TMDs at Small-x : Boer-Mulders Function

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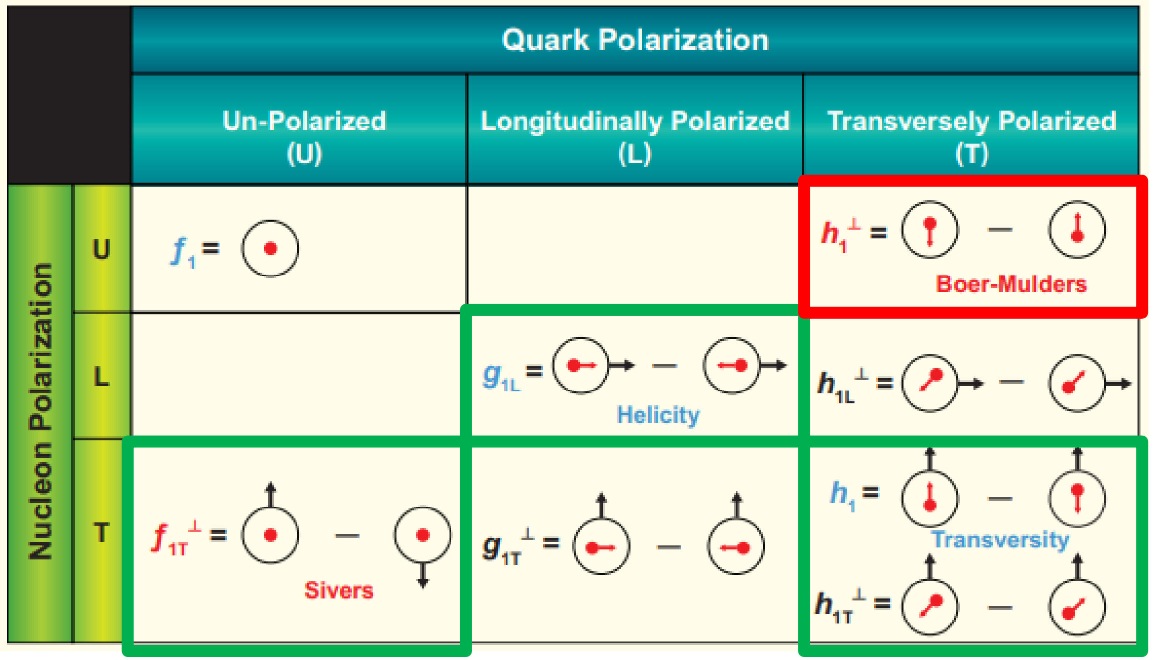
Quark TMDs

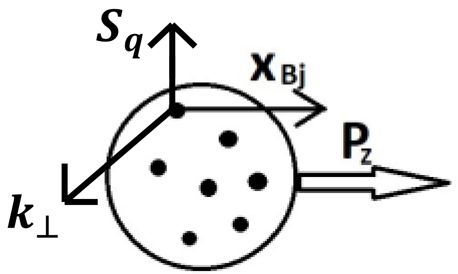


The leading twist quark TMDs give various correlations between the transverse momentum and polarizations of the quarks within a hadron with the polarization of the parent hadron

Their scale evolution in Q^2 is given by the CSS equations, but the small- x evolution is an ongoing effort

Boer-Mulders Function

Gives the correlation between the transverse momentum and transverse polarization of quarks in an unpolarized hadron



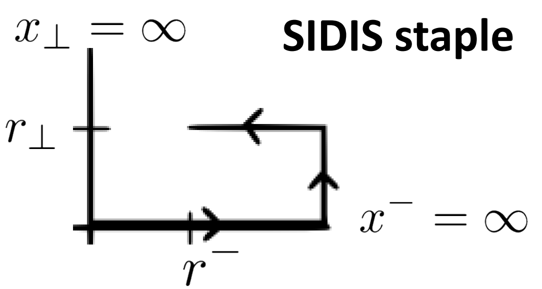
One of the two leading-twist T-odd quark TMDs along with the Sivers function

Defined through a nonlocal operator product

$$\frac{-\epsilon_T^{ij} k_T^i}{M_P} h_1^{\perp\, q}(x, k_T^2) = \frac{1}{(2\pi)^3} \int \mathrm{d}^2 r\, \mathrm{d}r^- \, e^{ik\cdot r} \, \langle P| \, \bar{\psi}(0) U[0,r] \frac{\gamma^5 \gamma^+ \gamma^j}{2} \psi(r) \, |P\rangle$$

$$\mathcal{U}[r,0] = \mathcal{P}\mathrm{exp}\left[ig \int_0^r \mathrm{d}x_\mu \, A^\mu(x) \right]$$

SIDIS staple



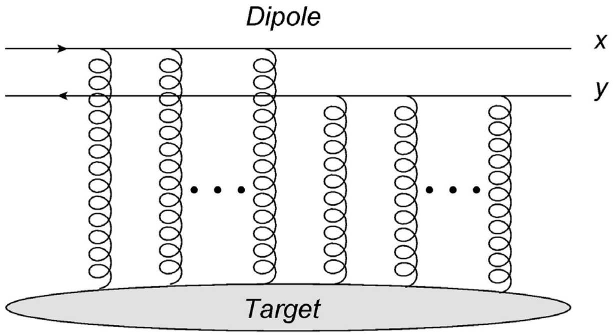
Small-x TMDs from polarized Wilson lines

Kovchegov, Sievert and Pitonyak (2015-2019) developed a general high-energy scattering operator formalism

Studied small- x evolution equations for the quark helicity TMD, gluon helicity TMD, quark transversity TMD, and quark Sivers function

