Reggeon Field Theory: from a Toy World to QCD

Michael Lublinsky

Ben-Gurion University of the Negev, Israel

Alex Kovner

University of Connecticut, USA

A. Kovner, E. Levin and M.L,
"Nuclei in the toy world: beyond the Pomeron in zero transverse dimensions," arXiv:2201.01551, in JHEP

A. Kovner, E. Levin, M. Li and M. Lublinsky,
"The JIMWLK evolution and the s-channel unitarity," arXiv:2006.15126, in JHEP

"Reggeon Field Theory and Self Duality: Making Ends Meet," arXiv:2007.12132, in JHEP

A. Kovner, E. Levin and M.L,
"QCD unitarity constraints on Reggeon Field Theory," arXiv:1605.03251, in JHEP

Motivation

- **QCD**$(s \to \infty) \longrightarrow$ **RFT**

  – Identify RFT Hilbert space and its basic degrees of freedom
  – Define algorithm for calculating the scattering amplitudes in terms of the RFT degrees of freedom.
  – Realise symmetries and QCD unitarity constraints on the RFT "Fock states"
  – Construct a, hopefully unique, RFT Hamiltonian consistent with the symmetries and constraints.
  – Check for the known limits: dense-dilute (JIMWLK)

- **Get intuition from zero-dimensional toy models**

- **The main ingredients:**

  RFT Degrees of freedom: Pomerons/dipoles in toy models and large N_c QCD;
  Wilson lines in full QCD

  s-channel unitarity = probabilistic interpretation of scattering amplitudes.
  t-channel unitarity = projectile-target symmetry (self-duality)

Summary of the results

- **In the world of toys**
  - **The Pomeron Hamiltonians containing only triple Pomeron vertex [BK($1 \to 2$) or Braun($1 \to 2$ and $2 \to 1$)] all violate the s-channel unitarity.**
  - **It is possible to construct a Pomeron Hamiltonian, which respects both the t- and s- channel unitarity conditions and reduces to BK in the dense-dilute limit.**
    Mueller & Salam (1996),
    Kovner and ML (2005),
    J. P. Blaizot, E. Iancu and D. N. Triantafyllopoulos (2006)
    Kovner, Levin, ML (2016)
  - **The Mueller-Salam Hamiltonian is not unique and presumably not adequate in the dense limit; Multiple emissions (multi-Pomeron vertices) are important and have to be included** Kovner, Levin, ML (2021).

- **In QCD**
  - **JIMWLK Hamiltonian violates the s-channel unitarity**
    Kovner, Levin, Li, ML (2020)
  - **It is possible to construct a (family of) self-dual RFTs, which all reduce to JIMWLK in the dense-dilute limit. Yet, it is not known if any of the RFTs respect the s-channel unitarity.**
    Kovner, Levin, Li, ML (2020)

1. Degrees of freedom: P **and** $\bar{P}$ **- Pomeron and its conjugate.**
$d = 1 - P$ **- target dipole;** $\bar{d} = 1 - \bar{P}$ **- projectile dipole.**

The commutation relations of P **and** $\bar{P}$ **are based on their perturbative identification:**

$$[\mathbf{P}, \bar{\mathbf{P}}] \; = \; -\,\gamma; \qquad \gamma \sim \mathbf{O}(\alpha_{\mathbf{s}}^{\mathbf{2}})$$

where γ **is the zero dimensional proxy for the dipole-dipole elastic scattering amplitude.**

2. S**-matrix for scattering of** $\bar{n}$ **dipoles of the projectile on** m **dipoles of the target**

$$\langle \mathbf{m}|\,\bar{\mathbf{n}}\,\rangle \; = \; \langle \mathbf{0}|\,(\mathbf{1} - \mathbf{P})^{\mathbf{m}}\,(\mathbf{1} - \bar{\mathbf{P}})^{\bar{\mathbf{n}}}\,|\mathbf{0}\rangle$$

where the left and right Pomeron "Fock space vacua" are defined by

$$\langle \mathbf{0}|\,\bar{\mathbf{P}} \; = \; \mathbf{P}\,|\mathbf{0}\rangle \; = \; \mathbf{0}; \qquad \langle \mathbf{0}|\,\bar{\mathbf{d}} \; = \; \langle \mathbf{0}|; \qquad \mathbf{d}\,|\mathbf{0}\rangle \; = \; |\mathbf{0}\rangle$$

States in RFT

$$\langle m| = \sum_i a_i \langle i|; \qquad 1 \geq a_i \geq 0; \qquad \sum_i a_i = 1$$

$$|\bar{n}\rangle = \sum_i \bar{a}_i |i\rangle; \qquad 1 \geq \bar{a}_i \geq 0; \qquad \sum_i \bar{a}_i = 1$$

Here a_i and $\bar{a}_i$ have the meaning of probabilities.

