# Structure functions from the renormalization group improved small $x$ equation 

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## Outline

- Motivation: why small x resummation ?
- CCSS resummation framework
- Setup for the structure function calculations
- Results for the $\mathrm{F}_{2}$ and $\mathrm{F}_{2 c}$
- Comparison with other models for unintegrated gluon density
- Outlook: impact factors and quarks

Work done in collaboration with Wanchen Li e-Print: 2201.10579

## Motivation for resummation

BFKL equation

$$
\begin{aligned}
& \frac{d f_{g}\left(x, k_{T}^{2}\right)}{d \ln 1 / x}=\frac{\alpha_{s} N_{c}}{\pi} \int d^{2} k_{T}^{\prime} \mathcal{K}\left(k_{T}, k_{T}^{\prime}\right) f_{g}\left(x, k_{T}^{\prime}\right) \\
& \alpha_{s} \mathcal{K}_{0}+\alpha_{s}^{2} \mathcal{K}_{1}+\ldots \\
& \omega_{P} \simeq \bar{\alpha}_{s} 4 \ln 2\left(1-6.5 \bar{\alpha}_{s}\right) \\
& \text { NLLx large as compared with LLx }
\end{aligned}
$$

## Why?

- Strong coupling is not a naturally small parameter in the Regge limit. Regge limit is nonperturbative.
- Compare DGLAP where the large scale implies small coupling.
- No momentum sum rule, since the evolution is local in $x$. In DGLAP: momentum sum rule satisfied at each order due to the non-locality of the evolution in $x$.
- Approximations in the phase space (multi-Regge kinematics, etc..) cannot/are only slowly recovered by the (fixed number of) the higher orders of expansion in the coupling constant.


## NLLx BFKL kernel

Representation of the kernel $\quad \mathcal{K}=\sum_{n=0}^{\infty} \bar{\alpha}_{s}^{n+1} \mathcal{K}_{n} \quad \bar{\alpha}_{s} \equiv \frac{N_{c} \alpha_{s}}{\pi}$
Mellin variables: $\quad \gamma \leftrightarrow \ln k_{T}^{2} \quad \omega \leftrightarrow \ln 1 / x$

## LLx kernel in Mellin space

$$
\chi_{0}(\gamma)=2 \psi(1)-\psi(\gamma)-\psi(1-\gamma)
$$

running coupling
triple poles
double poles

NLLx kernel in Mellin space

$$
\begin{align*}
\chi_{1}(\gamma)= & -\frac{b}{2}\left[\chi_{0}^{2}(\gamma)+\chi_{0}^{\prime}(\gamma)\right]-\frac{1}{4} \chi_{0}^{\prime \prime}(\gamma)-\frac{1}{4}\left(\frac{\pi}{\sin \pi \gamma}\right)^{2} \frac{\cos \pi \gamma}{3(1-2 \gamma)}\left(11+\frac{\gamma(1-\gamma)}{(1+2 \gamma)(3-2 \gamma)}\right) \\
& +\left(\frac{67}{36}-\frac{\pi^{2}}{12}\right) \chi_{0}(\gamma)+\frac{3}{2} \zeta(3)+\frac{\pi^{3}}{4 \sin \pi \gamma} \\
& -\sum_{n=0}^{\infty}(-1)^{n}\left[\frac{\psi(n+1+\gamma)-\psi(1)}{(n+\gamma)^{2}}+\frac{\psi(n+2-\gamma)-\psi(1)}{(n+1-\gamma)^{2}}\right] \tag{1}
\end{align*}
$$

## Origin of triple poles: kinematical constraint

Kinematical constraint can generate the triple poles when truncated to NLLs
Imposed on the transverse momenta in the ladder