Rewriting the TMD operator definitions at small- x yields modified dipole correlators

$$\mathrm{tr}[V_x V_y^\dagger] =$$



Standard eikonal dipole, *Balitsky (1996)*

$$V_x[b^-, a^-] = \mathcal{P}\exp\left[ig\int_{a^-}^{b^-} \mathrm{d}s^-\, A^+(s^-, x_\perp)\right]$$

*in $A^- = 0$ or $\partial_\mu A^\mu = 0$ gauge

$$V_x = V_x[\infty, -\infty]$$

Small-x TMDs from polarized Wilson lines

Simplify

- Rewrite operator definition in small- x limit using shockwave formalism
- Expand to a given order in eikonality
- Obtain expression for TMD in terms of 'polarized dipoles'

Evolve

- Calculate small- x gluon/quark emissions in dipole
- Take (for example) large- N_c limit to obtain closed equations

Solve

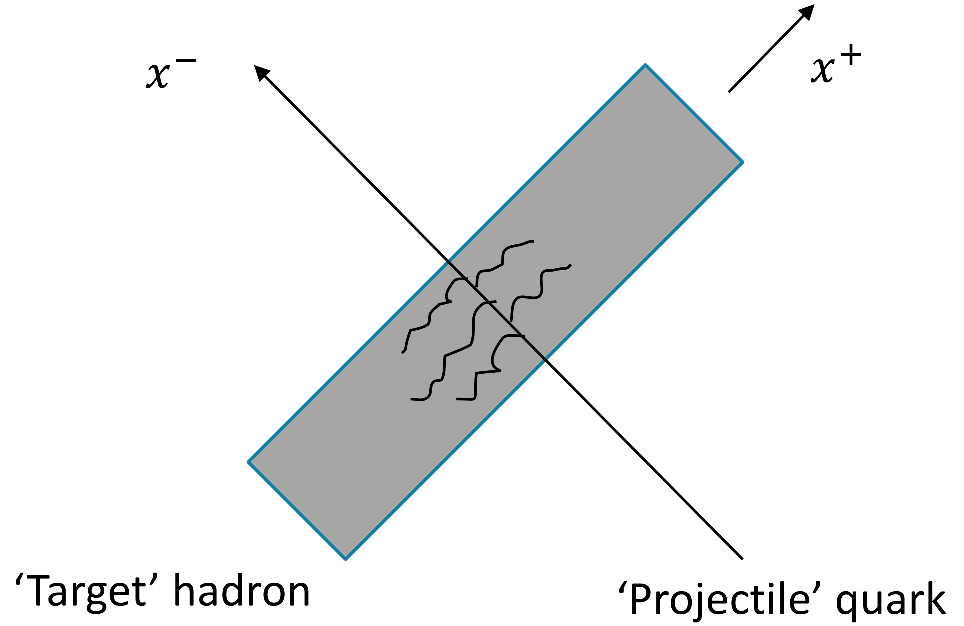
- Solve integral equations analytically (if possible) or numerically
- Plug evolved dipole back into TMD definition

Small- x → Shockwave formalism

Fourier factor picks out long range correlations in the x^- direction

$$e^{ixP^+r^-} \to \text{large } r^- \text{ for small } x$$

Hadron is very Lorentz contracted, so interactions in gauge link happen over short x^- lifetime inside the shockwave

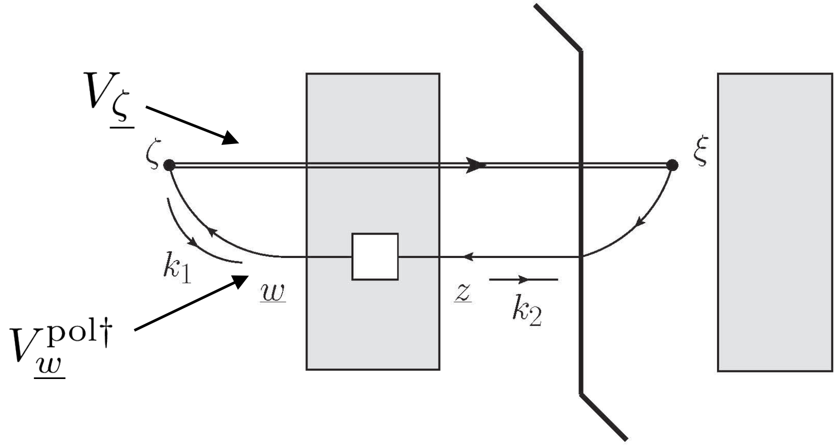


Small-x Boer-Mulders

The quark correlator can be rewritten in terms of Wilson lines

$$\frac{k_T^y}{M_P} h_1^{\perp\, q}(x, k_T^2) = -\frac{2p_1^+}{(2\pi)^3} \int \mathrm{d}^2\zeta_\perp \, \mathrm{d}^2 w_\perp \, \frac{\mathrm{d}^2 k_{1\perp}\, \mathrm{d}k_1^-}{(2\pi)^3} e^{i(\underline{k}+\underline{k}_1)\cdot(\underline{w}-\underline{\zeta})} \frac{\theta(k_1^-)}{(xp_1^+ k_1^- + \underline{k}_1^2)(xp_1^+ k_1^- + \underline{k}^2)}$$
$$\times \sum_{\chi_1,\chi_2} \bar{v}_{\chi_2}(k_2) \frac{\gamma^5\gamma^+\gamma^2}{2} v_{\chi_1}(k_1) \left\langle \mathrm{T}\, V_{\underline{\zeta}}^{ij}[\infty, -\infty] \bar{v}_{\chi_1}(k_1) \hat{V}_{\underline{w}}^{\mathrm{pol}\dagger\, ji} v_{\chi_2}(k_2) \right\rangle + c.c.,$$