3. Rapidity evolution of the S matrix

$$\langle m| \bar{n}\rangle_Y = \langle 0| (1 - P)^m e^{H(P,\bar{P})Y} (1 - \bar{P})^{\bar{n}} |0\rangle$$

The unitarity condition

$$\langle m| e^{HY} = \sum_i a_i(Y)\langle i|; \qquad 1 \geq a_i \geq 0; \qquad \sum_i a_i = 1$$

$$e^{HY} |\bar{n}\rangle = \sum_i \bar{a}_i(Y)|i\rangle; \qquad 1 \geq \bar{a}_i \geq 0; \qquad \sum_i \bar{a}_i = 1$$

$$\boxed{\textbf{Fan diagram (BK) Hamiltonian :}} \qquad \mathbf{H_{BK} = -\Delta\left[P\bar{P} - P\bar{P}^2\right]}$$

$\Delta \sim \alpha_s$ **determines the probability to emit a dipole in one step of the evolution**

Action on the dilute projectile (unitarity is preserved) (Δ is absorbed in δY)

$$\mathbf{e^{\delta Y\, H_{BK}}\, |\bar{n}\rangle \;\approx\; (1 - \delta Y\, \bar{n})\, |\bar{n}\rangle \;+\; \delta Y\, \bar{n}\, |\bar{n}+1\rangle}$$

Action on the dense target (unitarity is violated)

$$\mathbf{\langle m|\, e^{\delta Y\, H_{BK}} = (1 + \delta Y m)\langle m| - \delta Y m[1 + \gamma(m-1)]\langle m-1| + \delta Y \gamma m(m-1)\langle m-2|}$$

Braun's Hamiltonian (projectile-target symmetric or self-dual under $P \leftrightarrow \bar{P}$)

$$\mathbf{H_{Braun} = -\frac{1}{\gamma}\left[P\bar{P} - P\bar{P}^2 - \bar{P}P^2\right]}$$

Unitarity is violated both in the evolution of the projectile and target

The source of the problem is in the algebra

$$[\mathbf{P}, \bar{\mathbf{P}}] = -\gamma$$

This commutation relation implies that each dipole (either projectile or target) can scatter only once. In some sense, the dipoles disappear once they scatter. Unphysical! We should allow, say, for each projectile dipole to scatter multiply on target dipoles.

$$(1 - \mathbf{P})(1 - \bar{\mathbf{P}}) = [1 - \gamma]\,(1 - \bar{\mathbf{P}})(1 - \mathbf{P}) \qquad \longrightarrow \qquad \mathbf{d}\bar{\mathbf{d}} = e^{-\gamma}\bar{\mathbf{d}}\mathbf{d}$$

$$\langle \mathbf{m}|\bar{\mathbf{n}}\rangle = (1 - \gamma)^{\mathbf{m}\bar{\mathbf{n}}} = e^{-\gamma\,\mathbf{m}\,\bar{\mathbf{n}}}$$

$$\boxed{\textbf{Unitarized Toy Model :}} \qquad \mathbf{H_{UTM}} = -\frac{\Delta}{\gamma}\bar{\mathbf{P}}\mathbf{P}$$

$$\mathbf{e}^{\delta \mathbf{Y}\, \mathbf{H_{UTM}}} |\bar{\mathbf{n}}\rangle \;\approx\; (1 - \delta\mathbf{Y}\frac{\Delta}{\gamma}(1 - \mathbf{e}^{-\gamma\,\bar{\mathbf{n}}}))\,|\bar{\mathbf{n}}\rangle \;+\; \delta\mathbf{Y}\,\frac{\Delta}{\gamma}(1 - \mathbf{e}^{-\gamma\,\bar{\mathbf{n}}})\,|\bar{\mathbf{n}}+1\rangle$$

