

Pion parton distribution function in Minkowski space.

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Outline

- I. Pion as a fermion-antifermion bound state in Minkowski space.
- II. Nakanishi integral representation and LF projection.
- III. Pion Distribution function, charge radius and Electromagnetic Form Factor.
- IV. Pion image on the null-plane
- V. Conclusions and perspectives

Pion as a Quark-antiquark bound state

Bound state: Solve the Bethe-Salpeter equation

Most non-perturbative methods are formulated in Euclidean space

Wick Rotation: We have to be care with the presence of singularities.

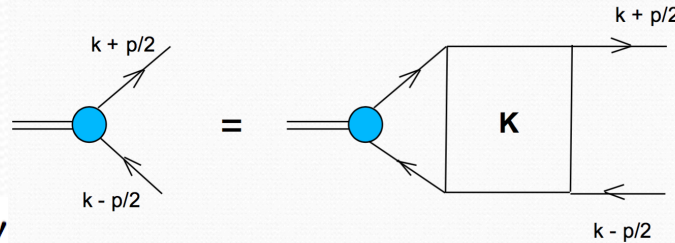
Our challenge is to work with a non-perturbative framework in Minkowski Space in order to access hadron structure observables defined on the light-front.

Minkowski solutions: Bethe-Salpeter

Tool: Integral representation

Quark-antiquark bound state - Pion

- Bethe-Salpeter equation (0^-) :



$$\Phi(k; P) = S\left(k + \frac{P}{2}\right) \int \frac{d^4 k'}{(2\pi)^4} S^{\mu\nu}(q) \Gamma_\mu(q) \Phi(k'; P) \hat{\Gamma}_\nu(q) S\left(k - \frac{P}{2}\right)$$

$$\hat{\Gamma}_\nu(q) = C \Gamma_\nu(q) C^{-1}$$

where we use: i) bare propagators for the quarks and gluons;
ii) ladder approximation

$$S(P) = \frac{i}{\not{P} - m + i\epsilon}$$

$$S^{\mu\nu}(q) = -i \frac{g^{\mu\nu}}{q^2 - \mu^2 + i\epsilon}$$

Quark-gluon vertex

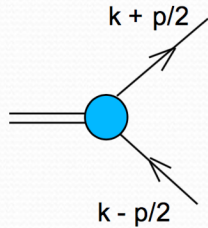
$$\Gamma^\mu = ig \frac{\mu^2 - \Lambda^2}{q^2 - \Lambda^2 + i\epsilon} \gamma^\mu$$

We consider only one of the Longitudinal components of the QGV

We set the value of the scale parameter (~ 300 MeV) from the combined analysis of Lattice simulations, the Quark-Gap Equation and Slanov-Taylor identity.

NIR for fermion-antifermion Bound State

BSA for a quark-antiquark bound state



$$\Phi(k, p) = \sum_{i=1}^4 S_i(k, p) \phi_i(k, p)$$

$$S_1 = \gamma_5 \quad S_2 = \frac{1}{M} \not{p} \gamma_5 \quad S_3 = \frac{k \cdot p}{M^3} \not{p} \gamma_5 - \frac{1}{M} \not{k} \gamma_5 \quad S_4 = \frac{i}{M^2} \sigma_{\mu\nu} p^\mu k^\nu \gamma_5$$

Using the *Nakanishi Integral Representation* for the scalar functions

$$\phi_i(k, p) = \int_{-1}^{+1} dz' \int_0^\infty d\gamma' \frac{g_i(\gamma', z')}{(k^2 + p \cdot k z' + M^2/4 - m^2 - \gamma' + i\epsilon)^3}$$

System of coupled integral equations

$$\int_{-1}^1 dz' \int_0^\infty d\gamma' \frac{g_i(\gamma', z')}{[k^2 + z' p \cdot k - \gamma' - \kappa^2 + i\epsilon]^3} = \sum_j \int_{-1}^1 dz' \int_0^\infty d\gamma' \mathcal{K}_{ij}(k, p; \gamma', z') g_j(\gamma', z')$$

Projecting BSE onto the LF hyper-plane $x^+=0$

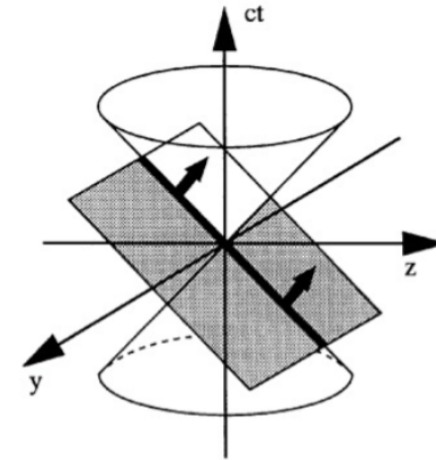
Light-Front variables

$$x^\mu = (x^+, x^-, \mathbf{x}_\perp)$$

LF-time $x^+ = x^0 + x^3$

$$x^- = x^0 - x^3$$

$$\mathbf{x}_\perp = (x^1, x^2)$$



Within the LF framework, the valence wf is obtained by integrating the BSA on κ (elimination of the relative LF time).

LF amplitudes

$$\psi_i(\gamma, \xi) = \int \frac{dk^-}{2\pi} \phi_i(k, p) = -\frac{i}{M} \int_0^\infty d\gamma' \frac{g_i(\gamma', z)}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2)\kappa^2]^2}$$

The coupled equation system is (NIR+LF projection, Karmanov & Carbonell 2010)

$$\int_0^\infty d\gamma' \frac{g_i(\gamma', z')}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2)\kappa^2]^2} = iMg^2 \sum_j \int_0^\infty d\gamma' \int_{-1}^1 dz' \mathcal{L}_{ij}(\gamma, z; \gamma' z') g_j(\gamma, z')$$

The Kernel contains singular contributions

Parton distribution function

WP, Ydrefors, Nogueira, Frederico and Salmè **PRD** 105, L071505 (2022).

The unpolarized transverse-momentum distribution (uTMD) reads

$$f_1(\gamma, \xi) = \frac{N_c}{4} \int d\phi_{\hat{\mathbf{k}}_\perp} \int \frac{dz^- d\mathbf{z}_\perp}{2(2\pi)^3} e^{i[\xi P^+ z^- / 2 - \mathbf{k}_\perp \cdot \mathbf{z}_\perp]} \langle P | \bar{\psi}_q \left(-\frac{1}{2} z \right) \gamma^+ \psi_q \left(\frac{1}{2} z \right) | P \rangle \big|_{z^+=0}$$

The **PDF** is the integral over the squared transverse momentum.

$$u(\xi) = \int_0^\infty d\gamma f_1(\gamma, \xi)$$

Considering the charge symmetry and Mandelstam framework

$$f_1(\gamma, \xi) = \frac{1}{(2\pi)^4} \frac{1}{8} \int_{-\infty}^{\infty} dk^+ \delta(k^+ + P^+/2 - \xi P^+) \int_{-\infty}^{\infty} dk^- \int_0^{2\pi} d\phi_{\hat{\mathbf{k}}_\perp} \left\{ \text{Tr} \left[S^{-1}(k - P/2) \bar{\Phi}(k, P) \frac{\gamma^+}{2} \Phi(k, P) \right] - \text{Tr} \left[S^{-1}(k + P/2) \Phi(k, P) \frac{\gamma^+}{2} \bar{\Phi}(k, P) \right] \right\}$$

LF Momentum Distributions

The fermionic field on the null-plane is given by:

$$\psi^{(+)}(\tilde{x}, x^+ = 0^+) = \int \frac{d\tilde{q}}{(2\pi)^{3/2}} \frac{\theta(q^+)}{\sqrt{2q^+}} \sum_{\sigma} \left[U^{(+)}(\tilde{q}, \sigma) b(\tilde{q}, \sigma) e^{i\tilde{q} \cdot \tilde{x}} + V^{(+)}(\tilde{q}, \sigma) d^{\dagger}(\tilde{q}, \sigma) e^{-i\tilde{q} \cdot \tilde{x}} \right]$$

where

$$U^{(+)}(\tilde{q}, \sigma) = \Lambda^+ u(\tilde{q}, \sigma) \quad , \quad V^{(+)}(\tilde{q}, \sigma) = \Lambda^+ v(\tilde{q}, \sigma) \quad \Lambda^{\pm} = \frac{1}{4} \gamma^{\mp} \gamma^{\pm}$$

Hence d^{\dagger} and b are the fermion creation/annihilation operators

The LF valence amplitude is the Fock component with the lowest number of constituents

$$\varphi_2(\xi, \mathbf{k}_{\perp}, \sigma_i; \mathbf{M}, \mathbf{J}^{\pi}, \mathbf{J}_z) = (2\pi)^3 \sqrt{N_c} \, 2\mathbf{p}^+ \sqrt{\xi(1-\xi)} \langle 0 | b(\tilde{\mathbf{q}}_2, \sigma_2) d(\tilde{\mathbf{q}}_1, \sigma_1) | \tilde{\mathbf{p}}, \mathbf{M}, \mathbf{J}^{\pi}, \mathbf{J}_z \rangle$$

where

$$\tilde{q}_1 \equiv \{q_1^+ = M(1-\xi), -\mathbf{k}_{\perp}\} \quad , \quad \tilde{q}_2 \equiv \{q_2^+ = M\xi, \mathbf{k}_{\perp}\}$$

$$\text{and } \xi = 1/2 + k^+/p^+$$

LF Momentum Distributions

LF valence amplitude in terms of BS amplitude is:

$$\varphi_2(\xi, \mathbf{k}_\perp, \sigma_i; \mathbf{M}, \mathbf{J}^\pi, \mathbf{J}_z) = \frac{\sqrt{N_c}}{p^+} \frac{1}{4} \bar{u}_\alpha(\tilde{q}_2, \sigma_2) \int \frac{dk^-}{2\pi} [\gamma^+ \Phi(k, p) \gamma^+]_{\alpha\beta} v_\beta(\tilde{q}_1, \sigma_1).$$

which can be decomposed into two spin contributions:

Anti-aligned configuration:

$$\psi_{\uparrow\downarrow}(\gamma, z) = \psi_2(\gamma, z) + \frac{z}{2} \psi_3(\gamma, z) + \frac{i}{M^3} \int_0^\infty d\gamma' \frac{\partial g_3(\gamma', z) / \partial z}{\gamma + \gamma' + z^2 m^2 + (1 - z^2) \kappa^2}$$

Aligned configuration: $\psi_{\uparrow\uparrow}(\gamma, z) = \psi_{\downarrow\downarrow}(\gamma, z) = \frac{\sqrt{\gamma}}{M} \psi_4(\gamma, z)$

with the LF amplitudes given by

$$\psi_i(\gamma, z) = -\frac{i}{M} \int_0^\infty d\gamma' \frac{g_i(\gamma', z)}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2) \kappa^2]^2}$$

Valence Probability

We can define the Valence Probability as

$$P_{val} = \frac{1}{(2\pi)^3} \sum_{\sigma_1 \sigma_2} \int_{-1}^1 \frac{dz}{(1-z^2)} \int d\mathbf{k}_{\perp} \left| \varphi_{n=2}(\xi, \mathbf{k}_{\perp}, \sigma_i; \mathbf{M}, \mathbf{J}^{\pi}, \mathbf{J}_z) \right|^2$$

where $z = 1 - 2\xi$

The probability of the LF-valence WF reads

$$P_{val} = \int_{-1}^1 dz \int_0^{\infty} \frac{d\gamma}{(4\pi)^2} \left[|\psi_{\uparrow\downarrow}(\gamma, z)|^2 + |\psi_{\uparrow\uparrow}(\gamma, z)|^2 \right]$$

where we decomposed it in terms of the aligned and anti-aligned LFWF

The contribution to the PDF from the LF-valence WF is

$$u_{val}(\xi) = \int_0^{\infty} \frac{d\gamma}{(4\pi)^2} \left[|\psi_{\uparrow\downarrow}(\gamma, z)|^2 + |\psi_{\uparrow\uparrow}(\gamma, z)|^2 \right]$$

Quantitative results: Static properties

WP, Ydrefors, A. Nogueira, Frederico and Salme **PRD 103** 014002 (2021).