The polarized Wilson line $V_{\underline{w}}^{\mathrm{pol}\dagger}$ makes the correlator a transverse polarized dipole



$$\supset \mathrm{tr}\left[V_{\underline{\zeta}} V_{\underline{w}}^{\mathrm{pol}\dagger}\right]$$

Small-x Boer-Mulders

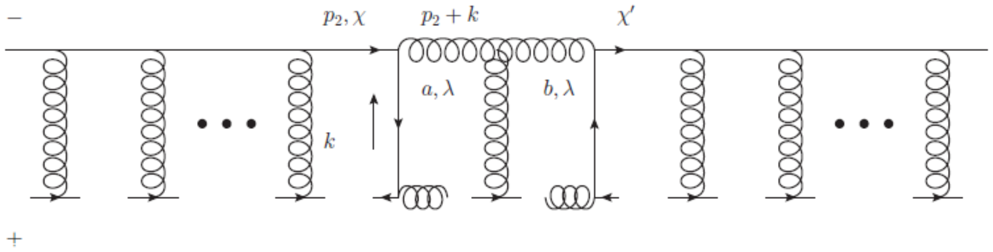
We find two sub-sub-eikonal polarized dipoles contributing in the massless quark limit

$$\frac{k_T^y}{M_P} h_1^{\perp q}(x, k_T^2) = \frac{-x 4 i N_c}{(2\pi)^3} \int d^2\zeta_\perp \, d^2 w_\perp \int\limits_{\frac{\Lambda^2}{s}}^{1} \frac{dz}{z} \int \frac{d^2 k_{1\perp}}{(2\pi)^3} \frac{e^{i(\underline{k}+\underline{k}_1)\cdot(\underline{w}-\underline{\zeta})}}{\underline{k}_1^2 \underline{k}^2} \left(\frac{1}{\underline{k}_1^2} + \frac{1}{\underline{k}^2} \right)$$

$$\times \left[(\underline{k}_1 \cdot \underline{k} - 2\underline{S} \cdot \underline{k}_1 \underline{S} \cdot \underline{k}) H^1_{\underline{w},\underline{\zeta}}(z) + (\underline{S} \times \underline{k}_1 \underline{S} \cdot \underline{k} + \underline{S} \cdot \underline{k}_1 \underline{S} \times \underline{k}) H^2_{\underline{w},\underline{\zeta}}(z) \right]$$

$$H^1_{\underline{w},\underline{\zeta}}(z) \equiv \frac{1}{2N_c} \mathrm{Re} \left\langle\!\left\langle \mathrm{T\,tr}\left[V_{\underline{\zeta}} V_{\underline{w}}^{\mathrm{T}\dagger} \right] - \mathrm{T\,tr}\left[V_{\underline{w}} V_{\underline{\zeta}}^{\mathrm{T}\dagger} \right] \right\rangle\!\right\rangle,$$

$$H^2_{\underline{w},\underline{\zeta}}(z) \equiv \frac{1}{2N_c} \mathrm{Im} \left\langle\!\left\langle \mathrm{T\,tr}\left[V_{\underline{\zeta}} V_{\underline{w}}^{\mathrm{T}\perp\dagger} \right] - \mathrm{T\,tr}\left[V_{\underline{w}} V_{\underline{\zeta}}^{\mathrm{T}\perp\dagger} \right] \right\rangle\!\right\rangle$$



Here $\langle\langle \ldots \rangle\rangle = zs\langle \ldots \rangle$ with z the internal longitudinal momentum fraction and s the center of mass energy squared

Flavor Singlet and Flavor Non-Singlet

The Boer-Mulders function is T-odd, and QCD is P even so we will look at the C-odd flavor non-singlet TMD

$$h_1^{\perp NS} = h_1^{\perp q} - h_1^{\perp \bar{q}}$$

$$\frac{k_T^y}{M_P} h_1^{\perp NS}(x, k_T^2) = \frac{-x 4 i N_c}{(2\pi)^3} \int \mathrm{d}^2\zeta_\perp \, \mathrm{d}^2 w_\perp \int\limits_{\frac{\Lambda^2}{s}}^{1} \frac{\mathrm{d}z}{z} \int \frac{\mathrm{d}^2 k_{1\perp}}{(2\pi)^3} \frac{e^{i(\underline{k}+\underline{k}_1)\cdot(\underline{w}-\underline{\zeta})}}{\underline{k}_1^2 \underline{k}^2} \left(\frac{1}{\underline{k}_1^2} + \frac{1}{\underline{k}^2} \right)$$

$$\times \left[(\underline{k}_1 \cdot \underline{k} - 2 \underline{S} \cdot \underline{k}_1 \underline{S} \cdot \underline{k}) H^{1NS}_{\underline{w},\underline{\zeta}}(z) + (\underline{S} \times \underline{k}_1 \underline{S} \cdot \underline{k} + \underline{S} \cdot \underline{k}_1 \underline{S} \times \underline{k}) H^{2NS}_{\underline{w},\underline{\zeta}}(z) \right]$$