Equivalently, defining probability to find n dipoles in the wavefunction

$$\frac{dP_n^{\textbf{UTM}}(Y)}{dY} = -\frac{\Delta}{\gamma}\left(1 - e^{-\gamma n}\right)P_n^{\textbf{UTM}}(Y) \;+\; \frac{\Delta}{\gamma}\left(1 - e^{-\gamma(n-1)}\right)P_{n-1}^{\textbf{UTM}}(Y)$$

- **This is the model by Mueller and Salam**
- **Both self-dual and s-channel unitary**
- **Reduces to BK in either dilute projectile or target limits**
- **In the dense limit violates the KNO scaling**

Yet, the model is NOT unique. Furthermore, at each step of the evolution it produces one dipole only. This is somewhat unphysical for dense systems of partons.

Unitarized Toy Model with Multiple emissions (UTMM)

$$\mathbf{H} \;=\; \mathbf{e}^{-\frac{\Delta}{\gamma}\bar{\mathbf{P}}\mathbf{P}} - \mathbf{1}$$

Several ordered in rapidity regimes to consider:

1) Very dilute: number of dipoles $\ll 1/\Delta \sim 1/\alpha_s$
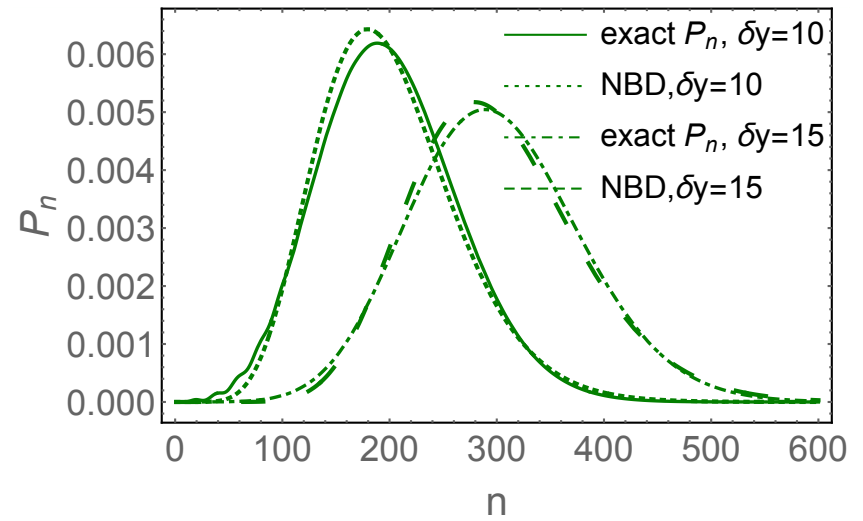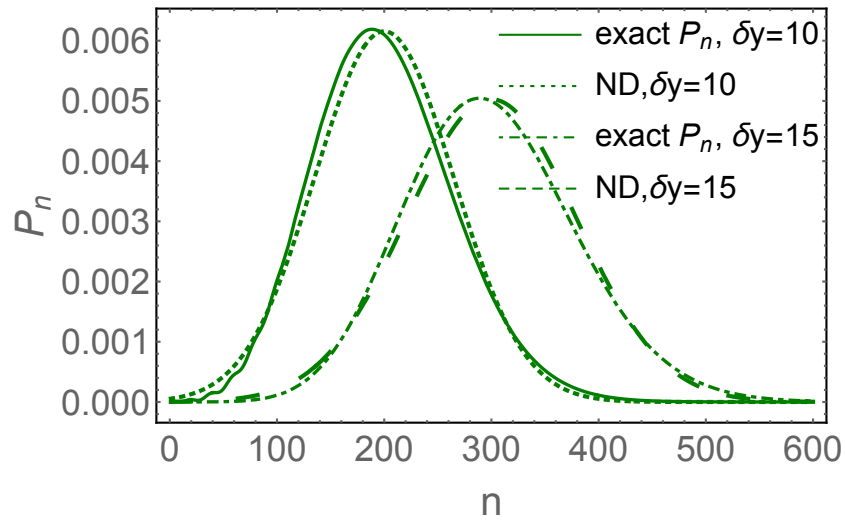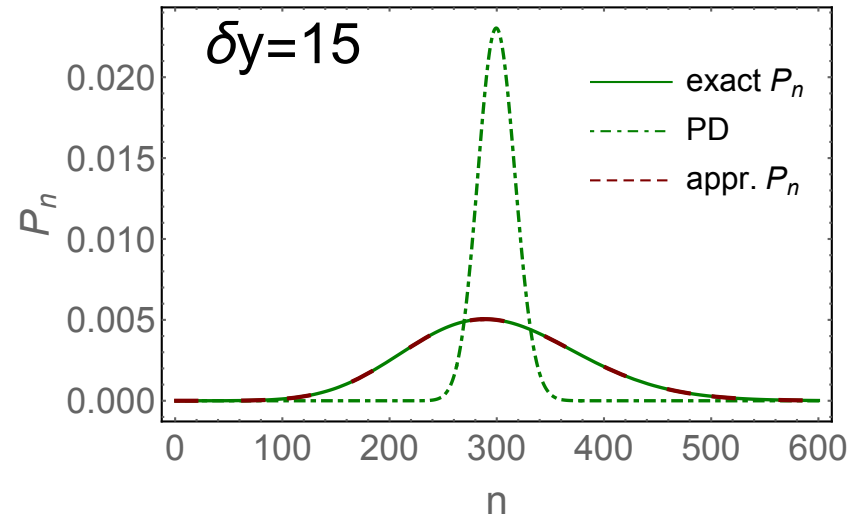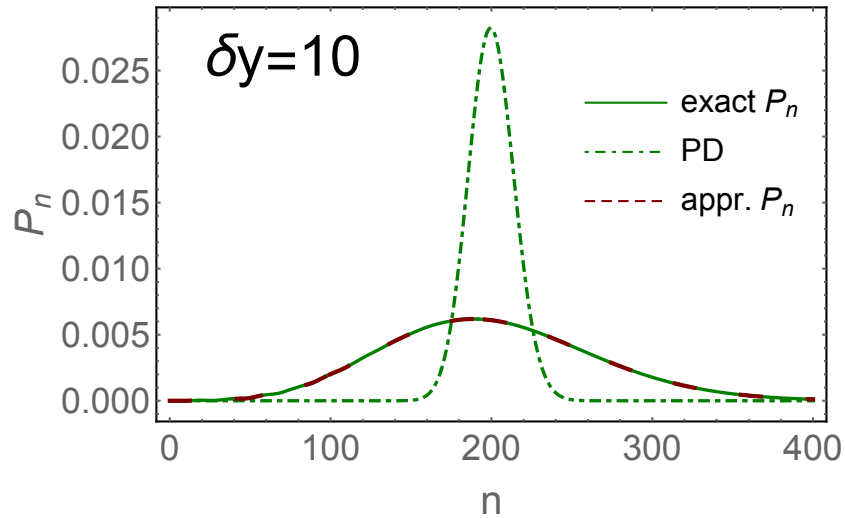
(BFKL physics)