| Set | m (MeV) | B/m | μ/m | Λ/m | P_{val} | $P_{\uparrow\downarrow}$ | $P_{\uparrow\uparrow}$ | f_π (MeV) |
|-------------|------------|-------------|------------|-------------|-------------|--------------------------|------------------------|---------------|
| I | 187 | 1.25 | 0.15 | 2 | 0.64 | 0.55 | 0.09 | 77 |
| II | 255 | 1.45 | 1.5 | 1 | 0.65 | 0.55 | 0.10 | 112 |
| III | 255 | 1.45 | 2 | 1 | 0.66 | 0.56 | 0.11 | 117 |
| IV | 215 | 1.35 | 2 | 1 | 0.67 | 0.57 | 0.11 | 98 |
| V | 187 | 1.25 | 2 | 1 | 0.67 | 0.56 | 0.11 | 84 |
| VI | 255 | 1.45 | 2.5 | 1 | 0.68 | 0.56 | 0.11 | 122 |
| VII | 255 | 1.45 | 2.5 | 1.1 | 0.69 | 0.56 | 0.12 | 127 |
| VIII | 255 | 1.45 | 2.5 | 1.2 | 0.70 | 0.57 | 0.13 | 130 |
| IX | 255 | 1.45 | 1 | 2 | 0.70 | 0.57 | 0.14 | 134 |
| X | 215 | 1.35 | 1 | 2 | 0.71 | 0.57 | 0.14 | 112 |
| XI | 187 | 1.25 | 1 | 2 | 0.71 | 0.58 | 0.14 | 96 |

The set VIII reproduces the pion decay constant

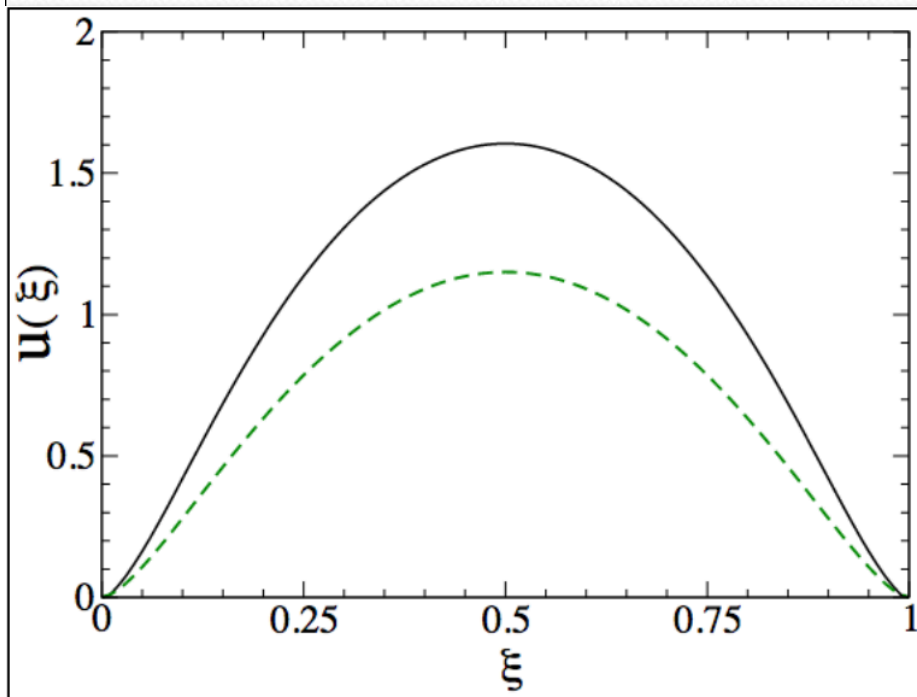
$$m_q = 255 \text{ MeV}, m_g = 637.5 \text{ MeV and } \Lambda = 306 \text{ MeV}$$

The contributions beyond the valence component are important, $\sim 30\%$

The Valence probability has a small variation for the range of parameters

Parton distribution function

WP, Ydrefors, Nogueira, Frederico and Salmè **PRD** 105, L071505 (2022).



Solid line: full calculation of the BSE at model scale

Dashed line: The LF valence contribution

The symmetry of the PDFs is related to the charge symmetry.

The full PDF is normalized to 1,
while the Valence PDF has norm 0.7

The difference of 30% is due to the presence of **higher Fock components** in the pion state.

$$|q\bar{q}; n \text{ gluons}\rangle$$

At the initial scale, for $\xi \rightarrow 1$, the exponent of $(1 - \xi)^{\eta_0}$ is $\eta_0 = 1.4$

Parton distribution function

WP, Ydrefors, Nogueira, Frederico and Salmè PRD 105, L071505 (2022).

Low order Mellin moments at scales $Q = 2.0 \text{ GeV}$ and $Q = 5.2 \text{ GeV}$.

| | BSE_2 | LQCD_2 | BSE_5 | LQCD_5 |
|-----------------------|----------------|-------------------|----------------|-------------------|
| $\langle x \rangle$ | 0.259 | 0.261 ± 0.007 | 0.221 | 0.229 ± 0.008 |
| $\langle x^2 \rangle$ | 0.105 | 0.110 ± 0.014 | 0.082 | 0.087 ± 0.009 |
| $\langle x^3 \rangle$ | 0.052 | 0.024 ± 0.018 | 0.039 | 0.042 ± 0.010 |
| $\langle x^4 \rangle$ | 0.029 | | 0.021 | 0.023 ± 0.009 |
| $\langle x^5 \rangle$ | 0.018 | | 0.012 | 0.014 ± 0.007 |
| $\langle x^6 \rangle$ | 0.012 | | 0.008 | 0.009 ± 0.005 |

$\langle x \rangle$ - Alexandrou et al PRD 103, 014508 (2021).

LQCD, $Q = 2.0 \text{ GeV}$:

$\langle x^2 \rangle$ and $\langle x^3 \rangle$ - Alexandrou et al PRD 104, 054504 (2021).

LQCD, $Q = 5.2 \text{ GeV}$: Alexandrou et al PRD 104, 054504 (2021)

Hadronic scale and effective charge for DGLAP
Cui et al EPJC 2020 80 1064

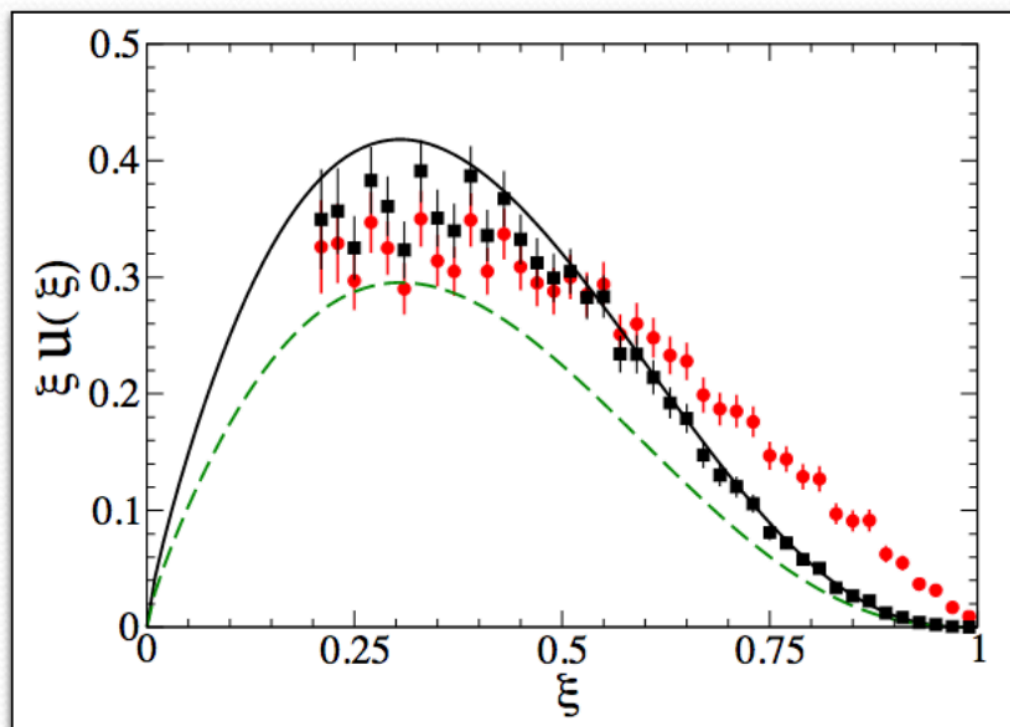
$$Q_0 = 0.330 \pm 0.030 \text{ GeV}$$

Within the error, we choose the $Q_0 = 0.360$ in order to fit the first Mellin moment

We used lowest order DGLAP equations for evolution

Parton distribution function

WP, Ydrefors, Nogueira, Frederico and Salmè **PRD** 105, L071505 (2022).



Solid line: full calculation of the BSE evolved from the initial scale $Q_0 = 0.360$ GeV to $Q = 5.2$ GeV

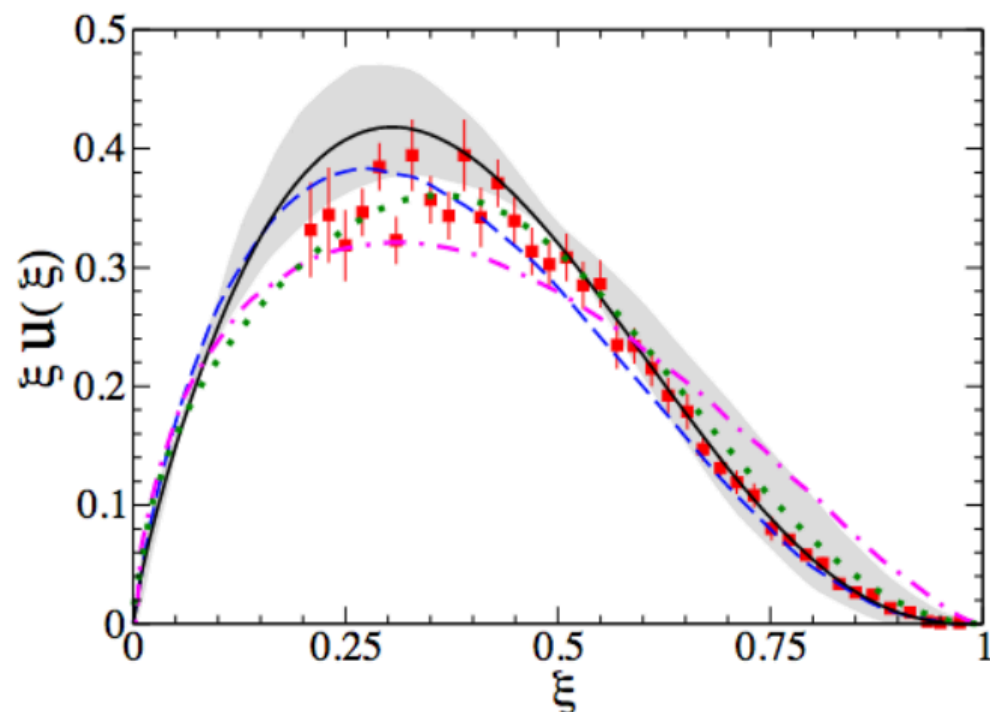
Dashed line: the evolved LF valence contribution

Full dots: experimental data from E615

Full squares: reanalyzed experimental data from Aicher, et al PRL 105, 252003 (2010). evolved to $Q = 5.2$ GeV

Parton distribution function

WP, Ydrefors, Nogueira, Frederico and Salmè **PRD** 105, L071505 (2022).



Solid line: full calculation of the BSE evolved from the initial scale $Q_0 = 0.360$ GeV to $Q = 5.2$ GeV

Dashed line: DSE calculation (Cui et al)

Dash-dotted line: DSE calculation with dressed quark-photon vertex from Bednar et al PRL 124, 042002 (2020).

