Heavy quarkonium production
via matching of High Energy Factorization
and NLO of Collinear Factorization

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Outline

1. $\eta_c$ inclusive hadroproduction: $z = \frac{M^2}{\hat{s}} \ll 1$, resummation of:

$$\alpha_s^n \ln^{n-1} \left(\frac{1}{z}\right)$$

2. $J/\psi$ inclusive photoproduction: $\eta = \frac{\hat{s} - M^2}{M^2} \gg 1$, resummation of $\alpha_s^n \ln^{n-1}(1 + \eta)$
Part 1: $\eta_c$ or $\eta_b$ inclusive hadroproduction
Perturbative instability of the $\eta_c$ total cross section

For the $p_T$-integrated cross section of $\eta_Q$ hadroproduction, the LO partonic subprocess is simply:

$$g + g \rightarrow Q\bar{Q} \left[ S_{0}^{(1)} \right].$$

The NLO correction can be computed in closed form [Kuhn, Mirkes, 93'; Petrelli et.al., 98'], and:
Collinear factorization for total CS for the state $m = 2S + 1$ $L_j^{(0)}$:

$$
\sigma^{[m]}(\sqrt{S}) = \int_{z_{\text{min}}}^{1} \frac{dz}{z} L_{ij}(z, \mu_F) \hat{\sigma}^{[m]}_{ij}(z, \mu_F, \mu_R),
$$

where $i, j = q, \bar{q}, g$, $z = M^2/\hat{s}$ and partonic luminosity:

$$
L_{ij}(z, \mu_F) = \int_{-y_{\text{max}}}^{+y_{\text{max}}} dy \tilde{f}_i \left( \frac{M}{\sqrt{S}z} e^y, \mu_F \right) \tilde{f}_j \left( \frac{M}{\sqrt{S}z} e^{-y}, \mu_F \right),
$$

with $\tilde{f}_j(x, \mu_F)$ - momentum density PDFs.

NLO coefficient function [Kuhn, Mirkes, 93'; Petrelli et.al., 98'] in the $z \to 0$ limit

$$
\hat{\sigma}^{[m]}_{ij} = \sigma^{[m]}_{\text{LO}} \left[ A_{0}^{[m]} \delta(1 - z) + C_{ij} \frac{\alpha_s(\mu_R)}{\pi} \left( A_{0}^{[m]} \ln \frac{M^2}{\mu_F^2} + A_{1}^{[m]} \right) + O(z\alpha_s, \alpha_s^2) \right],
$$

where $C_{gg} = 2C_A = 2N_c$, $C_{qg} = C_{gq} = C_F = (N_c^2 - 1)/(2N_c)$, $C_{q\bar{q}} = 0$ and $A_{1}^{[m]} < 0$. 
Resummed coefficient function

Small parameter: \( z = \frac{M^2}{\hat{s}} \).

**LLA** \( (\alpha_s^n \ln^{n-1} \frac{1}{z}) \) in High-Energy Factorization [Collins, Ellis, 91'; Catani, Ciafaloni, Hautmann, 91', 94']:

\[
\hat{\sigma}^{[m]}_{ij}, \text{HEF}(z, \mu_F, \mu_R) = \int_{-\infty}^{\infty} d\eta \int_{0}^{\infty} d\mathbf{q}_{T1}^2 d\mathbf{q}_{T2}^2 C_{gi} \left( \frac{M_T}{M} \sqrt{z} e^{-\eta}, \mathbf{q}_{T1}^2, \mu_F, \mu_R \right) \\
\times C_{gj} \left( \frac{M_T}{M} \sqrt{z} e^{-\eta}, \mathbf{q}_{T2}^2, \mu_F, \mu_R \right) \int_{0}^{2\pi} d\phi \frac{H^{[m]}(\mathbf{q}_{T1}^2, \mathbf{q}_{T2}^2, \phi)}{2} + O(z) + \text{NLL},
\]

The coefficient functions \( H^{[m]} \) are known at LO in \( \alpha_s \) [Hagler et.al, 2000; Kniehl, Vasin, Saleev 2006] for \( m = 1 S^{(1,8)}_0, 3 P^{(1,8)}_J, 3 S^{(8)}_1 \).