2) Intermediate: $1/\Delta \sim 1/\alpha_s \ll$ number of dipoles $\ll 1/\gamma \sim 1/\alpha_s^2$

multiple emissions are important but no saturation physics yet;

$$\frac{1}{\Delta}\ln\frac{1}{\Delta} \;<\; \mathbf{Y} \;<\; \frac{1}{\Delta}\ln\frac{1}{\gamma}$$

3) Dense: number of dipoles $\gg 1/\gamma$

Both multiple emissions and saturation physics are at play; $\mathbf{Y} > \frac{1}{\Delta}\ln\frac{1}{\gamma}$

P_n – **UTMM. PD is the Poisson distribution (UTM). ND and NBD denote the normal and negative binomial distributions.**
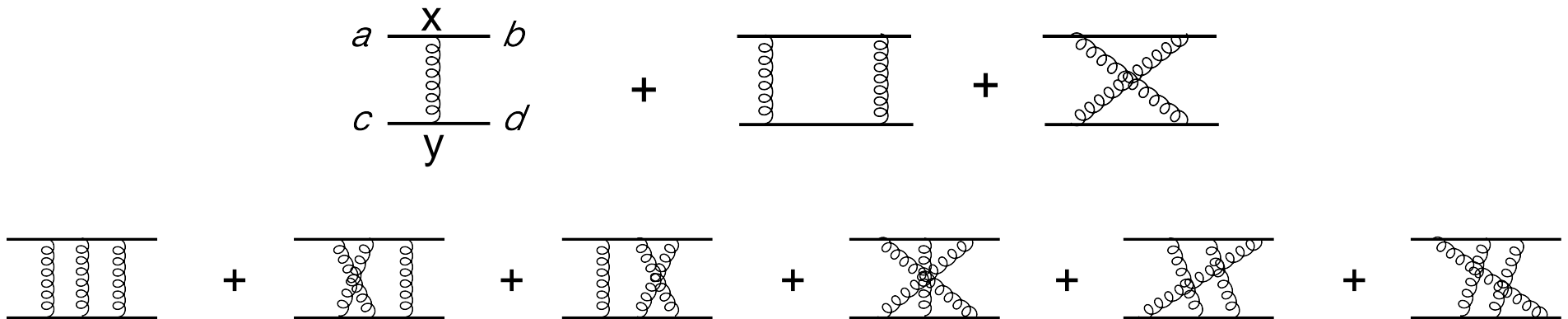
1. RFT degrees of freedom: Wilson line operator and its dual

$$\bar{\mathbf{U}}(\mathbf{x}) = \mathbf{e}^{\mathbf{T^a}\frac{\delta}{\delta\rho^{\mathbf{a}}(\mathbf{x})}} \; ; \qquad \mathbf{U}(\mathbf{x}) = \mathbf{e}^{\mathbf{igT^a}\int_{\mathbf{y}}\phi(\mathbf{x-y})\rho^{\mathbf{a}}(\mathbf{y})}$$

Here ρ is the color charge density and $\phi(\mathbf{x-y}) = \frac{\mathbf{g}}{\nabla^{\mathbf{2}}} = \frac{\mathbf{g}}{\mathbf{2\pi}}\ln|\mathbf{x-y}|$

The algebra encodes a gluon-gluon scattering and is very complicated

$$[\bar{\mathbf{U}}(\mathbf{x}), \mathbf{U}(\mathbf{y})] = \mathbf{1 + g\phi + ...}$$

2. Left and Right Fock vacuum states

$$\langle \mathbf{L}|\bar{\mathbf{U}}_{\mathbf{ab}} = \delta_{\mathbf{ab}}\langle \mathbf{L}|; \qquad \mathbf{U}_{\mathbf{ab}}|\mathbf{R}\rangle = \delta_{\mathbf{ab}}|\mathbf{R}\rangle$$

S-matrix

$$\mathbf{S_{if}} = \langle \mathbf{L}|\mathbf{U}^{\mathbf{a_1 b_1}}(\mathbf{x_1})\ldots\mathbf{U}^{\mathbf{a_N b_N}}(\mathbf{x_N})\bar{\mathbf{U}}^{\mathbf{c_1 d_1}}(\mathbf{y_1})\ldots\bar{\mathbf{U}}^{\mathbf{c_M d_M}}(\mathbf{y_M})|\mathbf{R}\rangle$$

RFT States:

$$|\mathbf{P}\rangle = \sum_{\mathbf{m},\{\bar{\mathbf{c}},\bar{\mathbf{d}};\bar{\mathbf{y}}\}} \mathbf{G_M^m}(\mathbf{Y}, \{\mathbf{c},\mathbf{d},\mathbf{y};\ \bar{\mathbf{c}},\bar{\mathbf{d}},\bar{\mathbf{y}}\}) \prod_{\mathbf{i=1}}^{\mathbf{m}} [\bar{\mathbf{U}}^{\bar{\mathbf{c}}_{\mathbf{i}}\bar{\mathbf{d}}_{\mathbf{i}}}(\bar{\mathbf{y}}_{\mathbf{i}})]|\mathbf{R}\rangle$$

$$|\mathbf{T}\rangle = \sum_{\mathbf{n},\{\bar{\mathbf{a}},\bar{\mathbf{b}};\bar{\mathbf{x}}\}} \mathbf{F_N^n}(\mathbf{Y}, \{\mathbf{a},\mathbf{b},\mathbf{x};\ \bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{x}}\})\langle \mathbf{L}|\prod_{\mathbf{i=1}}^{\mathbf{n}} [\mathbf{U}^{\bar{\mathbf{a}}_{\mathbf{i}}\bar{\mathbf{b}}_{\mathbf{i}}}(\bar{\mathbf{x}}_{\mathbf{i}})]$$