Dotted line: BLFQ colaboration, PLB 825, 136890 (2022)

Gray area: LQCD results

It is in agreement with PQCD, exponent greater than 2

Evolved $\xi u(\xi)$, for $\xi \rightarrow 1$, the exponent of $(1 - \xi)^{\eta_5}$ is $\eta_5 = 2.94$

LQCD: Alexandrou et al PRD 104, 054504 (2021) obtained 2.20 ± 0.64

Cui et al EPJA 58, 10 (2022) obtained 2.81 ± 0.08

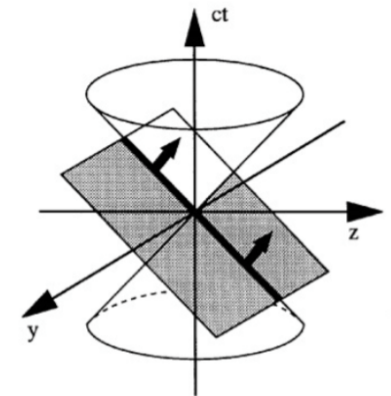
Pion image on the null-plane

The probability distribution of the quarks inside the pion, on the light-front, is evaluated in the space given by the Cartesian product of the Ioffe-time and the plane spanned by the transverse coordinates.

Our goal is to use the configuration space in order to have a more detailed information of the space-time structure of the hadrons.

The Ioffe-time is useful for studying the relative importance of short and long light-like distances. It is defined as:

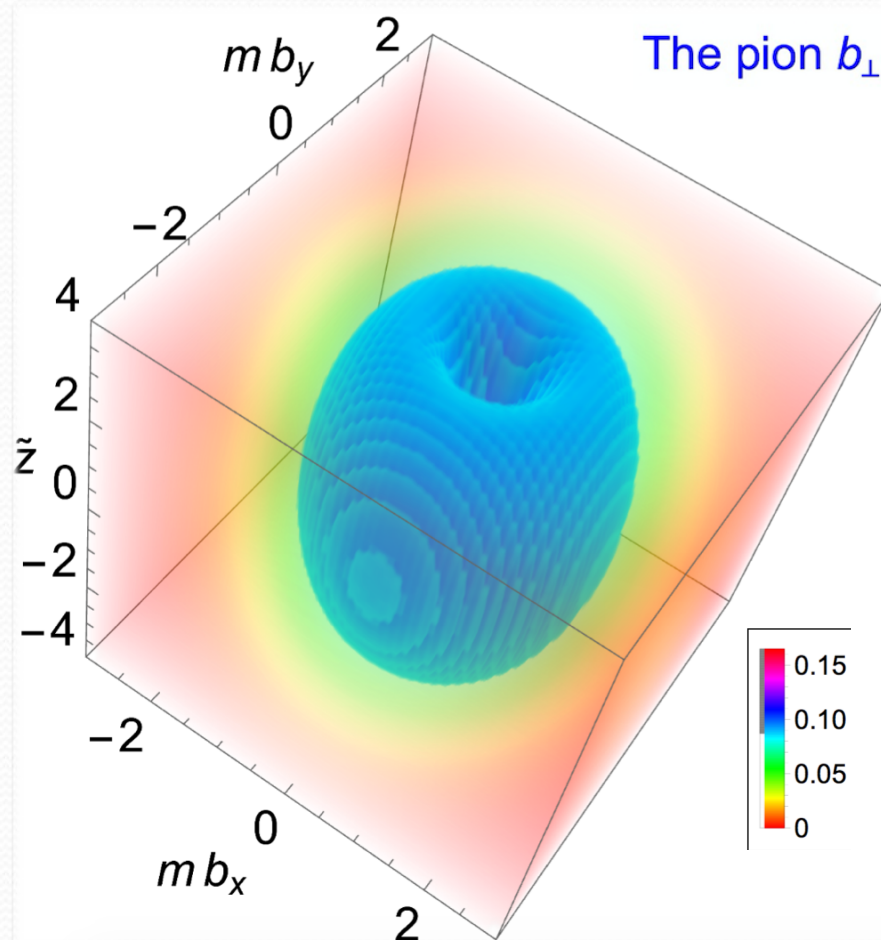
$$\tilde{z} = x \cdot P_{\text{target}} = x^- P_{\text{target}}^+ / 2 \quad \text{on the hyperplane } x^+ = 0$$



Pion image on the null-plane

We perform a Fourier transform of the valence wf

The space-time structure of the pion in terms of loffe-time $\tilde{z} = x^- p^+ / 2$ and the transverse coordinates $\{b_x, b_y\}$

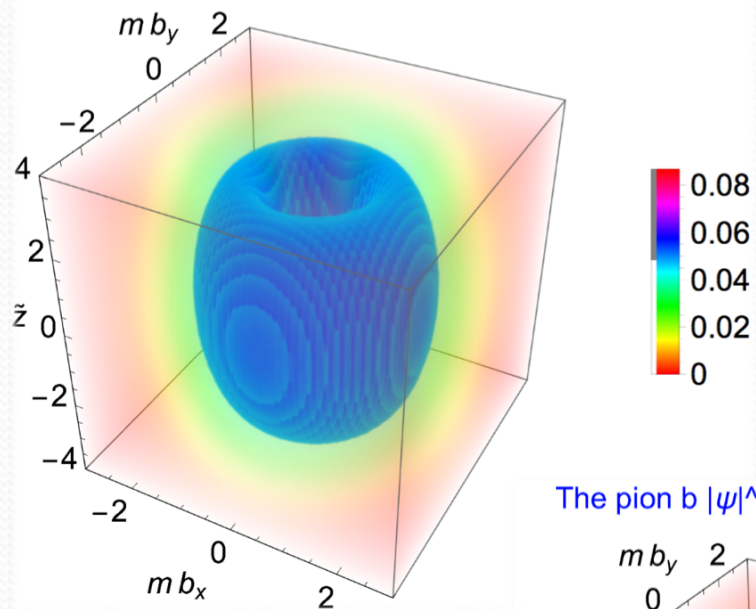


The pion $b_\perp |\psi|^2$ in the 3D loffe-space

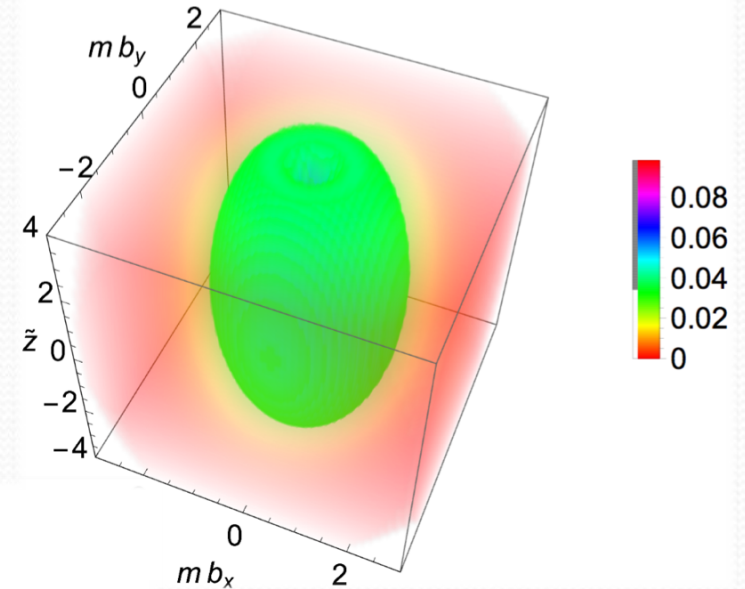
3D imaging of the Pion

3D Pion image: Spin configurations

Anti-aligned

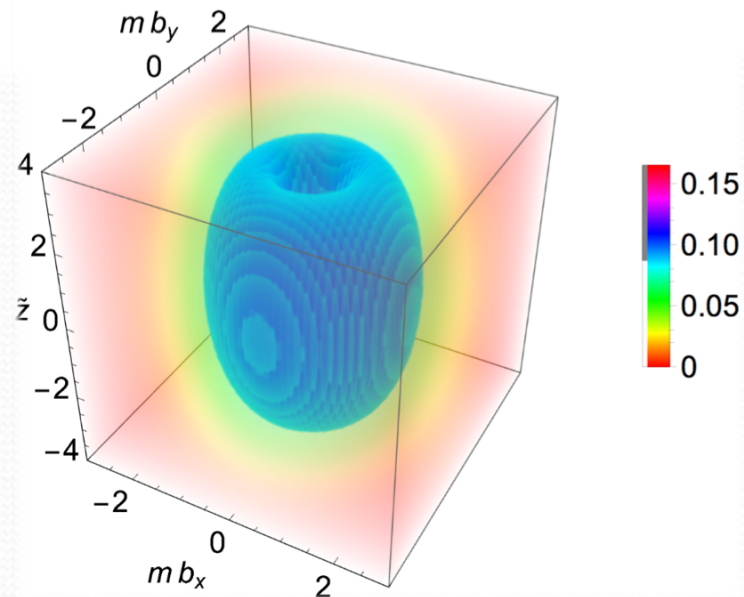


Aligned



The pion b $|\psi|^2$ in the 3D loffe-space

3D imaging of the Pion



Pion charge radius

Ydrefors, WP, Nogueira, Frederico and Salmè **PLB** 820, 136494 (2021).

Pion charge radius and its decomposition in valence and non valence contributions.

| Set | m | B/m | μ/m | Λ/m | P_{val} | f_π | r_π (fm) | r_{val} (fm) | r_{nval} (fm) |
|-----|-----|-------|---------|-------------|-----------|---------|--------------|----------------|-----------------|
| I | 255 | 1.45 | 2.5 | 1.2 | 0.70 | 130 | 0.663 | 0.710 | 0.538 |
| II | 215 | 1.35 | 2 | 1 | 0.67 | 98 | 0.835 | 0.895 | 0.703 |

where

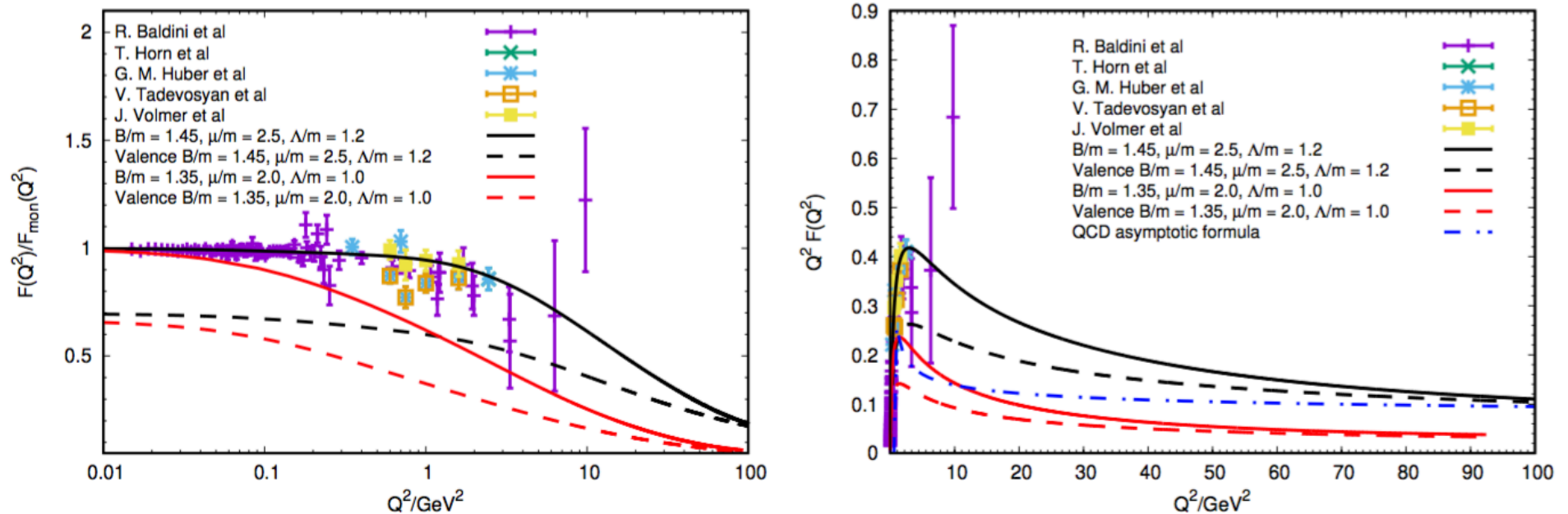
$$r_\pi^2 = -6dF(Q^2)/dQ^2|_{Q^2=0}$$

$$P_{val(nval)} r_{val(nval)}^2 = -6 dF_{val(nval)}(Q^2)/dQ^2|_{Q^2=0}$$

The set I is in fair agreement with the PDG value: $r_\pi^{PDG} = 0.659 \pm 0.004$ fm

Electromagnetic Form Factor

Ydrefors, WP, Nogueira, Frederico and Salmè **PLB** 820, 136494 (2021).



$$m_q = 255 \text{ MeV}, m_g = 637.5 \text{ MeV and } \Lambda = 306 \text{ MeV}$$

Good agreement with experimental data (black curve).