Kwiecinski, Martin, Sutton;
Anderson, Gustafson, Kharazziha, Samuelson
Related to the scale choice asymmetric case (eg. DIS)
symmetric case (eg.Mueller-
Navelet jets, $\gamma^{*} \gamma^{*}$ )

$$
\begin{aligned}
& k=\left(k^{+}, k^{-}, \mathbf{k}_{T}\right) \\
& k^{2}=k^{+} k^{-}-k_{T}^{2}
\end{aligned}
$$

Virtualities dominated by transverse components

$$
\left|k^{2}\right| \simeq k_{T}^{2}
$$

Kinematical constraint

$$
k_{T}^{\prime 2}<\frac{k_{T}^{2}}{z}
$$

Leads to the shift of the poles in the kernel

$$
\begin{aligned}
& \chi(\gamma, \omega)=2 \psi(1)-\psi(\gamma)-\psi(1-\gamma+\omega) \\
& \chi(\gamma, \omega)=2 \psi(1)-\psi\left(\gamma+\frac{\omega}{2}\right)-\psi\left(1-\gamma+\frac{\omega}{2}\right)
\end{aligned}
$$

## Shifts of poles

Shift of poles (symmetric case)

$$
\chi_{n}^{\omega}(\gamma)=\chi_{n L}^{\omega}\left(\gamma+\frac{\omega}{2}\right)+\chi_{n R}^{\omega}\left(1-\gamma+\frac{\omega}{2}\right)
$$

LL case with shifts $\quad \chi_{0}^{\omega}=2 \psi(1)-\psi\left(\gamma+\frac{\omega}{2}\right)-\psi\left(1-\gamma+\frac{\omega}{2}\right)$
Shift of poles reproduces highest poles up to NNLO in sYM (highest poles the same in QCD)

Exact result

$$
\begin{aligned}
& \chi_{1}^{s Y M}=-\frac{1}{2 \gamma^{3}}-1.79+\mathcal{O}(\gamma), \quad \begin{array}{c}
\text { Gromov, Levkovich-Maslyuk, Sizov;Velizhanin; } \\
\text { Caron-Huot, Herranen }
\end{array} \\
& \chi_{2}^{s Y M}=\frac{1}{2 \gamma^{5}}-\frac{\zeta(2)}{\gamma^{3}}-\frac{9 \zeta(3)}{4 \gamma^{2}}-\frac{29 \zeta(4)}{8 \gamma}+\mathcal{O}(1)
\end{aligned}
$$

From shifts

$$
\begin{aligned}
& \chi_{1}(\gamma)=-\frac{1}{2 \gamma^{3}}-\frac{0}{\gamma^{2}}+\ldots \\
& \chi_{2}(\gamma)=\frac{1}{2 \gamma^{5}}-\frac{0}{\gamma^{4}}+\ldots
\end{aligned}
$$

Highest poles reproduced, lack of next-to-highest poles.

## Resummation setup



$$
\begin{aligned}
& \chi_{0}^{\omega}=2 \psi(1)-\psi\left(\gamma+\frac{\omega}{2}\right)-\psi\left(1-\gamma+\frac{\omega}{2}\right) \\
& \chi_{\mathrm{c}}^{\omega}(\gamma)=\frac{A_{1}(\omega)}{\gamma+\frac{\omega}{2}}+\frac{A_{1}(\omega)}{1-\gamma+\frac{\omega}{2}},
\end{aligned}
$$

Resummed kernel

$$
\begin{aligned}
\tilde{\chi}_{1}(\gamma) & =\chi_{1}(\gamma)-\chi_{0}^{0}(\gamma)\left[\chi_{0}^{1}(\gamma)+\chi_{\mathrm{c}}^{0}(\gamma)\right]-\chi_{0}^{\mathrm{run}}(\gamma) \\
& =\chi_{1}(\gamma)+\frac{1}{2} \chi_{0}(\gamma) \frac{\pi^{2}}{\sin ^{2}(\pi \gamma)}-\chi_{0}(\gamma) \frac{A_{1}(0)}{\gamma(1-\gamma)}+\frac{b}{2}\left(\chi_{0}^{\prime}+\chi_{0}^{2}\right)
\end{aligned}
$$