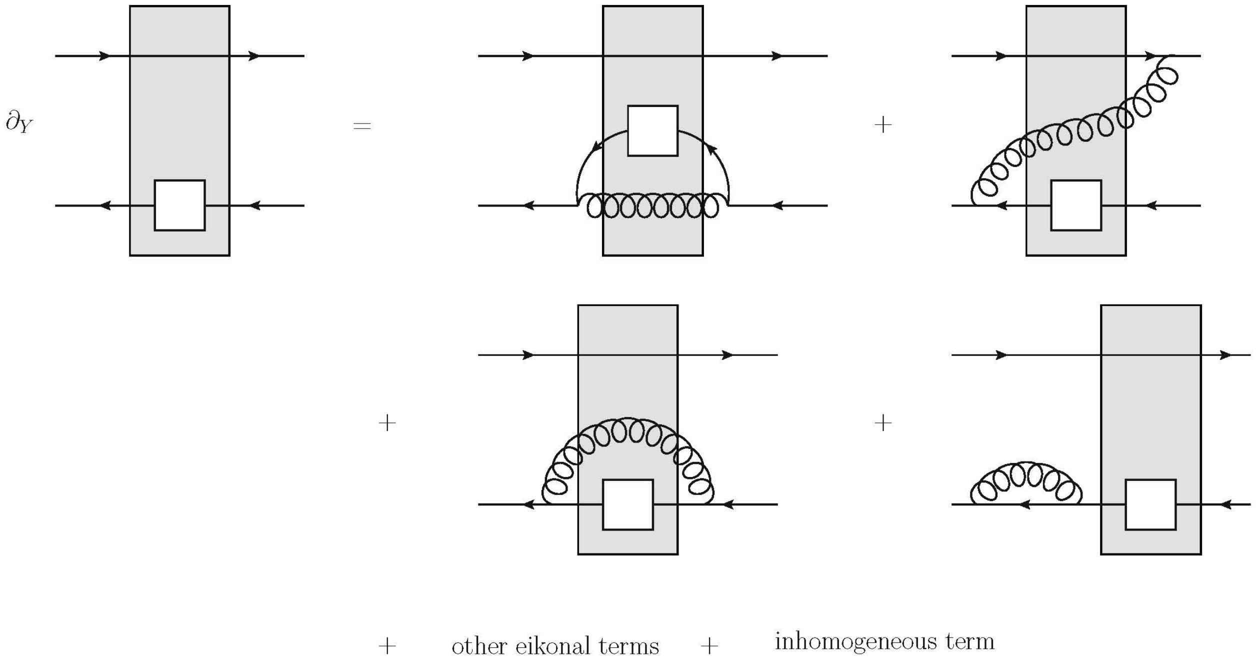
We have new dipoles defined as

$$H^{1NS}_{\underline{w},\underline{\zeta}}(z) \equiv \frac{1}{2N_c} \mathrm{Re} \left\langle\!\!\left\langle \mathrm{T}\,\mathrm{tr}\left[V_{\underline{\zeta}} V_{\underline{w}}^{\mathrm{T}\,\dagger} \right] - \mathrm{T}\,\mathrm{tr}\left[V_{\underline{w}}^{\mathrm{T}} V_{\underline{\zeta}}^{\dagger} \right] \right\rangle\!\!\right\rangle$$

$$H^{2NS}_{\underline{w},\underline{\zeta}}(z) \equiv \frac{1}{2N_c} \mathrm{Im} \left\langle\!\!\left\langle \mathrm{T}\,\mathrm{tr}\left[V_{\underline{\zeta}} V_{\underline{w}}^{\mathrm{T}\perp\,\dagger} \right] - \mathrm{T}\,\mathrm{tr}\left[V_{\underline{w}}^{\mathrm{T}\perp} V_{\underline{\zeta}}^{\dagger} \right] \right\rangle\!\!\right\rangle$$

Small-x Evolution

In the massless quark limit, only quark emissions and eikonal soft gluon emissions can contribute to evolution



Preliminary Linear Evolution Equations

$$\begin{aligned}
H^{1NS}_{\underline{1},\underline{0}}(z) =& H^{1NS\ (0)}_{\underline{1},\underline{0}}(z) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{\mathrm{d}z'}{z'} \int \mathrm{d}^2 x_2 \, \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \frac{1}{2} \mathrm{Re}\left[\left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{2}} V_{\underline{1}}^{\mathrm{T}\dagger} \right] \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{T}\dagger} \right] \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{1}}^{\mathrm{T}} V_{\underline{2}}^{\dagger} \right] \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{1}}^{\mathrm{T}} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle \right](z') \\
&+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{\mathrm{d}z'}{z'} \int \frac{\mathrm{d}^2 x_2}{x_{21}^2} \mathrm{Re}\left[\left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{1}} V_{\underline{2}}^{\mathrm{T}\dagger} \right] \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{2}}^{\mathrm{T}} V_{\underline{1}}^{\dagger} \right] \right\rangle\!\!\right\rangle \right](z'), \\
H^{2NS}_{\underline{1},\underline{0}}(z) =& H^{2NS\ (0)}_{\underline{1},\underline{0}}(z) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{\mathrm{d}z'}{z'} \int \mathrm{d}^2 x_2 \, \frac{x_{10}^2}{x_{21}^2 x_{20}^2} \frac{1}{2} \mathrm{Im}\left[\left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{2}} V_{\underline{1}}^{\mathrm{T}\perp\dagger} \right] \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{T}\perp\dagger} \right] - \left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{1}}^{\mathrm{T}} V_{\underline{2}}^{\dagger} \right] \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{1}}^{\mathrm{T}} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle \right](z') \\
&- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{\mathrm{d}z'}{z'} \int \frac{\mathrm{d}^2 x_2}{x_{21}^2} \mathrm{Im}\left[\left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{1}} V_{\underline{2}}^{\mathrm{T}\dagger} \right] \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{1}{N_c} \mathrm{T\,tr}\left[V_{\underline{2}}^{\mathrm{T}} V_{\underline{1}}^{\dagger} \right] \right\rangle\!\!\right\rangle \right](z')
\end{aligned}$$