For high Q^2 we obtain the valence dominance (dashed black curve)

Our results recover the pQCD for large Q^2 – Blue curve vs Black curve

Conclusions and Perspectives

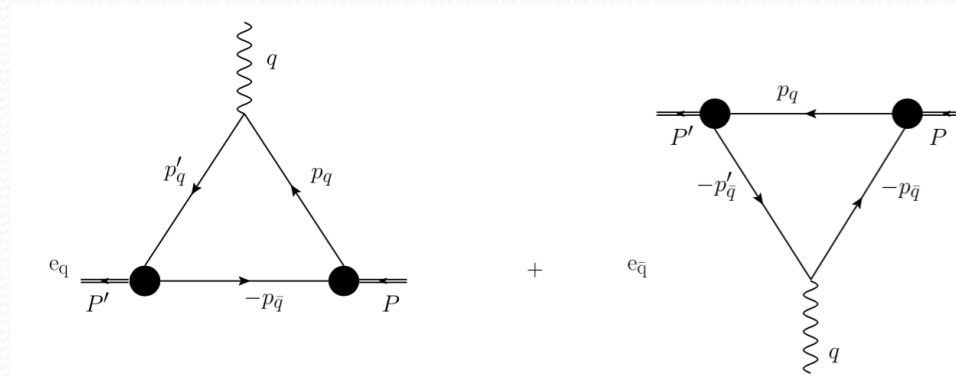
- We present a method for solving the fermionic BSE in Minkowski.
- We obtain the Parton Distribution function, charge radius and Electromagnetic Form Factor.
- Furthermore, the image of the pion in the configuration space has been constructed. This 3D imaging is in line with the goal of the future Electron Ion Collider.
- The beyond-valence contributions are important. The valence probability is of the order of 70%.
- We intend to calculate other Hadronic observables: TMD, GPD.
- Future plan is to include dressing functions for quark and gluon propagators and a more realistic quark-gluon vertex.



Backup

Covariant Electromagnetic Form Factor

Among the pion observables, the electromagnetic form factor plays a relevant role for accessing the inner pion structure, since it is related to the charge density in the so-called impact parameter space.



Adopting the Impulse approximation (bare photon vertex), we have

$$(p + p')^\mu F(Q^2) = -i \frac{N_c}{4M^2 + Q^2} \int \frac{d^4k}{(2\pi)^4} \text{Tr}[(-\not{k} - m) \bar{\Phi}_2(k_2; p') (\not{p} + \not{p}') \Phi_1(k_1; p)]$$

After using the NIR and computing the traces, one obtains

$$F(Q^2) = \frac{N_c}{32\pi^2} \sum_{ij} \int_0^\infty d\gamma \int_{-1}^1 dz g_j(\gamma, z) \int_0^\infty d\gamma' \int_{-1}^1 dz' g_i(\gamma', z') \int_0^1 dy y^2 (1-y)^2 \frac{c_{ij}}{M_{cov}^8}$$

Valence Electromagnetic Form Factors

The Valence contribution to the FF is obtained from the matrix elements of the component γ^+

$$F_{val}(Q^2) = \frac{N_c}{16\pi^3} \int d^2k_{\perp} \int_{-1}^1 dz \left[\psi_{\uparrow\downarrow}^*(\gamma', z) \psi_{\uparrow\downarrow}(\gamma, z) + \frac{\vec{k}_{\perp} \cdot \vec{k}'_{\perp}}{\gamma\gamma'} \psi_{\uparrow\uparrow}^*(\gamma', z) \psi_{\uparrow\uparrow}(\gamma, z) \right]$$

$$F_{val}(0) = p_{val}.$$

where $\vec{k}'_{\perp} = \vec{k}_{\perp} + \frac{1}{2}(1+z)\vec{q}_{\perp}$

Total FF (Drell-Yan Frame): $F(Q^2) = \sum_{n=2}^{\infty} F_n(Q^2) = F_{val}(Q^2) + F_{nval}(Q^2)$

where $F_n(Q^2)$ represents the contribution of the n-th Fock component

QCD Asymptotic Formula (Lepage & Brodsky, 1979):

$$Q^2 F_{asy}(Q^2) = 8\pi\alpha_s(Q^2) f_{\pi}^2$$

Running coupling constant - PDG

Pion Decay Constant

In terms of the BS amplitude, we can write the Pion Decay Constant as:

$$i p^\mu f_\pi = N_c \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[\gamma^\mu \gamma^5 \Phi(p, k)]$$

Contracting with p_μ and using the BSA decomposition we have

$$i M^2 f_\pi = -4 M N_c \int \frac{d^4 k}{(2\pi)^4} \phi_2(k, p)$$

which can be expressed as

$$f_\pi = i \frac{\pi N_c}{(2\pi)^3} \int_0^\infty d\gamma \int_{-1}^1 dz \psi_{\uparrow\downarrow}(\gamma, z)$$

Nakanishi Integral Representation

- Nakanishi representation: Generalization of the Källén-Lehman integral representation (two point functions) for n-point functions.

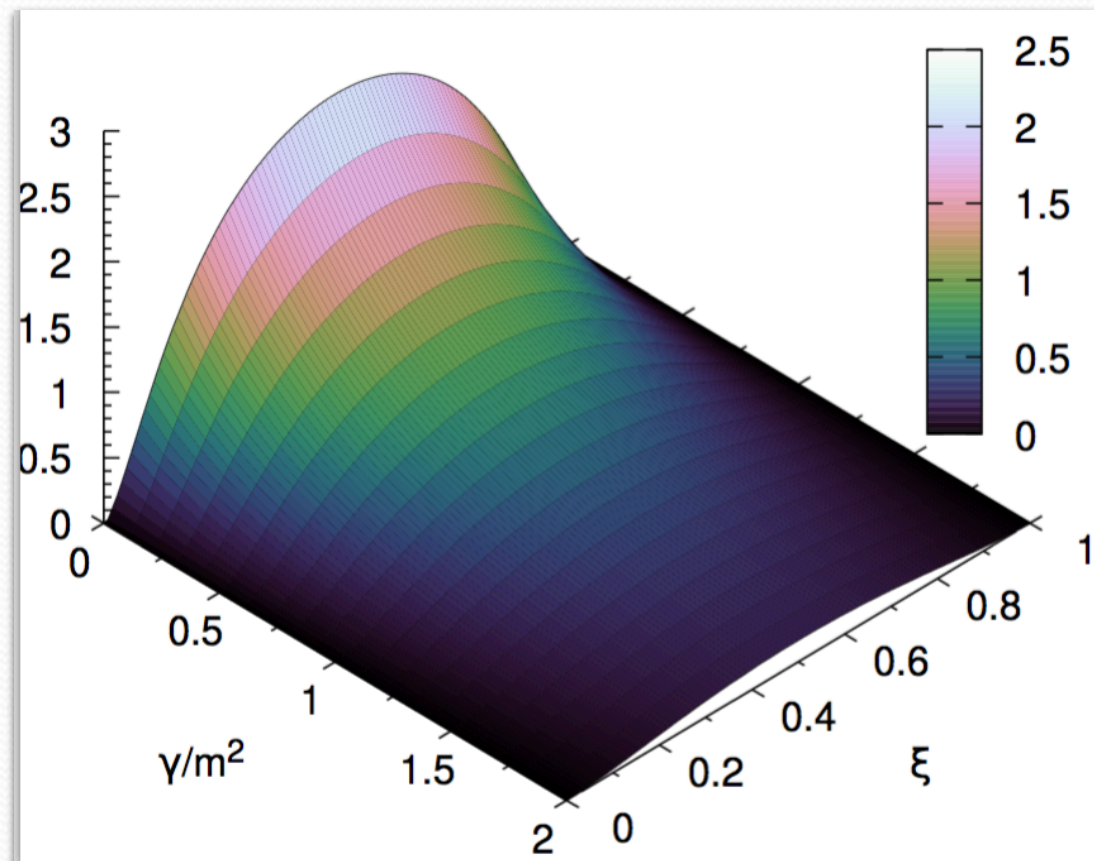
Bethe-Salpeter amplitude

$$\Phi(k, p) = \int_{-1}^1 dz' \int_0^\infty d\gamma' \frac{g(\gamma', z')}{(\gamma' + \kappa^2 - k^2 - p \cdot k z' - i\epsilon)^3}$$

BSE in Minkowski space with NIR

Transverse Momentum Distribution

Ydrefors, WP, Nogueira, Frederico and Salmè **Preliminary**



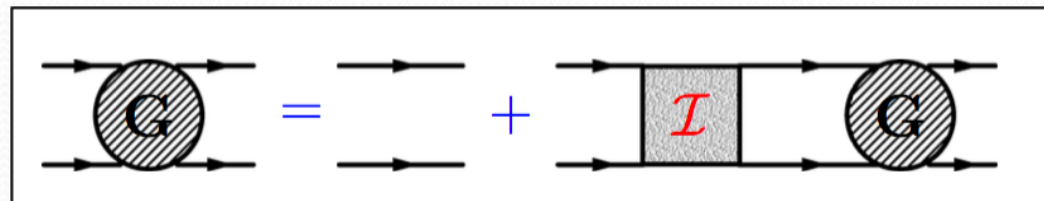
Pion Bound State

We start from the four-point Green function

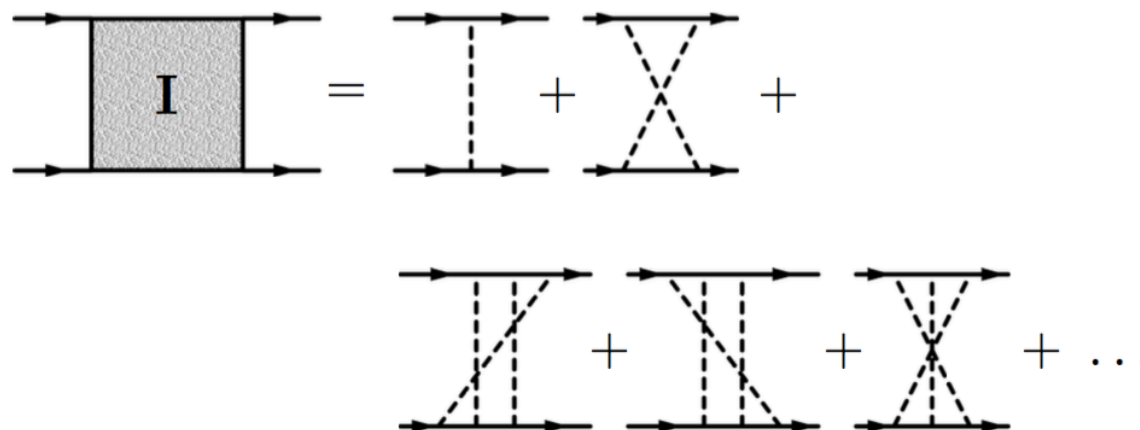
$$G(x_1, x_2; y_1, y_2) = \langle 0 | T \{ \phi_1(x_1) \phi_2(x_2) \phi_1^+(y_1) \phi_2^+(y_2) \} | 0 \rangle$$

which is a solution of the integral equation

$$G = G_0 + G_0 \mathcal{I} G$$



$\mathcal{I} \equiv$ kernel given by the infinite sum of irreducible Feynmann graphs



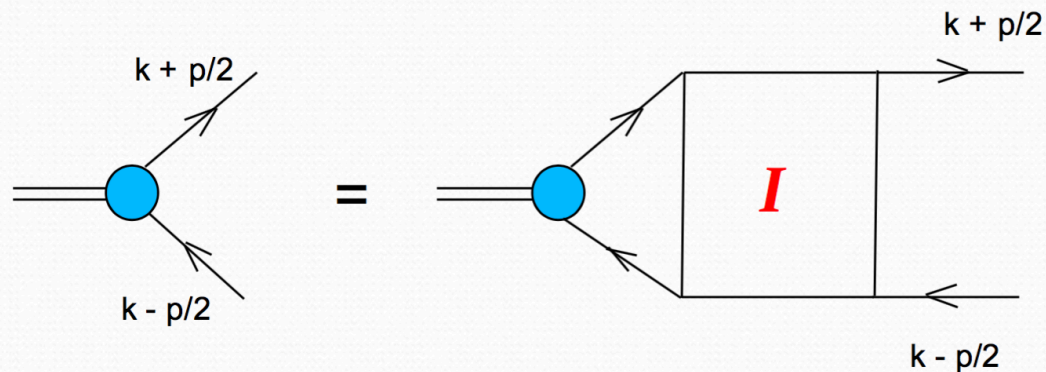
Iterations produce all the expected contributions

Bethe-Salpeter Equation

Close to the bound-state pole we obtain the BSE

$$\phi(k; p_B) = G_0(k; p_B) \int d^4k' \mathcal{I}(k, k'; p_B) \phi(k'; p_B)$$

BSA in configuration space: $\phi(x_1, x_2; p_B) = \langle 0 | T \{ \phi_1(x_1) \phi_2(x_2) \} | p_B \rangle$



The two-body irreducible Kernel of the four-point Green function

Challenge: To solve the BSE in Minkowski space

NIR for two-fermions

WP, Frederico, Salmè, Viviani, **PRD94** (2016) 071901

We can single out the singular contributions

For two-fermion BSE

$$C_j = \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} (k^-)^j S(k^-, v, z, z', \gamma, \gamma')$$

with $j=1,2,3$ and in the worst case

$$S(k^-, v, z, z', \gamma, \gamma') \sim \frac{1}{[k^-]^2} \quad \text{for } k^- \rightarrow \infty$$

Then one can not close the arc at the infinity .