Additional subtraction needed to satisfy the momentum sum rule.
Solution to the evolution equation done in momentum space

## CCSS evolution

Ciafaloni, Colferai, Salam, AS
Formulated in momentum space: solving integral equation
Form of the convolution: three main parts

$$
\mathcal{K}_{0}^{\mathrm{kc}}\left(z ; \boldsymbol{k}, \boldsymbol{k}^{\prime}\right) \stackrel{z, \boldsymbol{q}}{\otimes} f\left(\frac{x}{z}, k^{\prime}\right)+\mathcal{K}_{0}^{\mathrm{coll}}\left(z ; k, k^{\prime}\right) \stackrel{z, k^{\prime}}{\otimes} f\left(\frac{x}{z}, k^{\prime}\right)+\mathcal{K}_{1}^{\operatorname{subtr}}\left(z ; k, k^{\prime}\right) \stackrel{z, k^{\prime}}{\otimes} f\left(\frac{x}{z}, k^{\prime}\right)
$$

$\underline{\text { LLx with kinematical constraint }}$

$$
\begin{aligned}
& \mathcal{K}_{0}^{\mathrm{kc}}\left(z ; \boldsymbol{k}, \boldsymbol{k}^{\prime}\right) \stackrel{z, \boldsymbol{q}}{\otimes} f\left(\frac{x}{z}, k^{\prime}\right) \\
&=\int_{x}^{1} \frac{d z}{z} \int \frac{d^{2} \boldsymbol{q}}{\pi \boldsymbol{q}^{2}} \bar{\alpha}_{s}\left(\boldsymbol{q}^{2}\right)\left[f\left(\frac{x}{z},|\boldsymbol{k}+\boldsymbol{q}|\right) \Theta\left(\frac{k^{2}}{z}-k^{\prime 2}\right)-\Theta(k-q) f\left(\frac{x}{z}, k\right)\right]
\end{aligned}
$$

DGLAP evolution with non-singular part of the splitting function

$$
\tilde{P}_{g g}^{(0)}=P_{g g}^{(0)}-\frac{1}{z}
$$

$$
\begin{aligned}
& \mathcal{K}_{0}^{\text {coll }}\left(z ; k, k^{\prime}\right) \stackrel{z, k^{\prime}}{\otimes} f\left(\frac{x}{z}, k^{\prime}\right)=\int_{x}^{1} \frac{d z}{z} \int_{0}^{k^{2}} \frac{d k^{\prime 2}}{k^{2}} \bar{\alpha}_{s}\left(k^{2}\right) z \tilde{P}_{g g}(z) f\left(\frac{x}{z}, k^{\prime}\right) \\
&+\int_{x}^{1} \frac{d z}{z} \int_{k^{2}}^{k^{2} / z} \frac{d k^{\prime 2}}{k^{\prime 2}} \bar{\alpha}_{s}\left(k^{\prime 2}\right) z \frac{k^{\prime 2}}{k^{2}} \tilde{P}_{g g}\left(z \frac{k^{\prime 2}}{k^{2}}\right) f\left(\frac{x}{z}, k^{\prime}\right)
\end{aligned}
$$