Dipole Symmetry Ansatz

We can deduce the form of the dipoles from the definition of the Boer-Mulders function

$$\frac{k_T^y}{M_P} h_1^{\perp NS}(x, k_T^2) = \frac{-x4iN_c}{(2\pi)^3} \int \mathrm{d}^2\zeta_\perp \, \mathrm{d}^2 w_\perp \int\limits_{\frac{\Lambda^2}{s}}^{1} \frac{\mathrm{d}z}{z} \int \frac{\mathrm{d}^2 k_{1\perp}}{(2\pi)^3} \frac{e^{i(\underline{k}+\underline{k}_1)\cdot(\underline{w}-\underline{\zeta})}}{\underline{k}_1^2 \underline{k}^2} \left(\frac{1}{\underline{k}_1^2} + \frac{1}{\underline{k}^2} \right)$$
$$\times \left[(\underline{k}_1 \cdot \underline{k} - 2\underline{S} \cdot \underline{k}_1 \underline{S} \cdot \underline{k}) H^{1NS}_{\underline{w},\underline{\zeta}}(z) + (\underline{S} \times \underline{k}_1 \underline{S} \cdot \underline{k} + \underline{S} \cdot \underline{k}_1 \underline{S} \times \underline{k}) H^{2NS}_{\underline{w},\underline{\zeta}}(z) \right]$$

The RHS must be antisymmetric under $\underline{w} \leftrightarrow \underline{\zeta}$, so we can argue

$$\int \mathrm{d}^2 b_\perp \, H^{1NS}_{\underline{w},\underline{\zeta}}(z) = (\underline{w} - \underline{\zeta}) \times \underline{S} \; H^{1NS}\Big((\underline{w} - \underline{\zeta})^2, z \Big),$$
$$\int \mathrm{d}^2 b_\perp \, H^{2NS}_{\underline{w},\underline{\zeta}}(z) = (\underline{w} - \underline{\zeta}) \cdot \underline{S} \; H^{2NS}\Big((\underline{w} - \underline{\zeta})^2, z \Big)$$

Simplified Equations

This ansatz reduces the equations to only contain a term from eikonal, unpolarized gluon emissions

The two equations also become identical

$$H^{NS}(x_{10}^2, z) = H^{NS(0)}(x_{10}^2, z) - \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{\mathrm{d}z'}{z'} \int \frac{\mathrm{d}x_{21}^2}{x_{21}^2} \Gamma^{NS}(x_{10}^2, x_{21}^2, z'),$$

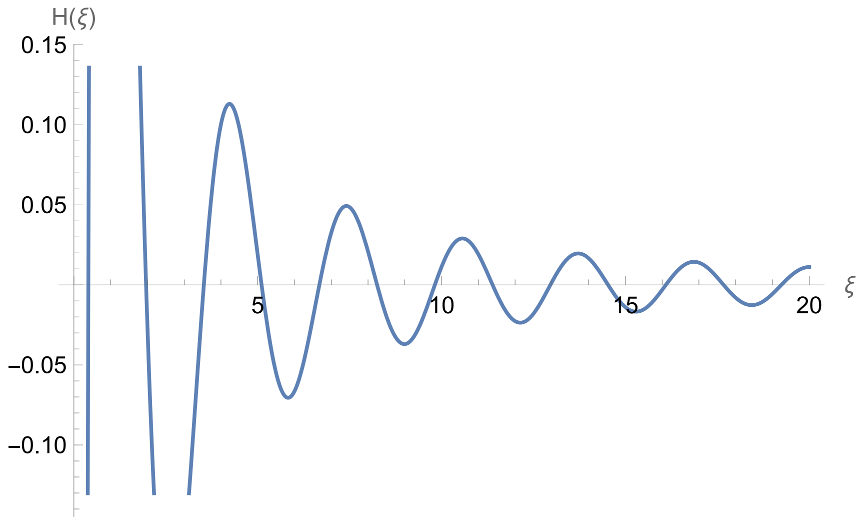
$$\Gamma^{NS}(x_{10}^2, x_{21}^2, z') = H^{NS(0)}(x_{10}^2, z') - \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{\mathrm{d}z''}{z''} \int \frac{\mathrm{d}x_{32}^2}{x_{32}^2} \Gamma^{NS}(x_{10}^2, x_{32}^2, z'')$$

Analytic Solution

Taking some approximations, we can solve this equation analytically and find

$$H^{NS}(x_{10}^2, z) \propto \frac{1}{\left[\sqrt{\frac{\alpha_s N_c}{2\pi}}\ln\left(zsx_{10}^2\right)\right]^{3/2}}\cos\left(2\sqrt{\frac{\alpha_s N_c}{2\pi}}\ln\left(zsx_{10}^2\right) - \frac{3\pi}{4}\right)$$

$$= \frac{1}{\xi^{3/2}}\cos\left(2\xi - \frac{3\pi}{4}\right)$$



Small-x Boer-Mulders

Plugging in this solution, the oscillations are washed out

$$\frac{k_T^y}{M_P} h_1^{\perp NS}(x, k_T^2) = \frac{-4xiN_c}{(2\pi)^3} \int \mathrm{d}^2\zeta_\perp \, \mathrm{d}^2 w_\perp \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{\mathrm{d}z}{z} \int \frac{\mathrm{d}^2 k_{1\perp}}{(2\pi)^3} \frac{e^{i(\underline{k}+\underline{k}_1)\cdot(\underline{w}-\underline{\zeta})}}{\underline{k}_1^2 \underline{k}^2} \left(\frac{1}{\underline{k}_1^2} + \frac{1}{\underline{k}^2} \right)$$

$$\times \left[(\underline{k}_1 \cdot \underline{k} - 2 \, \underline{S} \cdot \underline{k}_1 \, \underline{S} \cdot \underline{k}) \, H^{1NS}\Big((\underline{w} - \underline{\zeta})^2, z \Big) + (\underline{S} \times \underline{k}_1 \, \underline{S} \cdot \underline{k} + \underline{S} \cdot \underline{k}_1 \, \underline{S} \times \underline{k}) \, H^{2NS}\Big((\underline{w} - \underline{\zeta})^2, z \Big) \right]$$

$$\boxed{\sim \left(\frac{1}{x} \right)^{-1}}$$

We find that evolution leaves the naïve sub-sub-eikonal scaling unchanged!