The severity of the singularities (power j), does not depend on the Kernel

We calculate the singular contribution using

$$\int_{-\infty}^{\infty} dx \frac{1}{[\beta x - y \mp i\epsilon]^2} = \pm (2\pi)i \frac{\delta(\beta)}{[-y \mp i\epsilon]} \quad \text{Yan PRD 7 (1973) 1780}$$

Numerical Method

Basis expansion for the Nakanishi weight function

$$g_i(\gamma, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} w_{mn}^i G_{2m+r_i}^{\lambda_i}(z) \mathcal{J}_n(\gamma)$$

Gegenbauer polynomials

$$G_n^{\lambda}(z) = (1 - z^2)^q \Gamma(\lambda) \sqrt{\frac{n!(n + \lambda)}{2^{1-2\lambda} \pi \Gamma(n + 2\lambda)}} C_n^{\lambda}(z)$$

Laguerre polynomials

$$\mathcal{J}_n(\gamma) = \sqrt{a} L_n(a\gamma) e^{-a\gamma/2}$$

We obtain a discrete generalized eigenvalue problem

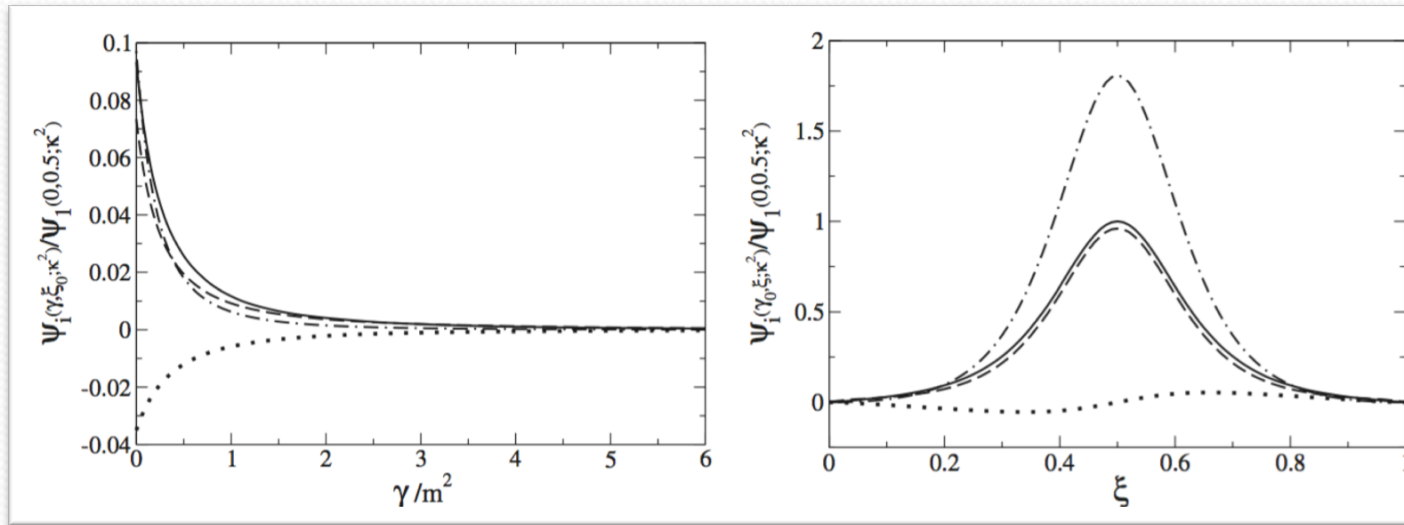
$$C \mathbf{w} = g^2 D \mathbf{w}$$

We used ~ 44 Laguerre polynomials and 44 Gegenbauer

Vector Exchange: LF amplitudes

WdP, Frederico, Salme, Viviani and Pimentel – EPJC 77 (2017) 764

Weak Binding



Strong Binding

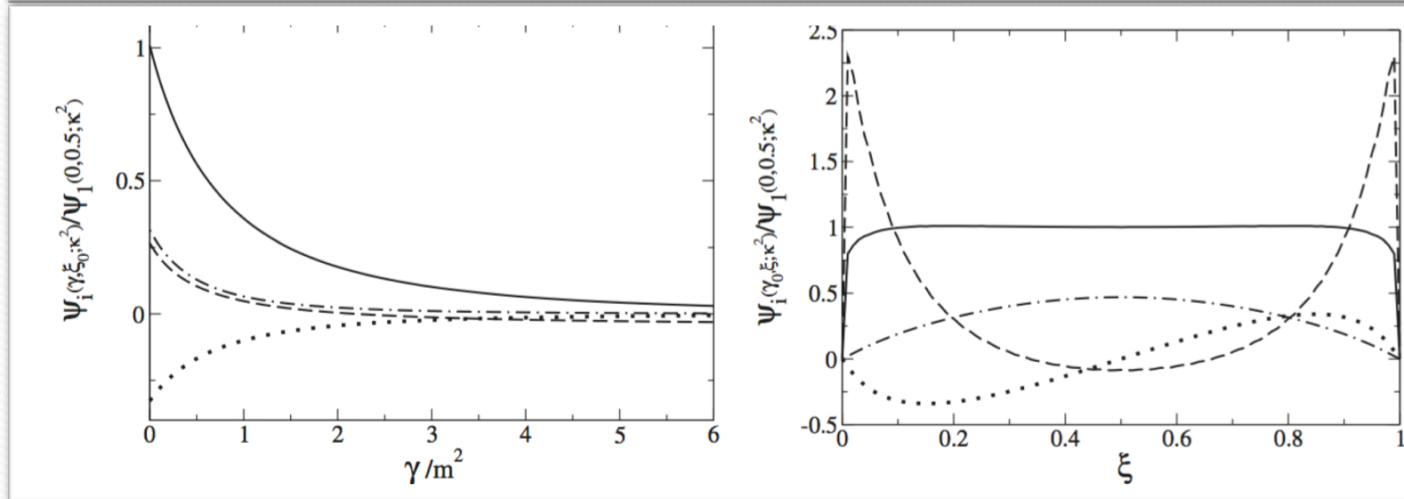


Fig. 3 LF amplitudes for weak ($B/m = 0.1$) and strong binding ($B/m = 1.0$) with mass $\mu/m = 0.15$. Solid line: ψ_1 . Dashed line: ψ_2 . Dotted line: ψ_3 . Dot-Dashed line: ψ_4 .

$$z = -2k^+/M$$

$$0 < \xi = (1 - z)/2 < 1$$

Normalization

In order to calculate hadronic properties, we need to properly normalize the BSA

$$Tr \left[\int \frac{d^4 k}{(2\pi)^4} \bar{\Phi}(k, p) \frac{\partial}{\partial p'^\mu} \{ S^{-1}(k + p'/2) \Phi(k, p) S^{-1}(k - p'/2) \} \Big|_{p'=p; p^2=M^2} \right] = -i 2p_\mu$$

Using the BSA expansion and performing the Dirac traces, we have

$$i \int \frac{d^4 k}{(2\pi)^4} \left[\phi_1 \phi_1 + \phi_2 \phi_2 + b \phi_3 \phi_3 + b \phi_4 \phi_4 - 4 b \phi_1 \phi_4 - 4 \frac{m}{M} \phi_2 \phi_1 \right] = 1$$

From the NIR, we obtain

$$\begin{aligned} & \frac{3}{32\pi^2} \int_{-1}^{+1} dz' \int_0^\infty d\gamma' \int_{-1}^{+1} dz \int_0^\infty d\gamma \int_0^1 dv \frac{v^2(1-v)^2}{\left[\kappa^2 + \frac{M^2}{4} \lambda^2 + \gamma' v + \gamma(1-v) \right]^4} \\ & \times \left\{ g_1(\gamma', z') g_1(\gamma, z) + g_2(\gamma', z') g_2(\gamma, z) - 4 \frac{m}{M} g_2(\gamma', z') g_1(\gamma, z) \right. \\ & + \frac{\left[\kappa^2 + \frac{M^2}{4} \lambda^2 + \gamma' v + \gamma(1-v) \right]}{2M^2} \\ & \times \left. [g_3(\gamma', z') g_3(\gamma, z) + g_4(\gamma', z') g_4(\gamma, z) - 4g_1(\gamma', z') g_4(\gamma, z)] \right\} = -1 \end{aligned}$$

Dressing the Quark

- Dressed quark propagators defined for time and space-like momentum.
- Dynamical Chiral Symmetry Breaking

Solution of the Schwinger-Dyson Eq. in Minkowski-Space.

Also discussed in

Sauli, Nucl. Phys. 689A, 467 (2001), JHEP 0302, 001 (2003)

Bicudo, Phys. Rev. D 69, 074003 (2004).

Mezrag & Salmè, EPJC 81, 34 (2021).

The model:

Rainbow-Ladder, Pauli Villars regularization,
massive effective gluon.