## CCSS evolution: ctd.

## NLLx BFKL with subtractions included

$$
\begin{aligned}
\int_{x}^{1} \frac{d z}{z} & \int d k^{\prime 2} \bar{\alpha}_{s}^{2}\left(k_{>}^{2}\right) \tilde{K}_{1}\left(k, k^{\prime}\right) f\left(\frac{x}{z}, k^{\prime}\right)=\frac{1}{4} \int_{x}^{1} \frac{d z}{z} \int d k^{\prime 2} \bar{\alpha}_{s}^{2}\left(k_{>}^{2}\right)\{ \\
& \left(\frac{67}{9}-\frac{\pi^{2}}{3}\right) \frac{1}{\left|k^{\prime 2}-k^{2}\right|}\left[f\left(\frac{x}{z}, k^{\prime 2}\right)-\frac{2 k_{<}^{2}}{\left(k^{\prime 2}+k^{2}\right)} f\left(\frac{x}{z}, k^{2}\right)\right]+ \\
& {\left[-\frac{1}{32}\left(\frac{2}{k^{\prime 2}}+\frac{2}{k^{2}}+\left(\frac{1}{k^{\prime 2}}-\frac{1}{k^{2}}\right) \log \left(\frac{k^{2}}{k^{\prime 2}}\right)\right)+\frac{4 \operatorname{Li}_{2}\left(1-k_{<}^{2} / k_{>}^{2}\right)}{\left|k^{\prime 2}-k^{2}\right|}\right.} \\
& -4 A_{1}(0) \operatorname{sgn}\left(k^{2}-k^{\prime 2}\right)\left(\frac{1}{k^{2}} \log \frac{\left|k^{\prime 2}-k^{2}\right|}{k^{\prime 2}}-\frac{1}{k^{\prime 2}} \log \frac{\left|k^{\prime 2}-k^{2}\right|}{k^{2}}\right) \\
& -\left(3+\left(\frac{3}{4}-\frac{\left(k^{\prime 2}+k^{2}\right)^{2}}{32 k^{\prime 2} k^{2}}\right)\right) \int_{0}^{\infty} \frac{d y}{k^{2}+y^{2} k^{\prime 2}} \log \left|\frac{1+y}{1-y}\right| \\
& \left.\left.+\frac{1}{k^{\prime 2}+k^{2}}\left(\frac{\pi^{2}}{3}+4 \operatorname{Li}_{2}\left(\frac{k_{<}^{2}}{k_{>}^{2}}\right)\right)\right] f\left(\frac{x}{z}, k^{\prime}\right)\right\} \\
& +\frac{1}{4} 6 \zeta(3) \int_{x}^{1} \frac{d z}{z} \bar{\alpha}_{s}^{2}\left(k^{2}\right) f\left(\frac{x}{z}, k\right) .
\end{aligned}
$$

+ additional subtractions to guarantee consistency with NLO DGLAP and momentum sum rule


## Structure function calculation

Evaluate structure function from $\mathrm{k}_{\mathrm{T}}$ factorization

$$
\begin{aligned}
F_{2}\left(x, Q^{2}\right) & =\sum_{q} e_{q}^{2} S_{q}\left(x, Q^{2}\right) \\
S_{q}\left(x, Q^{2}\right) & =\int_{x}^{1} \frac{d z}{z} \int \frac{d k^{2}}{k^{2}} S_{\mathrm{box}}^{q}\left(z, m_{q}^{2}, k^{2}, Q^{2}\right) f\left(\frac{x}{z}, k^{2}\right)
\end{aligned}
$$

Argument of the gluon density incorporating exact kinematics

$$
z=\left[1+\frac{\kappa^{\prime 2}+m_{q}^{2}}{\beta(1-\beta) Q^{2}}+\frac{k^{2}}{Q^{2}}\right]^{-1}
$$

Higher order terms in impact factor
Beyond the dipole model
Bialas, Peschanski, Navelet


## Setup for structure function calculation

Three contributions: depending on the ordering of quark and gluon momenta.
Introduce a cutoff parameter: $\quad k_{0}^{2} \simeq 1 \mathrm{GeV}^{2}$
Soft Pomeron
$k^{2}, \kappa^{\prime 2}<k_{0}^{2}$

$$
S^{(a)}=S_{u}^{P}+S_{d}^{P}+S_{s}^{P}
$$

$$
S_{u}^{\mathbb{P}}=S_{d}^{\mathbb{P}}=2 S_{s}^{\mathbb{P}}=C_{\mathbb{P}} x^{-\lambda}(1-x)^{8}
$$