The \( H^{[m]} \) is a tree-level “squared matrix element” of the 2 \( \rightarrow \) 1-type process:

\[
R_+(\mathbf{q}_{T1}, \mathbf{q}^+_1) + R_-(\mathbf{q}_{T2}, \mathbf{q}^-_2) \rightarrow c\bar{c}[m].
\]
LLA evolution w.r.t. $\ln 1/z$

In the LL($\ln 1/z$)-approximation, the $Y = \ln 1/z$-evolution equation for collinearly un-subtracted $\tilde{C}$-factor has the form:

$$
\tilde{C}(x, q_T) = \delta(1-z)\delta(q_T^2) + \hat{\alpha}_s \int_x^1 \frac{dz}{z} \int d^{2-2\epsilon} k_T K(k_T^2, q_T^2) \tilde{C} \left( \frac{x}{z}, q_T - k_T \right)
$$

with $\hat{\alpha}_s = \alpha_s C_A / \pi$ and

$$
K(k_T^2, p_T^2) = \delta^{(2-2\epsilon)}(k_T) \left( \frac{p_T^2}{\epsilon} \right)^{-\epsilon} \frac{(4\pi)^\epsilon \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} + \frac{1}{\pi(2\pi)^{-2\epsilon}k_T^2}.
$$

It is convenient to go from $(z, q_T)$-space to $(N, x_T)$-space:

$$
\tilde{C}(N, x_T) = \int d^{2-2\epsilon} q_T e^{ix_T q_T} \int_0^1 dx \ x^{N-1} \tilde{C}(x, q_T),
$$

because:

- Mellin convolutions over $z$ turn into products: $\int \frac{dz}{z} \rightarrow \frac{1}{N}$

- Large logs map to poles at $N = 0$: $\alpha_s^{k+1} \ln^k \frac{1}{z} \rightarrow \frac{\alpha_s^{k+1}}{N^{k+1}}$

- All collinear divergences are contained inside $\tilde{C}$ in $x_T$-space.
Exact LL solution

In \((N, q_T)\)-space, subtracted \(C\), which resums all terms \(\propto (\hat{\alpha}_s/N)^n\) has the form:

\[
C(N, q_T, \mu_F) = R(\gamma_{gg}(N, \alpha_s)) \frac{\gamma_{gg}(N, \alpha_s)}{q_T^2} \left( \frac{q_T^2}{\mu_F^2} \right)^{\gamma_{gg}(N, \alpha_s)},
\]

where \(\gamma_{gg}(N, \alpha_s)\) is the solution of [Jaroszewicz, 82']:

\[
\frac{\hat{\alpha}_s}{N} \chi(\gamma_{gg}(N, \alpha_s)) = 1, \text{ with } \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma),
\]

where \(\psi(\gamma) = d\ln \Gamma(\gamma)/d\gamma\) – Euler’s \(\psi\)-function. The first few terms:

\[
\gamma_{gg}(N, \alpha_s) = \frac{\hat{\alpha}_s}{N} + 2\zeta(3) \frac{\hat{\alpha}_s^4}{N^4} + 2\zeta(5) \frac{\hat{\alpha}_s^6}{N^6} + \ldots
\]

The function \(R(\gamma)\) is

\[
R(\gamma_{gg}(N, \alpha_s)) = 1 + O(\alpha_s^3).
\]
Does this work?

The resummation has to reproduce the $A_1^{[m]}$ NLO coefficient when expanded up to NLO in $\alpha_s$. And it does. We have performed expansion up to NNLO:

<table>
<thead>
<tr>
<th>State</th>
<th>$A_0^{[m]}$</th>
<th>$A_1^{[m]}$</th>
<th>$A_2^{[m]}$</th>
<th>$B_2^{[m]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1S_0$</td>
<td>1</td>
<td>$-1$</td>
<td>$\frac{\pi^2}{6}$</td>
<td>$\frac{\pi^2}{6}$</td>
</tr>
<tr>
<td>$^3S_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\frac{\pi^2}{6}$</td>
</tr>
<tr>
<td>$^3P_0$</td>
<td>1</td>
<td>$-\frac{43}{27}$</td>
<td>$\frac{\pi^2}{6} + \frac{2}{3}$</td>
<td>$\frac{\pi^2}{6} + \frac{40}{27}$</td>
</tr>
<tr>
<td>$^3P_1$</td>
<td>0</td>
<td>$\frac{5}{54}$</td>
<td>$-\frac{1}{9}$</td>
<td>$-\frac{2}{9}$</td>
</tr>
<tr>
<td>$^3P_2$</td>
<td>1</td>
<td>$-\frac{53}{36}$</td>
<td>$\frac{\pi^2}{6} + \frac{1}{2}$</td>
<td>$\frac{\pi^2}{6} + \frac{11}{9}$</td>
</tr>
</tbody>
</table>

for e.g.