Schwinger-Dyson equation in Rainbow ladder truncation

In collaboration with Duarte, Frederico, Ydrefors

Bare vertices, massive vector boson, Pauli-Villars regulator

The rainbow ladder Schwinger-Dyson equation in **Minkowski** space is

$$S_f^{-1}(k) = \not{k} - \overline{m}_0 + ig^2 \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu S_f(k-q) \gamma_\nu D^{\mu\nu}(q)$$

The massive gauge boson is given by

$$D^{\mu\nu}(q) = \frac{1}{q^2 - m_\sigma^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1-\xi)q^\mu q^\nu}{q^2 - \xi m_\sigma^2 + i\epsilon} \right]$$

$\xi = 0$ (Landau Gauge) & $\xi = 1$ (Feynman Gauge)

The dressed fermion propagator is

$$S_f(k) = \frac{1}{k - m_B + kA_f(k^2) - B_f(k^2) + i\epsilon}$$

Fermion Schwinger-Dyson equation (Rainbow ladder)

Self-Energies Integral representations

$$A_f(k^2) = \int_0^\infty d\gamma \frac{\rho_A(\gamma)}{k^2 - \gamma + i\epsilon}$$

$$B_f(k^2) = \int_0^\infty d\gamma \frac{\rho_B(\gamma)}{k^2 - \gamma + i\epsilon}$$

Vector and scalar Self-Energy densities

Fermion propagator – Integral representation

$$S_f = R \frac{\not{k} + \bar{m}_0}{k^2 - \bar{m}_0^2 + i\epsilon} + \not{k} \int_0^\infty d\gamma \frac{\rho_v(\gamma)}{k^2 - \gamma + i\epsilon} + \int_0^\infty d\gamma \frac{\rho_s(\gamma)}{k^2 - \gamma + i\epsilon}$$

physical mass

Vector and scalar spectral densities

$$\begin{aligned} \not{k}A(k^2) - B(k^2) = & ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu S(k-q)\gamma_\nu}{q^2 - m_\sigma^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1 - \xi) q^\mu q^\nu}{q^2 - \xi m_\sigma^2 + i\epsilon} \right] \\ & - ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu S(k-q)\gamma_\nu}{q^2 - \Lambda^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1 - \xi) q^\mu q^\nu}{q^2 - \xi \Lambda^2 + i\epsilon} \right] \end{aligned}$$

Gauge fixing

Pauli-Villars
regulator

Fermion Schwinger-Dyson equation (Rainbow ladder)

- Parameters: $\alpha = \frac{g^2}{4\pi}$, Λ , m_σ , \overline{m}_0 .
- Self energy densities: $\rho_A(\gamma) = -\text{Im}[A(\gamma)]/\pi$ and $\rho_B(\gamma) = -\text{Im}[B(\gamma)]/\pi$.
- Solutions of DSE obtained writing the trivial relation $S_f^{-1}S_f = 1$ in a suitable form:

$$\frac{R}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty d\gamma \frac{\rho_v(\gamma)}{k^2 - \gamma + i\epsilon} = \frac{1 + A_f(k^2)}{k^2(1 + A_f(k^2))^2 - (m_B + B_f(k^2))^2 + i\epsilon}$$
$$\frac{R\overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty d\gamma \frac{\rho_s(\gamma)}{k^2 - \gamma + i\epsilon} = \frac{m_B + B_f(k^2)}{k^2(1 + A_f(k^2))^2 - (m_B + B_f(k^2))^2 + i\epsilon}$$

Fermion Schwinger-Dyson equation (Rainbow ladder)

Fermion DSE solution

$$\begin{aligned}\rho_A(p^2) &= R \left[\mathcal{K}_A^{\xi=1}(p^2, \bar{m}_0^2; m_\sigma^2) + \frac{1}{m_\sigma^2} \mathcal{K}_A^\xi(p^2, \bar{m}_0^2; m_\sigma^2) \right] \\ &+ \int_{s_\xi^{\text{thres}}}^{\infty} ds \rho_v(s) \left[\mathcal{K}_A^{\xi=1}(p^2, s; m_\sigma^2) + \frac{1}{m_\sigma^2} \mathcal{K}_A^\xi(p^2, s; m_\sigma^2) \right] \\ &- [m_\sigma \rightarrow \Lambda]\end{aligned}$$

$$\begin{aligned}\rho_B(p^2) &= R\bar{m}_0 \left[\mathcal{K}_B^{\xi=1}(p^2, \bar{m}_0^2; m_\sigma^2) + \frac{1}{m_\sigma^2} \mathcal{K}_B^\xi(p^2, \bar{m}_0^2; m_\sigma^2) \right] \\ &+ \int_{s_\xi^{\text{thres}}}^{\infty} ds \rho_v(s) \left[\mathcal{K}_B^{\xi=1}(p^2, s; m_\sigma^2) + \frac{1}{m_\sigma^2} \mathcal{K}_B^\xi(p^2, s; m_\sigma^2) \right] \\ &- [m_\sigma \rightarrow \Lambda]\end{aligned}$$

$$s_\xi^{\text{thres}} = (\bar{m}_0 + \xi m_\sigma)^2$$

$$f_A(p^2) = 1 + P \int_{s^{\text{thres}}}^{\infty} ds \frac{\rho_A(s)}{p^2 - s}$$

$$f_B(p^2) = m_B + P \int_{s^{\text{thres}}}^{\infty} ds \frac{\rho_B(s)}{p^2 - s}$$

$$\begin{aligned}d(p^2) &= \left[p^2 f_A^2(p^2) - \pi^2 p^2 \rho_A^2(p^2) - f_B^2(p^2) + \pi^2 \rho_B^2(p^2) \right]^2 \\ &+ 4\pi^2 \left[p^2 \rho_A(p^2) f_A(p^2) - \rho_B(p^2) f_B(p^2) \right]^2\end{aligned}$$

Connection Formulas

$$\begin{aligned}\rho_v(p^2) &= -2 \frac{f_A(p^2)}{d(p^2)} [p^2 \rho_A(p^2) f_A(p^2) - \rho_B(p^2) f_B(p^2)] \\ &+ \frac{\rho_A(p^2)}{d(p^2)} [p^2 f_A^2(p^2) - \pi^2 p^2 \rho_A^2(p^2) - f_B^2(p^2) + \pi^2 \rho_B^2(p^2)] \\ \rho_s(p^2) &= -2 \frac{f_B(p^2)}{d(p^2)} [p^2 \rho_A(p^2) f_A(p^2) - \rho_B(p^2) f_B(p^2)] \\ &+ \frac{\rho_B(p^2)}{d(p^2)} [p^2 f_A^2(p^2) - \pi^2 p^2 \rho_A^2(p^2) - f_B^2(p^2) + \pi^2 \rho_B^2(p^2)]\end{aligned}$$

Comparison with Un-Wick rotated results

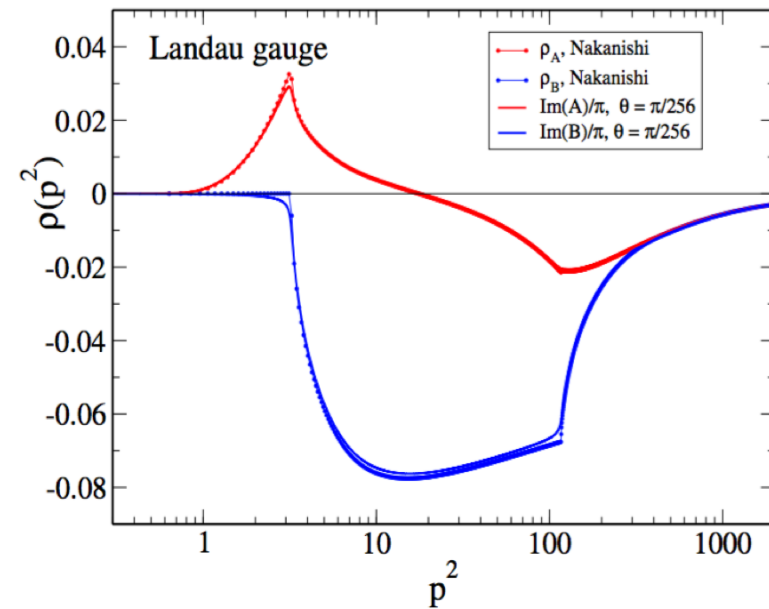
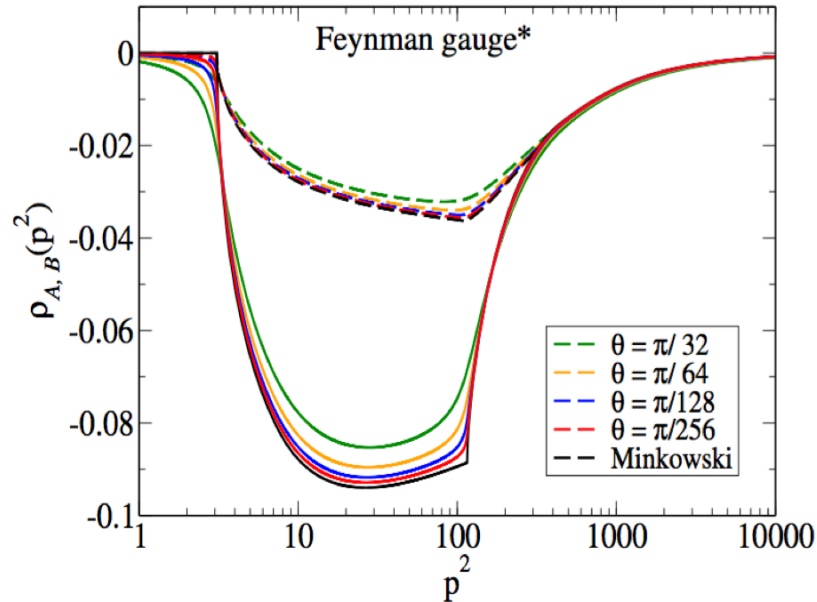
In collaboration with Duarte, Frederico, Ydrefors, Maris and Jia

- From Euclidean space formulation, in increments of δ : $p \rightarrow e^{-i\delta} p$

The integration path in SDE is deformed into the complex plane

- Minkowski space: $\delta = \pi/2$, or in a more convenient notation $\Theta = \pi/2 - \delta$.

$$\theta = 0, \begin{cases} p_0^2 = 0, & \vec{p}^2 > 0 & \text{spacelike region} \\ p_0^2 > 0, & \vec{p}^2 = 0 & \text{timelike region} \end{cases}$$



*S. Jia et al., Proceedings of HADRON-2019, arXiv:1912.00063, T. Frederico et al., Proceedings of NTSE-2018, arXiv:1905.00703.

$$m_0 = 0.5, \mu = 1.0, \Lambda = 10.0, \text{ and } \alpha = 0.5$$

Dynamical Chiral Symmetry Breaking

Strong coupling regime

Fit to Lattice Landau Gauge Results

Oliveira, Silva, Skullerud & Sternbeck, PRD 99 (2019) 094506

Quark propagator

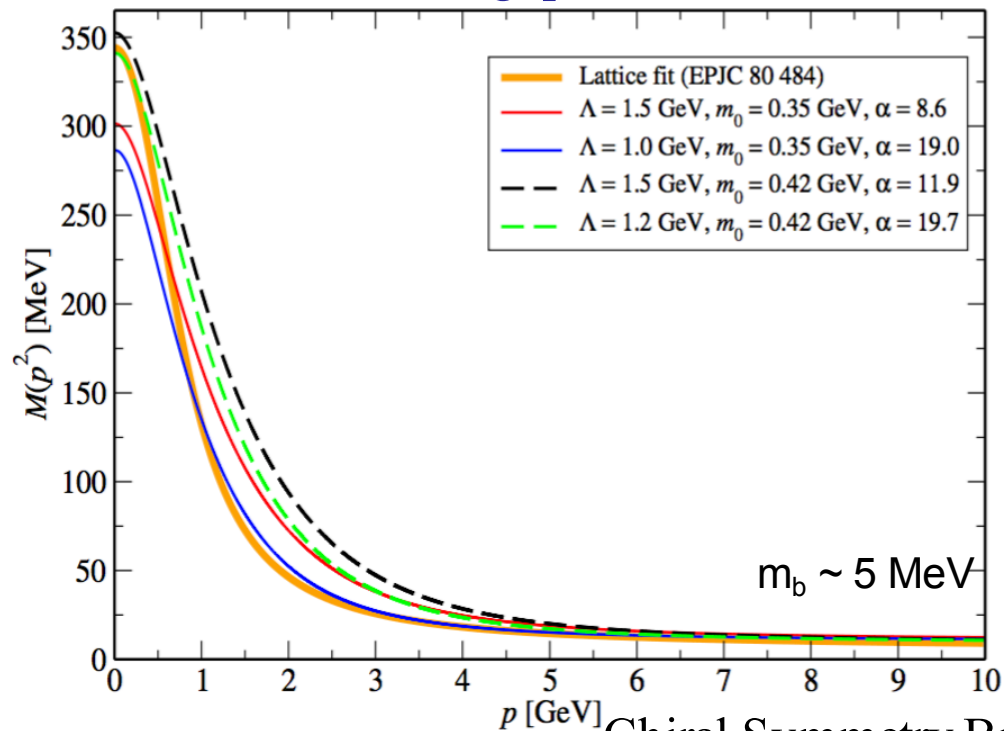
Phenomenological Model

In collaboration with Duarte, Frederico, Ydrefors

We can calibrate the model to reproduce Lattice Data for $M(p^2)$

Running quark mass

$$M^2(p^2) = \frac{B^2(p^2)}{A^2(p^2)}$$
$$Z(p^2) = \frac{1}{A(p^2)}$$



Chiral Symmetry Breaking

The next step is to use this solution to obtain the

Fermion-Antifermion bound state

Uniqueness of the Nakanish Representation

Nakanishi proposed that the weight function is unique. It means that if both LHS and RHS have the same integral operator, they can be extracted

$$\int_0^\infty d\gamma' \frac{g_1(\gamma', z')}{(\gamma + \gamma' + z^2 m^2 + (1 - z^2) \kappa^2)^2} = \int_0^\infty d\gamma' \frac{g_2(\gamma', z')}{(\gamma + \gamma' + z^2 m^2 + (1 - z^2) \kappa^2)^2}$$
$$\Rightarrow g_1(\gamma', z') = g_2(\gamma', z')$$