Conclusions

We have preliminary equations for the large-N_c linearized small-x evolution equations for the Boer-Mulders function

Solving these equations for the flavor non-singlet Boer-Mulders function yields exact sub-sub-eikonal scaling

$$\boxed{h_1^{\perp\ NS} \sim \left(\frac{1}{x}\right)^{-1}}$$

Similar results to both the eikonal and sub-eikonal pieces of the T-odd Sivers TMD

$$f_{1T}^{\perp q}(x, k_T^2) = C_O(x, k_T^2)\frac{1}{x} + C_1(k_T^2)\left(\frac{1}{x}\right)^{0+\text{possible new corrections}} + \ldots$$

Perhaps T-odd protects from linear evolution power corrections?

Backup Slides

Polarized Wilson Lines in Boer-Mulders Dipoles

$$V_{\underline{w}}^{\mathrm{T}} \supset \frac{g^2 (p_1^+)^2}{8 s^2} \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} \mathrm{d}z_2^- \, V_{\underline{w}}[\infty, z_2^-] t^b \psi_\beta(z_2^-, \underline{w}) U_{\underline{w}}^{ba}[z_2^-, z_1^-] \left[(i\gamma^5 \underline{S} \cdot \overleftarrow{\underline{D}} - \underline{S} \times \overleftarrow{\underline{D}}) \gamma^+ \gamma^- + (i\gamma^5 \underline{S} \cdot \underline{D} - \underline{S} \times \underline{D}) \gamma^- \gamma^+ \right]_{\alpha\beta} \bar{\psi}_\alpha(z_1^-, \underline{w}) t^a V_{\underline{w}}[z_1^-, -\infty],$$

$$V_{\underline{w}}^{\mathrm{T}\perp} \supset \frac{g^2 (p_1^+)^2}{8 s^2} \int\limits_{-\infty}^{\infty} \mathrm{d}z_1^- \int\limits_{z_1^-}^{\infty} \mathrm{d}z_2^- \, V_{\underline{w}}[\infty, z_2^-] t^b \psi_\beta(z_2^-, \underline{w}) U_{\underline{w}}^{ba}[z_2^-, z_1^-] \left[(i\underline{S} \cdot \overleftarrow{\underline{D}} - \gamma^5 \underline{S} \times \overleftarrow{\underline{D}}) \gamma^+ \gamma^- + (i\underline{S} \cdot \underline{D} - \gamma^5 \underline{S} \times \underline{D}) \gamma^- \gamma^+ \right]_{\alpha\beta} \bar{\psi}_\alpha(z_1^-, \underline{w}) t^a V_{\underline{w}}[z_1^-, -\infty]$$

Eikonal power counting

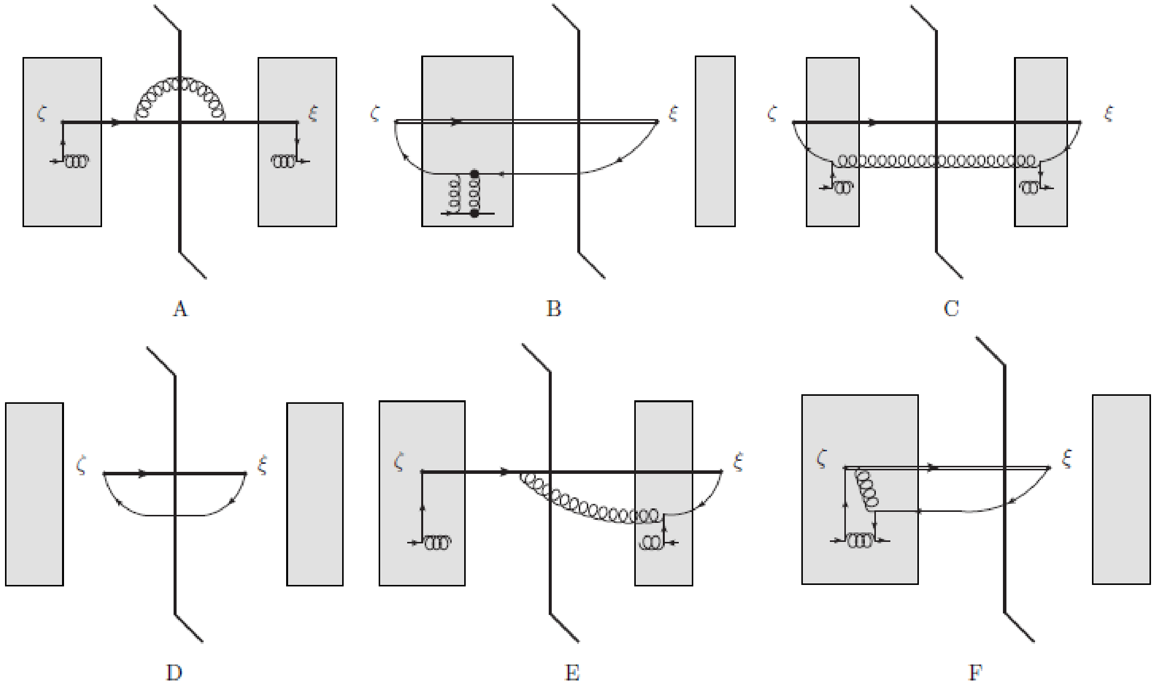
We can expand in powers of x or equivalently inverse powers of CM energy s