Collinear contribution with the input gluon $k^{2}<k_{0}^{2}<\kappa^{\prime 2}$

$$
\begin{aligned}
S^{(b)} & =\int_{x}^{1} \frac{\mathrm{~d} z}{z} S_{\text {box }}^{(b)}\left(z, k^{2}=0, Q^{2}\right) \int_{0}^{k_{0}^{2}} \frac{\mathrm{~d} k^{2}}{k^{2}} f\left(\frac{x}{z}, k^{2}\right) \\
& =\int_{x}^{1} \frac{\mathrm{~d} z}{z} S_{\text {box }}^{(b)}\left(z, k^{2}=0, Q^{2}\right) \frac{x}{z} g\left(\frac{x}{z}, k_{0}^{2}\right),
\end{aligned}
$$

Perturbative contribution
$k^{2}, \kappa^{2}>k_{0}^{2}$

$$
S_{q}\left(x, Q^{2}\right)=\int_{x}^{1} \frac{d z}{z} \int \frac{d k^{2}}{k^{2}} S_{\mathrm{box}}^{q}\left(z, m_{q}^{2}, k^{2}, Q^{2}\right) f\left(\frac{x}{z}, k^{2}\right)
$$

Total contribution

$$
S_{q}^{(a)}+S_{q}^{(b)}+S_{q}^{(c)}
$$

## Results: $\mathrm{F}_{2}$ and $\mathrm{F}_{2 \mathrm{c}}$



Excellent description of the data on structure functions $\mathrm{F}_{2}$ and $\mathrm{F}_{2 \mathrm{c}}$

## Structure function $\mathrm{F}_{2}$ : contributions

## Breakdown of contributions

(a) soft pomeron
(b) collinear contribution with initial gluon
(c) perturbative $\mathrm{k}_{\mathrm{T}}$ factorization



As expected : dominance of perturbative only at large $\mathrm{Q}^{2}$.
Still, non-perturbative background at significant level: at $\mathrm{x}=10^{-3}, \mathrm{Q}^{2}=150 \mathrm{GeV}^{2}$ pert. / total $=70 \%$

## Extracted unintegrated gluon density $f\left(x, k^{2}\right)$

## CCSS

KS linear
KS nonlinear


Extracted unintegrated gluon density compared with Kutak-Sapeta (KS) model (LL BFKL+kinemaical constraint + DGLAP, with and without saturation)

Consistent results with KS linear, somewhat steeper.
Expected deviation between the linear and nonlinear at small x and $\mathrm{k}^{2}$

## Ratio of CCSS/KS

## ratio: CCSS to KS linear <br> ratio: CCSS to KS nonlinear



Ratio of extracted unintegrated gluon density to KS model, with and without saturation Slightly steeper behavior for CCSS

Expected deviation between the linear and nonlinear at small x and $\mathrm{k}^{2}$

## Effective intercept




Effective intercept of the unintegrated gluon density

$$
\lambda_{\mathrm{eff}}=\frac{\partial \ln f\left(x, k^{2}\right)}{\partial \ln 1 / x}
$$

About 0.3-0.4 depending on the value of $x$ and $k^{2}$
Clear difference with respect to the nonlinear case

## Summary and outlook

- Small $x$ evolution requires resummation.
- Used CCSS framework to evaluate the unintegrated gluon density
- The solution to the integral equation in momentum space with LLx + NLLx BFKL + resummation+ subtractions
- Use exact kinematics in the high energy factorization formula: higher order terms from kinematics
- Three component model for the fitting of structure function. Charm treated perturbatively.
- Very good description of structure function inclusive and charm
- Unintegrated gluon consistent with other extractions, slightly higher intercept, interesting for studies with saturation


## Next directions:

- Impact factors: NLO + resummation and matching with CCSS (in progress)
- Including quarks through the matrix formulation (needs numerical implementation)