Stieltjes Transformation

$$G(x) = \int_0^\infty \frac{F(y)}{(y + x)^2} dy$$

We can relate the kernel with a integral in the complex plane

$$F(y) = \frac{y}{2\pi} \int_{-\pi}^{+\pi} d\phi e^{i\phi} G(y e^{i\phi})$$

Extreme Binding Energy (B=2m)

Example: Fermion-Antifermion Bound State with massless vector exchange

The BSE in the limit of Extreme Binding Energy (M=0) is:

$$\int_0^\infty d\gamma' \frac{g_1(\gamma', z)}{(\gamma' + \gamma + m^2)^2} = \frac{(\mu^2 - \Lambda^2)^2}{8\pi^2} g^2 \int_0^\infty d\gamma' \int_{-1}^{+1} dz' \int_0^1 dv v^2 (1-v)^2 g_1(\gamma', z') \times$$
$$\times \frac{\theta(k_D^+)}{(1+z)} \frac{[3D_2(\gamma, z; \gamma', z', v) + (1-v)(\mu^2 - \Lambda^2)]}{[D_2(\gamma, z; \gamma', z', v) + (1-v)(\mu^2 - \Lambda^2)]^3 [D_2(\gamma, z; \gamma', z', v)]^2} + [z \rightarrow -z; z' \rightarrow -z'],$$

with

$$D_2(\gamma, z; \gamma', z', v) =$$
$$-\frac{v}{(1+z)} m^2 + m^2 v z + m^2 z' (1-v) + (1-v)(1+z')\gamma + (1+z)\gamma' + (1-v)(1+z)\mu^2$$

Extreme Binding Energy ($B = 2m$)

Using Feynman parametrization, Dirac delta properties and Uniqueness we have

$$g_1(\gamma'', z) = 3 \frac{(\mu^2 - \Lambda^2)^2}{2\pi^2} g^2 \int_0^\infty d\gamma' \int_{-1}^{+1} dz' \int_0^1 dv v^2 (1-v)^2 g_1(\gamma', z') \frac{1}{(1+z)} \times \\ \left[\theta(k_D^+) \int_0^1 d\xi \xi^2 (1-\xi) \left(\frac{4}{6} \delta'(\gamma'' - \alpha_3 + m^2) + \frac{\alpha_5}{24} \delta'''(\gamma'' - \alpha_3 + m^2) \right) + [z \rightarrow -z, z' \rightarrow -z'] \right].$$

Solving numerically we obtain $g^2=68$ (fundamental state) , which is consistent with the solution of the BSE for B close to $2m$

$$g(\gamma, z) = \int_{-1}^{+1} dz' \int_0^\infty d\gamma' W(\gamma, z, \gamma', z', v) g(\gamma', z')$$

Definition of $W(\gamma, z; \gamma', z'; v)$

Stieltjes Transformation

We can compare with the Uniqueness method

The BSE is written as

$$\int_0^\infty \frac{g(\gamma', z) d\gamma'}{[\gamma' + \gamma + z^2 m^2 + (1 - z^2) \kappa^2]^2} = \int_{-1}^{+1} dz' \int_0^\infty d\gamma' V(\gamma, z; \gamma', z', v) g(\gamma', z')$$

Stieltjes Transformation:

$$g(\gamma, z) = \int_{-1}^{+1} dz' \int_0^\infty d\gamma' N(\gamma, z; \gamma', z', v) g(\gamma', z')$$

$$N(\gamma, z; \gamma', z', v) = \frac{\gamma}{2\pi} \int_{-\pi}^{+\pi} d\phi e^{i\phi} V(\gamma e^{i\phi} - m^2 z^2 - (1 - z^2) \kappa^2, z; \gamma', z')$$

Uniqueness

$$g(\gamma, z) = \int_{-1}^{+1} dz' \int_0^\infty d\gamma' W(\gamma, z, \gamma', z', v) g(\gamma', z')$$

Stieltjes Transformation

Stieltjes Transformation:

$$g(\gamma, z) = \int_{-1}^{+1} dz' \int_0^{\infty} d\gamma' N(\gamma, z; \gamma', z', v) g(\gamma', z')$$

$$N(\gamma, z; \gamma', z', v) = \frac{\gamma}{2\pi} \int_{-\pi}^{+\pi} d\phi e^{i\phi} V(\gamma e^{i\phi} - m^2 z^2 - (1 - z^2)\kappa^2, z; \gamma', z')$$

Uniqueness

$$g(\gamma, z) = \int_{-1}^{+1} dz' \int_0^{\infty} d\gamma' W(\gamma, z, \gamma', z', v) g(\gamma', z')$$

For the following values

$$g^2 = 68, \gamma = 6, \gamma' = 0.4, z = 0.6, z' = 0.7, m = 1$$

The Kernels are

$$N(\gamma, z; \gamma', z'; v) = 0.361861$$

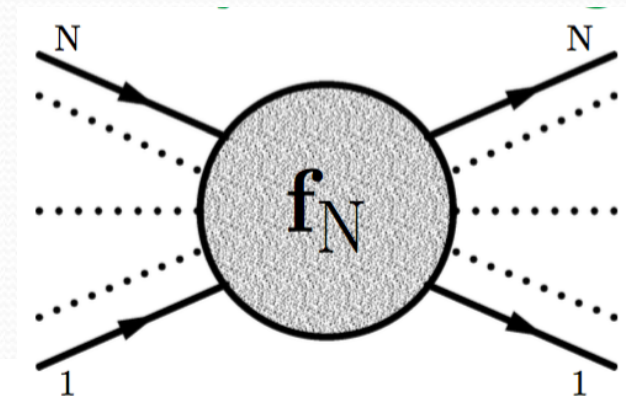
$$W(\gamma, z; \gamma', z'; v) = 0.361861$$

Nakanishi Integral Representation

Let's take a connected Feynman diagram (G) with N external momenta p_i , n internal propagators with momenta l_j and masses m_j and k loops.

The transition amplitude is given by (scalar theory)

$$f_G(p_i) = \prod_{r=1}^k \int d^4 q_r \frac{1}{(l_1^2 - m_1^2 + i\epsilon) \cdots (l_n^2 - m_n^2 + i\epsilon)}$$



Feynman parametrization $\frac{1}{A_1 \dots A_n} = (n-1)! \prod_{i=1}^n \int_0^1 d\alpha_i \frac{\delta(1 - \sum \alpha_i)}{\sum_{i=1}^n \alpha_i A_i}$

$$l_j = \sum_{r=1}^k b_{jr} q_r + \sum_{i=1}^N c_{ji} p_i$$

We obtain

$$f_G(p_i) = \frac{(i\pi)^k (n-2k-1)!}{(n-1)!} \prod_{i=1}^n \int_0^1 d\alpha_i \frac{\delta(\sum \alpha_i - 1)}{U^2 (\sum_{ii'} e_{ii'} p_i p_{i'} - \sum_{i=1}^n \alpha_i m_j^2 + i\epsilon)^{n-2k}}$$

The denominator is a linear combination of the scalar product of the external momenta and the masses.

The coefficients and the exponent $(n-2k)$ depends on the particular Feynman diagram.

Nakanishi Integral Representation

After some change of variables we can write

$$f_G(p_i) = \prod_h \int_0^1 dz_h \int_0^\infty d\chi \frac{\delta(1 - \sum_i z_i) \phi_G^{(n-2k)}(z, \chi)}{(\sum_i z_i s_i - \chi + i\epsilon)^{n-2k}}$$

Performing integration by parts, we have the integral representation

$$f_G(p_i) = \prod_h \int_0^1 dz_h \int_0^\infty d\chi \frac{\delta(1 - \sum_i z_i) \phi_G^{(1)}(z, \chi)}{(\sum_i z_i s_i - \chi + i\epsilon)}$$

where

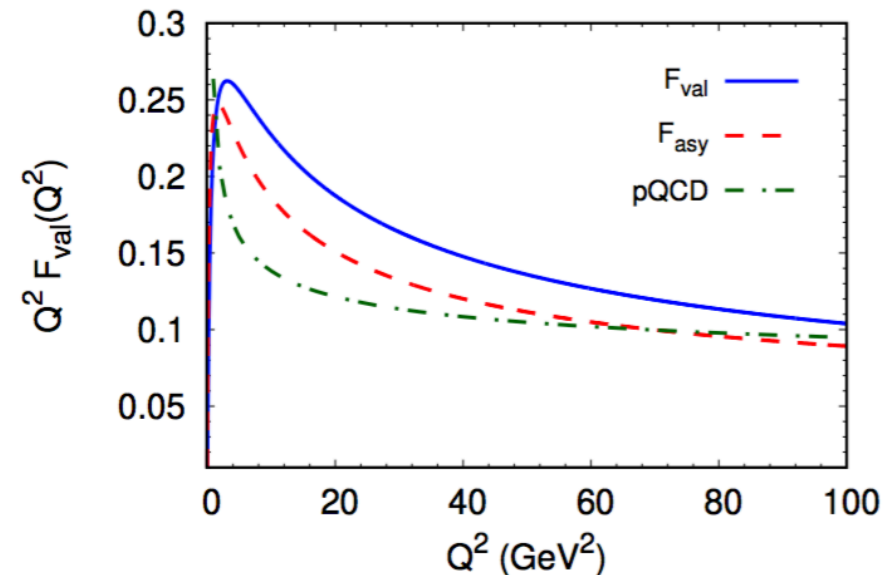
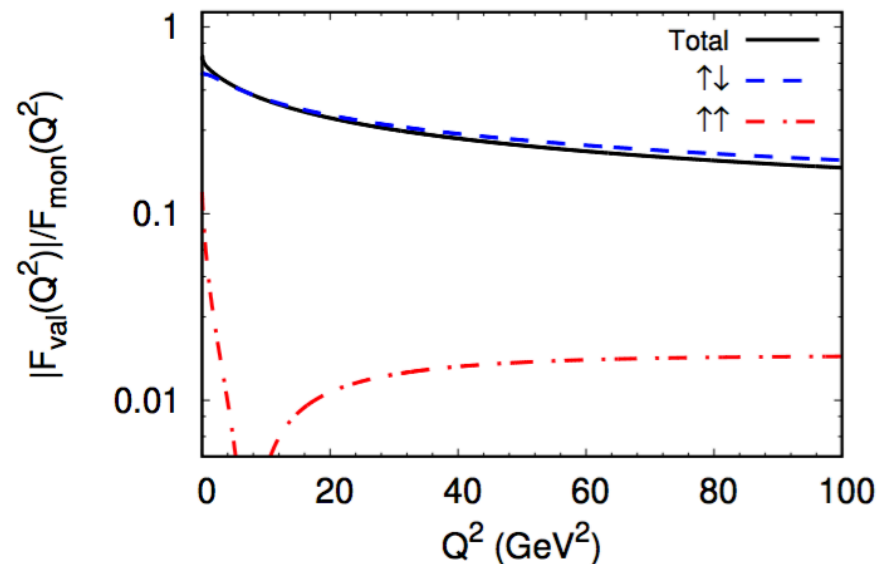
$$\phi_G^{(1)}(\chi, z_h) = (-1)^{n-2k-1} \frac{\partial^{n-2k-1}}{\partial \chi^{n-2k-1}} \phi_G^{(n-2k)}(\chi, z_h)$$

The dependence upon the details of the diagram moves from the **denominator** to the **numerator**. We obtain the **same** formal expression for the denominator of **any** diagram.

Spin configurations contributions

Ydrefors, WP, Nogueira, Frederico and Salmè **PLB** 820, 136494 (2021).

Within the BSE approach we can calculate the contribution to the valence FF from the 2 different spin configurations present in the pion.



For zero momentum transfer, the pure relativistic Spin-aligned configuration contributes with 20%.

Zero in spin-aligned FF is due to relativistic spin-orbit coupling that produces the term $\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}'$, which flips the sign around $Q^2 \sim 8 \text{ GeV}^2$

For large Q^2 , the difference between the exact formula, the asymptotic expression and pQCD becomes small.