Eikonal distributions $q(x, k_T) \sim \frac{1}{x}$, no COM energy suppression

Sub-eikonal distributions $q(x, k_T) \sim x^0, \frac{1}{s}$ energy suppression

Sub-sub-eikonal distributions $q(x, k_T) \sim x, \frac{1}{s^2}$ energy suppression

Small-x TMD diagrams



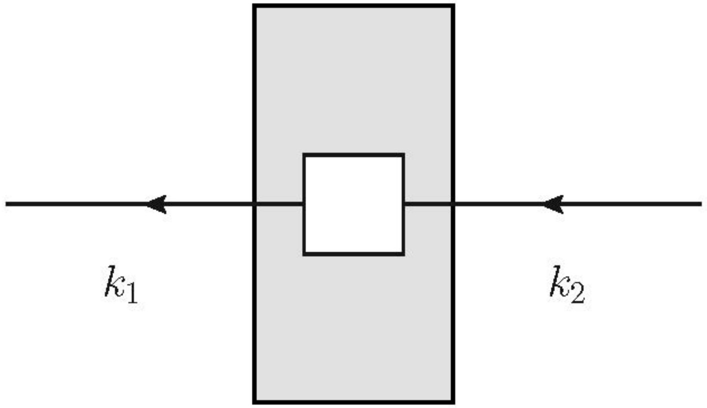
Fairly general analysis shows that only class B diagrams contribute to spin dependent TMDs at sub-eikonal order

Sub-eikonal corrections

The (anti)quark propagator through the shockwave can include sub-eikonal corrections to allow for spin-dependence

From the helicity and transversity TMDs, at least need sub-sub-eikonal corrections for general leading-twist quark TMD

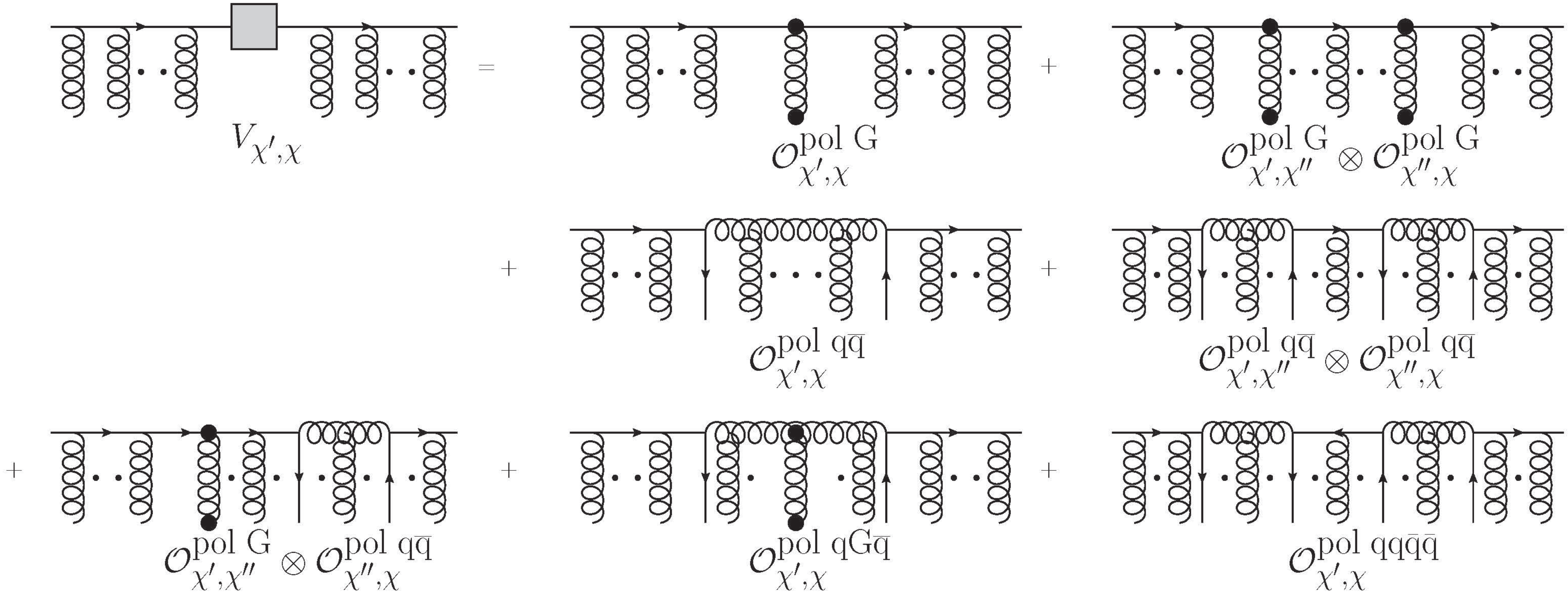
$$\bar{\psi}(\zeta)\,\psi(\xi) \supset$$



$$\not{k}_1 \hat{V}^{\dagger}_{\underline{w}} \not{k}_2$$

$$\begin{aligned}\hat{V}^{\dagger}_{\underline{w}} &= \delta_{\chi,\chi'} V^{\dagger}_{\underline{w}} + \text{sub-eikonal corrections}\\ &= \delta_{\chi,\chi'} V^{\dagger}_{\underline{w}} + V^{\text{pol}\dagger}_{\underline{w}}\end{aligned}$$