Projectile and Target unitarity condition:

$$1 > \mathbf{G_M^m}(\mathbf{Y}, \{\mathbf{c},\mathbf{c},\mathbf{y};\ \bar{\mathbf{c}},\bar{\mathbf{c}},\bar{\mathbf{y}}\}) > 0; \qquad \sum_{\mathbf{m}}\sum_{\bar{\mathbf{c}}}\int \{\mathbf{d}\bar{\mathbf{y}}\}\mathbf{G_M^m}(\mathbf{Y}, \{\mathbf{c},\mathbf{c},\mathbf{y};\ \bar{\mathbf{c}},\bar{\mathbf{c}},\bar{\mathbf{y}}\}) = 1$$

$$1 > \mathbf{F_N^n}(\mathbf{Y}, \{\mathbf{a},\mathbf{a},\mathbf{x};\ \bar{\mathbf{a}},\bar{\mathbf{a}},\bar{\mathbf{x}}\}) > 0; \qquad \sum_{\mathbf{n}}\sum_{\bar{\mathbf{a}}}\int \{\mathbf{d}\bar{\mathbf{x}}\}\mathbf{F_N^n}(\mathbf{Y}, \{\mathbf{a},\mathbf{a},\mathbf{x};\ \bar{\mathbf{a}},\bar{\mathbf{a}},\bar{\mathbf{x}}\}) = 1$$

3. Energy evolution with RFT Hamiltonian $H_{RFT}[U, \bar{U}]$.

$$S_{if}(Y) = \langle L|U^{a_1 b_1}(x_1) \dots U^{a_N b_N}(x_N) e^{YH_{RFT}[U,\bar{U}]} \bar{U}^{c_1 d_1}(y_1) \dots \bar{U}^{c_M d_M}(y_M)|R\rangle$$

Projectile evolution

$$\bar{U}^{c_1 d_1}(y_1) \dots \bar{U}^{c_M d_M}(y_M)|R\rangle \to e^{YH_{RFT}[U,\bar{U}]} \bar{U}^{c_1 d_1}(y_1) \dots \bar{U}^{c_M d_M}(y_M)|R\rangle$$

$$= \sum_{m,\{\bar{c},\bar{d};\bar{y}\}} G_M^m(Y, \{c, d, y; \ \bar{c}, \bar{d}, \bar{y}\}) \prod_{i=1}^{m} [\bar{U}^{\bar{c}_i \bar{d}_i}(\bar{y}_i)]|R\rangle$$

Projectile unitarity condition is satisfied by H_{JIMWLK}

Target evolution

$$\langle L|U^{a_1 b_1}(x_1) \dots U^{a_N b_N}(x_N) \to \langle L|U^{a_1 b_1}(x_1) \dots U^{a_N b_N}(x_N) e^{YH_{RFT}}$$

$$= \sum_{n,\{\bar{a},\bar{b};\bar{x}\}} F_N^n(Y, \{a, b, x; \ \bar{a}, \bar{b}, \bar{x}\}) \langle L| \prod_{i=1}^{n} [U^{\bar{a}_i \bar{b}_i}(\bar{x}_i)]$$

Target unitarity condition is violated by H_{JIMWLK}!

$$H_{\text{JIMWLK}} = \frac{\alpha_s}{2\pi^2} \int_{x,y,z} \frac{(\mathbf{x}-\mathbf{z})\cdot(\mathbf{y}-\mathbf{z})}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2} \left[2\mathcal{J}_L^a(\mathbf{x})\mathcal{J}_R^b(\mathbf{y})\bar{U}^{ab}(\mathbf{z}) - \mathcal{J}_L^a(\mathbf{x})\mathcal{J}_L^a(\mathbf{y}) - \mathcal{J}_R^a(\mathbf{x})\mathcal{J}_R^a(\mathbf{y}) \right].$$

$$[\mathcal{J}_L^a(\mathbf{x}), \bar{U}^{mn}(\mathbf{y})] = -(\mathbf{T}^a\bar{U}(\mathbf{y}))^{mn}\delta(\mathbf{x}-\mathbf{y}),$$
$$[\mathcal{J}_R^a(\mathbf{x}), \bar{U}^{mn}(\mathbf{y})] = -(\bar{U}(\mathbf{y})\mathbf{T}^a)^{mn}\delta(\mathbf{x}-\mathbf{y}).$$