Kernels

$$\mathcal{K}_A^{\xi=1}(a, m_\sigma, \gamma) = \frac{2g^2}{(4\pi)^2} \frac{\Theta[\gamma - (a + m_\sigma)^2]}{\gamma} \sqrt{m_\sigma^4 - 2m_\sigma^2(\gamma + a^2) + (\gamma - a^2)^2}$$

$$- \frac{g^2}{(4\pi)^2} (1 + \xi) \int_0^1 d\alpha \alpha \Theta[\alpha_1(1 - \alpha_1)\gamma - \alpha_1 a^2 - m_\sigma^2(1 - \alpha_1)]$$

$$\mathcal{K}_A^\xi(a, m_\sigma, \gamma) = \frac{g^2}{(4\pi)^2 m_\sigma^2} \int_0^1 d\alpha_1 [3\gamma\alpha_1^2 + \alpha_1(a^2 - \xi m_\sigma^2 - \gamma) - \xi m_\sigma^2]$$

$$\times \Theta[\alpha_1(1 - \alpha_1)\gamma - \alpha_1 a^2 - \xi m_\sigma^2(1 - \alpha_1)] \Theta[m_\sigma^2(1 - \alpha_1) + \alpha_1 a^2 - \alpha_1(1 - \alpha_1)\gamma]$$

$$\mathcal{K}_B^{\xi=1}(a, m_\sigma, \gamma) = \frac{(3 + \xi)g^2}{(4\pi)^2} \frac{\Theta[\gamma - (a + m_\sigma)^2]}{\gamma} \sqrt{a^4 - 2a^2(\gamma + m_\sigma^2) + (\gamma - m_\sigma^2)^2}$$

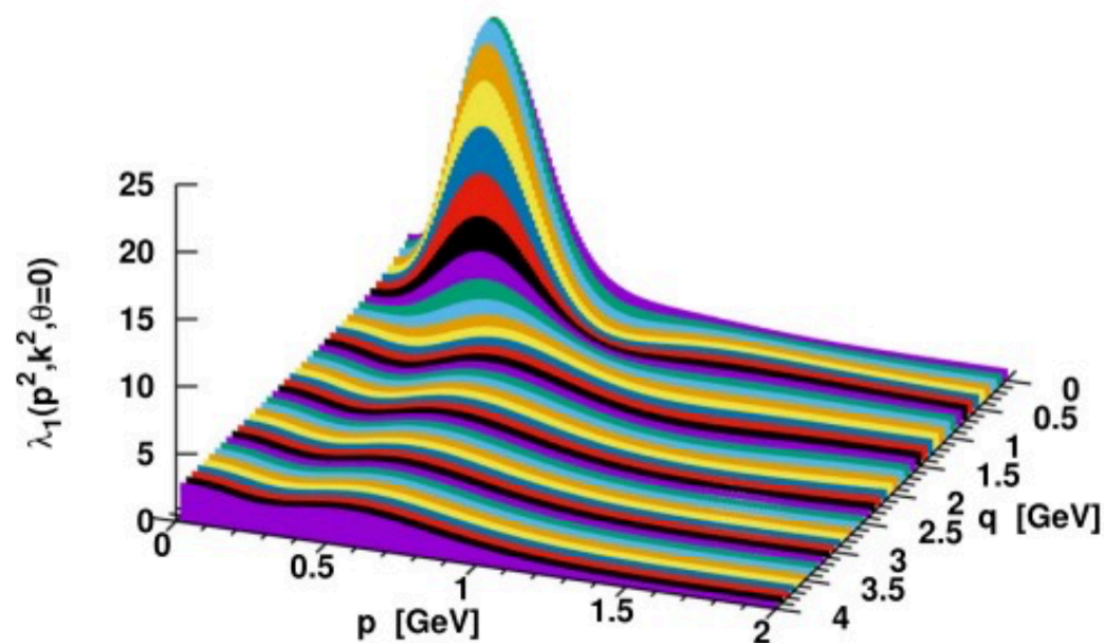
$$\mathcal{K}_B^\xi(a, m_\sigma, \gamma) = \frac{g^2 \xi}{(4\pi)^2} \int_0^1 d\alpha_1 [m_\sigma^2(1 - \alpha_1) - \gamma(1 - \alpha_1)\alpha_1 + \alpha_1 a^2]$$

$$\times \Theta[\gamma(1 - \alpha_1)\alpha_1 - \alpha_1(a^2 - \xi m_\sigma^2) - \xi m_\sigma^2]$$

$$R^{-1} = 1 + \int_{\gamma^{\text{thres}}}^{\infty} d\gamma \frac{\rho_A(\gamma)}{\bar{m}_0^2 - \gamma} - 2\bar{m}_0^2 P \int_{\gamma^{\text{thres}}}^{\infty} d\gamma' \frac{\rho_A(\gamma')}{(\bar{m}_0^2 - \gamma')^2} + 2\bar{m}_0 P \int_{\gamma^{\text{thres}}}^{\infty} d\gamma' \frac{\rho_B(\gamma')}{(\bar{m}_0^2 - \gamma')^2} \quad \gamma^{\text{thres}} = (\bar{m}_0 + \xi m_\sigma)^2$$

Common parameters: $\Lambda = 10$, $m_\sigma = 1$, $\alpha = 0.5$.

| ξ | R | m_o | m_B |
|-------------|-------|-------|-------|
| 1 (Feynman) | 0.884 | 0.759 | 0.5 |
| 0 (Landau) | 1.05 | 0.797 | 0.5 |



$$\alpha \sim 0.22$$

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<https://doi.org/10.1140/epjc/s10052-020-8037-0>

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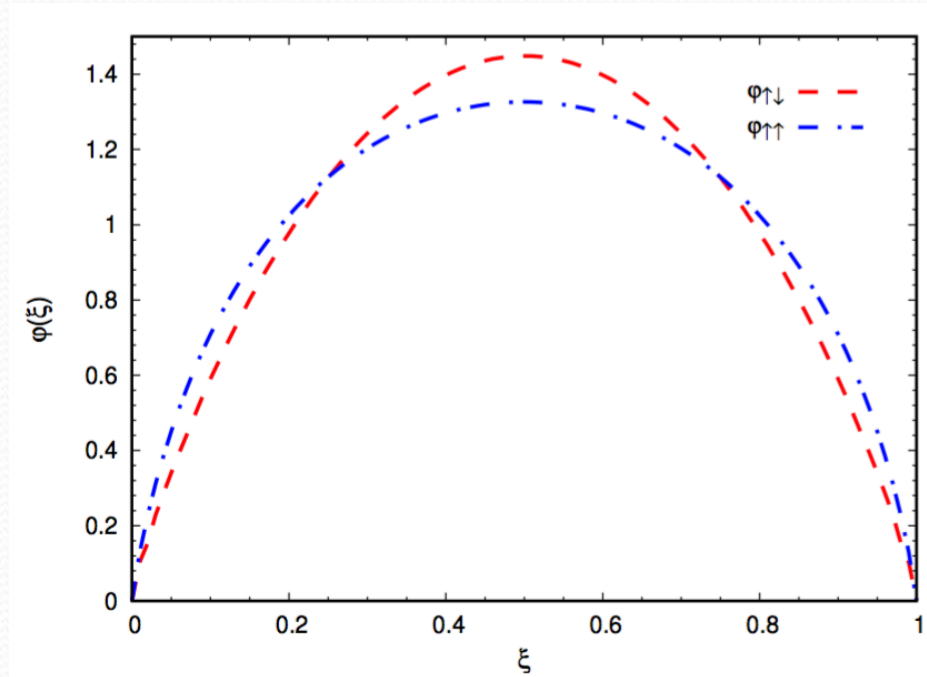
The soft-gluon limit and the infrared enhancement of the quark-gluon vertex

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Pion Distribution Amplitude



The spin components of the DA, defined by

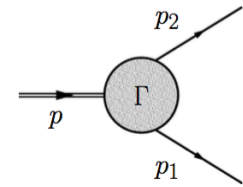
$$\phi_{\uparrow\downarrow(\uparrow\uparrow)}(\xi) = \frac{\int_0^\infty d\gamma \psi_{\uparrow\downarrow(\uparrow\uparrow)}(\gamma, z)}{\int_0^1 d\xi \int_0^\infty d\gamma \psi_{\uparrow\downarrow(\uparrow\uparrow)}(\gamma, z)}$$

Aligned component (blue) more wide than the anti-aligned one (red).

Nakanishi Integral Representation

To represent the BSA, we consider the constituent particles with momentum p_1 , p_2 and the bound-state with momentum p .

$$p = p_1 + p_2 \quad k = (p_1 - p_2)/2$$



$$f_3(p_i) = \prod_h \int_0^1 dz_h \delta(\sum_h z_h - 1) \int_{0^-}^{\infty} d\chi \frac{\phi_3^{(1)}(\chi, z_h)/(z_1 + z_2)}{(k^2 + p \cdot k \frac{(z_1 - z_2)}{(z_1 + z_2)} + \frac{M^2}{4} \frac{(z_1 + z_2 + 4z_3) - \chi}{(z_1 + z_2)} + i\epsilon)}$$

Using the identities

$$1 = \int d\gamma' \delta(\gamma' + \left(\frac{\frac{M^2}{4} (z_1 + z_2 + 4z_3) - \chi}{(z_1 + z_2)} \right))$$

$$1 = \int_{-1}^1 dz' \delta(z' - \left(\frac{z_1 - z_2}{z_1 + z_2} \right))$$

we obtain the NIR

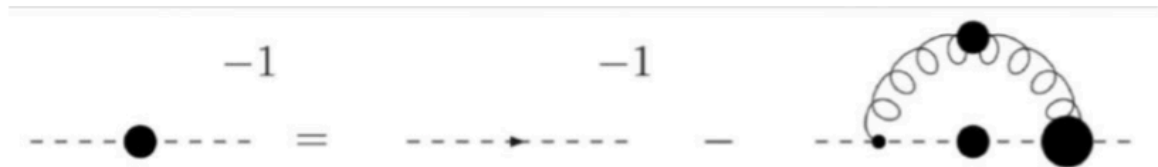
$$f_3(p, k) = \int d\gamma' \int_{-1}^1 dz' \frac{g^{(1)}(\gamma', z')}{k^2 + z' p \cdot k - \gamma' + i\epsilon}$$

where

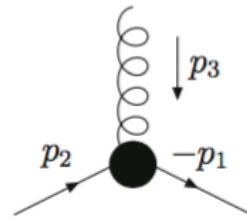
$$g^{(1)}(\gamma', z') = \prod_h \int_0^1 dz_h \delta(\sum_h z_h - 1) \int_{0^-}^{\infty} d\chi \\ \times \frac{\phi_3^{(1)}(\chi, z_h)}{(z_1 + z_2)} \delta(z' - \left(\frac{z_1 - z_2}{z_1 + z_2} \right)) \delta(\gamma' + \left(\frac{\frac{M^2}{4} (z_1 + z_2 + 4z_3) - \chi}{(z_1 + z_2)} \right))$$

Quark-Gluon Vertex

Schwinger-Dyson eq. Quark propagator



Quark-gluon vertex



$$\Gamma_{\mu}^a(p_1, p_2, p_3) = g t^a \Gamma_{\mu}(p_1, p_2, p_3)$$

$$\Gamma_\mu(p_1, p_2, p_3) = \Gamma_\mu^{(L)}(p_1, p_2, p_3) + \Gamma_\mu^{(T)}(p_1, p_2, p_3)$$

Longitudinal component

$$\Gamma_{\mu}^{\text{L}}(p_1, p_2, p_3) = -i \left(\lambda_1 \gamma_{\mu} + \lambda_2 (\not{p}_1 - \not{p}_2) (p_1 - p_2)_{\mu} \right. \\ \left. + \lambda_3 (p_1 - p_2)_{\mu} + \lambda_4 \sigma_{\mu\nu} (p_1 - p_2)^{\nu} \right)$$

quark-gluon vertex from factors

- Slanov-Taylor identity & Quark-Ghost Kernel
- Padé approximants
- Error minimization $\sim 2-4\%$
- simulating annealing

$\alpha_s = 0.22$ and all propagators renormalised at $\mu = 4.3 \text{ GeV}$