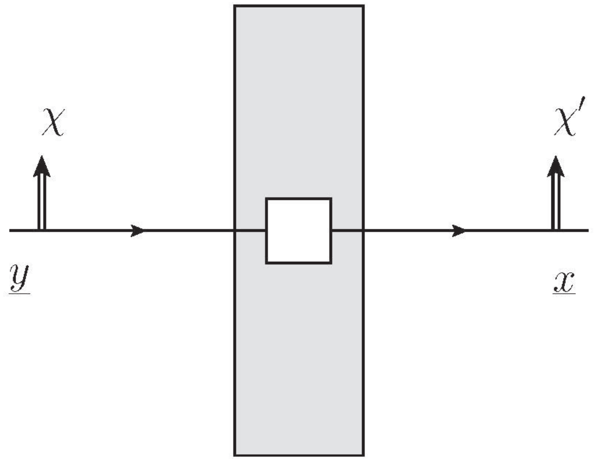
General polarized Wilson line



General polarized Wilson line

Full **sub-sub-eikonal** polarized fundamental Wilson line for TMDs which depend on the proton's transverse spin

$$
\begin{aligned}
V_{\underline{x},\underline{y};\chi',\chi} &= V_{\underline{x}}\,\delta^2(\underline{x}-\underline{y})\,\delta_{\chi,\chi'} + \int_{-\infty}^{\infty} \mathrm{d}z^-\, d^2z\; V_{\underline{x}}[\infty,z^-]\,\delta^2(\underline{x}-\underline{z})\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi',\chi}(z^-,\underline{z})\,V_{\underline{y}}[z^-,-\infty]\,\delta^2(\underline{y}-\underline{z}) \\
&+ \int_{-\infty}^{\infty} \mathrm{d}z_1^-\, d^2z_1 \int_{z_1^-}^{\infty} \mathrm{d}z_2^-\, d^2z_2 \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_2^-]\,\delta^2(\underline{x}-\underline{z}_2)\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi',\chi''}(z_2^-,\underline{z}_2)\,V_{\underline{z}_1}[z_2^-,z_1^-]\,\delta^2(\underline{z}_2-\underline{z}_1) \\
&\times \mathcal{O}^{\mathrm{pol\ G}}_{\chi'',\chi}(z_1^-,\underline{z}_1)\,V_{\underline{y}}[z_1^-,-\infty]\,\delta^2(\underline{y}-\underline{z}_1) + \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} dz_2^-\; V_{\underline{x}}[\infty,z_2^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi',\chi}(z_2^-,z_1^-;\underline{x},\underline{y})\,V_{\underline{y}}[z_1^-,-\infty] \\
&+ \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} \mathrm{d}z_2^- \int_{z_2^-}^{\infty} \mathrm{d}z_3^- \int_{z_3^-}^{\infty} \mathrm{d}z_4^-\, d^2z \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_4^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi',\chi''}(z_4^-,z_3^-;\underline{x},\underline{z})\,V_{\underline{z}}[z_3^-,z_2^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi'',\chi}(z_2^-,z_1^-;\underline{z},\underline{y})\,V_{\underline{y}}[z_1^-,-\infty] \\
&+ \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} \mathrm{d}z_2^- \int_{z_2^-}^{\infty} \mathrm{d}z_3^- \int_{z_3^-}^{\infty} \mathrm{d}z_4^-\; V_{\underline{x}}[\infty,z_4^-]\,\mathcal{O}^{\mathrm{pol\ qq\bar{q}\bar{q}}}_{\chi',\chi}(z_4^-,z_3^-,z_2^-,z_1^-;\underline{x})\,V_{\underline{y}}[z_1^-,-\infty]\delta^2(\underline{x}-\underline{y}) \\
&+ \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} dz_2^-\, d^2z_2 \int_{z_2^-}^{\infty} dz_3^-\; V_{\underline{x}}[\infty,z_3^-]\,\delta^2(\underline{z}_2-\underline{x})\mathcal{O}^{\mathrm{pol}\ \ \mathrm{qG\bar{q}}}_{\chi',\chi}(z_1^-,z_2^-,z_3^-;\underline{x},\underline{z}_2)\delta^2(\underline{z}_2-\underline{y})V_{\underline{y}}[z_1^-,-\infty] \\
&+ \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} \mathrm{d}z_2^- \int_{z_2^-}^{\infty} \mathrm{d}z_3^-\, d^2z \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_3^-]\,\delta^2(\underline{x}-\underline{z})\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi',\chi''}(z_3^-;\underline{z})\,V_{\underline{z}}[z_3^-,z_2^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi'',\chi}(z_2^-,z_1^-;\underline{z},\underline{y})\,V_{\underline{y}}[z_1^-,-\infty] \\
&+ \int_{-\infty}^{\infty} \mathrm{d}z_1^- \int_{z_1^-}^{\infty} \mathrm{d}z_2^- \int_{z_2^-}^{\infty} \mathrm{d}z_3^-\, d^2z \sum_{\chi''=\pm 1} V_{\underline{x}}[\infty,z_3^-]\,\mathcal{O}^{\mathrm{pol\ q\bar{q}}}_{\chi',\chi''}(z_3^-,z_2^-;\underline{x},\underline{z})\,V_{\underline{z}}[z_2^-,z_1^-]\,\mathcal{O}^{\mathrm{pol\ G}}_{\chi'',\chi}(z_1^-;\underline{z})\,V_{\underline{y}}[z_1^-,-\infty]\,\delta^2(\underline{y}-\underline{z})
\end{aligned}
$$



cf. *Altinoluk et al* (2020), *Chirilli* (2021) sub-eikonal propagator