2}{6} + \ln^2 \left(\frac{B\mu^2 e^{2\gamma_E}}{-A_{\mathbf{b}}} \right) + 2 \text{Li}_2(1 + A_{\mathbf{b}}) + 2 \ln(B\mu^2 e^{2\gamma_E}) \ln \frac{(n \cdot v_1)(\mathbf{v}_2 \cdot \mathbf{b})}{(n \cdot v_2)(\mathbf{v}_1 \cdot \mathbf{b})} \right] \right. \\ &\quad \left. + C_F \ln(B\mu^2 e^{2\gamma_E}) \left[\ln(B\mu^2 e^{2\gamma_E}) - 2 \ln \zeta_2 + 2 \ln \left(\frac{2(n \cdot v_2)}{(v_1 \cdot v_2)(n \cdot v_2)} \right) - \frac{\pi^2}{6} + 4 \ln(-i \mathbf{v}_1 \cdot \hat{\mathbf{b}}) \right] \right\} + \mathcal{O}(a_s^2), \end{aligned}$$

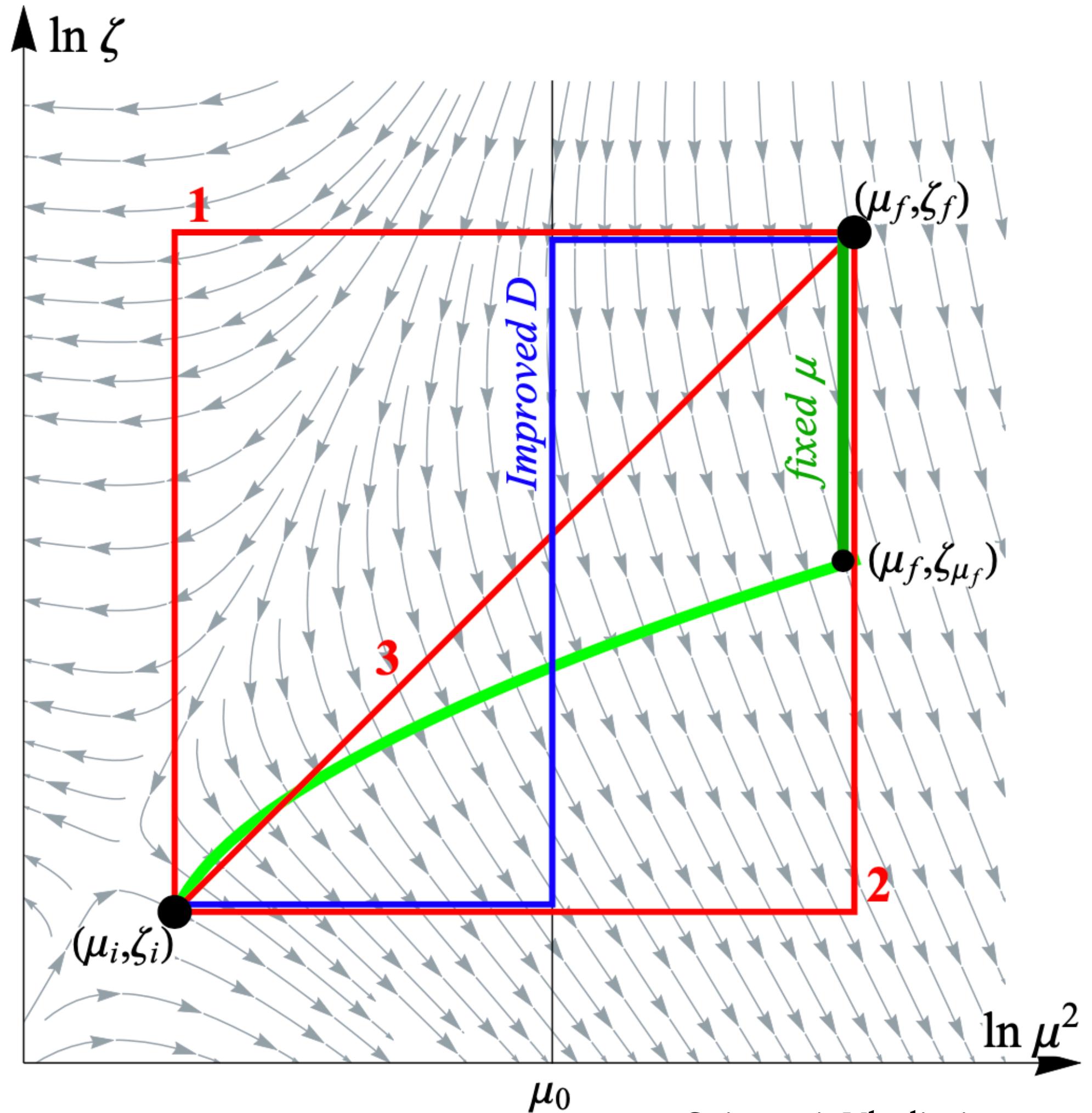
Renormalization
function

$$Z_{\gamma g}^S(\mathbf{b}, \mu, \zeta_2) = 1 + a_s \left\{ C_F \left[\frac{4}{\epsilon^2} + \frac{4}{\epsilon} \ln \left(\frac{B\mu^2 e^{2\gamma_E}}{-A_{\mathbf{b}}} \right) \right] - C_A \frac{2}{\epsilon} \ln \zeta_2 \right\} + \mathcal{O}(a_s^2)$$

$$\begin{aligned} Z_{\gamma f}^S(\mathbf{b}, \mu, \zeta_2) &= 1 + a_s \left\{ C_A \left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon} \left(\ln \frac{(n \cdot v_1)(\mathbf{v}_2 \cdot \mathbf{b})}{(n \cdot v_2)(\mathbf{v}_1 \cdot \mathbf{b})} \right) + \ln \left(\frac{B\mu^2 e^{2\gamma_E}}{-A_{\mathbf{b}}} \right) \right] \right. \\ &\quad \left. + \frac{2}{\epsilon} C_F \left[\ln(B\mu^2 e^{2\gamma_E}) - \ln \zeta_2 + \ln \left(\frac{2(n \cdot v_2)}{(v_1 \cdot v_2)(n \cdot v_2)} \right) + 2 \ln(-i \mathbf{v}_1 \cdot \hat{\mathbf{b}}) \right] \right\} + \mathcal{O}(a_s^2) \end{aligned}$$

Evolution, double-scale evolution

fixed μ evolution



Describe evolution of functions
depending on two scales