$\mathbf{SU_L(N) \times SU_R(N)}$:

$$[\mathcal{J}_L^a(\mathbf{x}), \mathcal{J}_L^b(\mathbf{y})] = if^{abc}\mathcal{J}_L^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}),$$

$$[\mathcal{J}_R^a(\mathbf{x}), \mathcal{J}_R^b(\mathbf{y})] = -if^{abc}\mathcal{J}_R^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})$$

$$[\mathcal{J}_L^a(\mathbf{x}), \mathcal{J}_R^a(\mathbf{y})] = 0$$

H_{JIMWLK} **is obviously not symmetric (self-dual) under** $U \leftrightarrow \bar{U}$ **(interchange of projectile and target)**

$$[\mathbf{U^{mn}(y)}, \mathcal{I}_\mathbf{L}^\mathbf{a}(\mathbf{x})] = -(\mathbf{T^a U(y)})^\mathbf{mn}\delta(\mathbf{x-y})\,,$$

$$[\mathbf{U^{mn}(y)}, \mathcal{I}_\mathbf{R}^\mathbf{a}(\mathbf{x})] = -(\mathbf{U(y)T^a})^\mathbf{mn}\delta(\mathbf{x-y})\,.$$

Another $\mathbf{SU_L(N)\times SU_R(N)}$:

$$[\mathcal{I}_\mathbf{L}^\mathbf{a}(\mathbf{x}), \mathcal{I}_\mathbf{L}^\mathbf{b}(\mathbf{y})] = -\mathbf{if^{abc}}\mathcal{I}_\mathbf{L}^\mathbf{c}(\mathbf{x})\delta(\mathbf{x-y})\,,$$
$$[\mathcal{I}_\mathbf{R}^\mathbf{a}(\mathbf{x}), \mathcal{I}_\mathbf{R}^\mathbf{b}(\mathbf{y})] = \mathbf{if^{abc}}\mathcal{I}_\mathbf{R}^\mathbf{c}(\mathbf{x})\delta(\mathbf{x-y})$$

$$[\mathcal{I}_\mathbf{L}^\mathbf{a}(\mathbf{x}), \mathcal{I}_\mathbf{R}^\mathbf{a}(\mathbf{y})] = \mathbf{0}$$

Because $[\bar{U}, U] \neq 0$, the algebra $[\mathcal{J}_{L,R}, \mathcal{I}_{L,R}] \neq 0$ and is very complicated. So far, we know how to deal with these algebras perturbatively only.

Wilson line like operators in the *fundamental representation*

$$\mathbf{V_L(x)} = \mathrm{Exp}\left\{\mathbf{i}\int_y \mathbf{g}\phi(\mathbf{x}-\mathbf{y})\mathbf{t^e}\mathcal{J_L^e(y)}\right\}$$

$$\mathbf{V_R(x)} = \mathrm{Exp}\left\{-\mathbf{i}\int_y \mathbf{g}\phi(\mathbf{x}-\mathbf{y})\mathbf{t^e}\mathcal{J_R^e(y)}\right\}$$

$$\mathbf{\bar{V}_L(x)} = \mathrm{Exp}\left\{\mathbf{i}\int_y \mathbf{g}\phi(\mathbf{x}-\mathbf{y})\mathbf{t^e}\mathcal{I_L^e(y)}\right\}$$

$$\mathbf{\bar{V}_R(x)} = \mathrm{Exp}\left\{-\mathbf{i}\int_y \mathbf{g}\phi(\mathbf{x}-\mathbf{y})\mathbf{t^e}\mathcal{I_R^e(y)}\right\}.$$

A Self-dual Hamiltonian

$$\mathbf{H_{RFT}} = \frac{1}{\pi\mathbf{g^2}}\int \mathbf{d^2x}\,\partial^2[\mathbf{\bar{V}_L^{\beta\gamma}(x)}\mathbf{\bar{V}_R^{\delta Y\alpha}(x)}]\mathbf{V_L^{\alpha\beta}(x)}\mathbf{V_R^{\gamma\delta}(x)}$$

Reduces to JIMWLK in the dense-dilute limit

**Looks very similar to the "diamond action" of
Y. Hatta, E. Iancu, L. McLerran, A. Stasto, D. Triantafyllopoulos, NPA 764, 423 (2006)
but in fact different because it is built of Wilson lines obeying different algebra.**