\[
\hat{\sigma}_{gg, \text{HEF}}^m (z \to 0) = \sigma_{\text{LO}}^m \left\{ A_0^{[m]} \delta (1 - z) + \frac{\alpha_s}{\pi} 2C_A \left[ A_1^{[m]} + A_0^{[m]} \ln \frac{M^2}{\mu_F^2} \right] \right. \\
\left. + \left( \frac{\alpha_s}{\pi} \right)^2 \ln \frac{1}{z} \cdot C_A^2 \left[ 2A_2^{[m]} + B_2^{[m]} + 4A_1^{[m]} \ln \frac{M^2}{\mu_F^2} + 2A_0^{[m]} \ln^2 \frac{M^2}{\mu_F^2} \right] + O(\alpha_s^3) \right\},
\]
Matching with NLO of CF

The HEF works only at \( z \ll 1 \), misses power corrections \( O(z) \), while NLO CF is exact in \( z \), but only NLO in \( \alpha_s \). **We need to match them.**

**Simplest prescription:** just subtract the overlap at \( z \ll 1 \):

\[
\sigma^{[m]}_{\text{NLO+HEF}} = \sigma^{[m]}_{\text{LO CF}} + \int_{z_{\text{min}}}^{1} \frac{dz}{z} \left[ \tilde{\sigma}^{[m],ij}_{\text{HEF}}(z) \right. \\
+ \left. \hat{\sigma}^{[m],ij}_{\text{NLO CF}}(z) - \hat{\sigma}^{[m],ij}_{\text{NLO CF}}(0) \right] \mathcal{L}_{ij}(z),
\]

**Or introduce smooth weights:**

\[
\sigma^{[m]}_{\text{NLO+HEF}} = \sigma^{[m]}_{\text{LO CF}} + \int_{z_{\text{min}}}^{1} dz \left\{ \left[ \tilde{\sigma}^{[m],ij}_{\text{HEF}}(z) \frac{\mathcal{L}_{ij}(z)}{z} \right] w^{ij}_{\text{HEF}}(z) \\
+ \left[ \hat{\sigma}^{[m],ij}_{\text{NLO CF}}(z) \frac{\mathcal{L}_{ij}(z)}{z} \right] (1 - w^{ij}_{\text{HEF}}(z)) \right\},
\]
Inverse error weighting method

In the InEW method [Echevarria, et.al., 2018] the weights are calculated from estimates of the error of each contribution:

$$w_{\text{HEF}}^{ij}(z) = \frac{[\Delta \sigma_{\text{HEF}}^{ij}(z)]^{-2}}{[\Delta \sigma_{\text{HEF}}^{ij}(z)]^{-2} + [\Delta \sigma_{\text{CF}}^{ij}(z)]^{-2}}.$$ 

- For $\Delta \sigma_{\text{CF}}$ we take the NNLO $\alpha_s^2 \ln \frac{1}{z}$ term of $\hat{\sigma}(z)$ predicted by HEF,
- For $\Delta \sigma_{\text{HEF}}$ we take the $\alpha_s O(z)$ part of the NLO CF result for $\hat{\sigma}(z)$.
- In both cases, stability against $O(\alpha_s^2)$ (constant in $z$, unknown) corrections is checked
Matching plots

Plots of the integrand of the total cross section (gg channel) as function of $z = M^2/\hat{s}$:

\[ \sqrt{s} = 100 \text{ GeV, } gg \]

\[ \sqrt{s} = 7000 \text{ GeV, } gg \]
Matched results for $\eta_c$
Matched results for $\eta_b$
The PDF dependence

\[ \eta_c \]

NLO+HEF, InEW matching, M=3 GeV, $^3S_0(1)$-state
channels: $gg+qg+\bar{q}g+q\bar{q}+q\bar{q}+q\bar{q}$

\[ \eta_b \]

NLO+HEF, InEW matching, M=9.4 GeV, $^3S_0(1)$-state
channels: $gg+qg+\bar{q}g+q\bar{q}+q\bar{q}+q\bar{q}$
Part 2: $J/\psi$ inclusive photoproduction
\[ \gamma(q) + p(P) \rightarrow J/\psi(p) + X \text{ @ NLO} \]

The LO CS partonic process is

\[ \gamma + g \rightarrow c\bar{c} \left[ S_1^{(1)} \right] + g. \]

The CS contribution is \( > 50\% \) of \( p_T\)-diff. cross section even at NLO. For \( p_T\)-integrated CS one has:

CT18NLO, \( z_{\text{max}} = 0.9 \)

\(< O[3S_1^{(1)}] >= 1.82 \text{ GeV}^2 + 15\% \text{ FD} \)
Why?

\[ \frac{d\sigma_{\gamma p}}{dz} = \frac{M^2}{S_{\gamma p}} \frac{S_{\gamma p}/M^2 - 1}{\int_0^1 d\eta f_i \left( \frac{M^2}{S_{\gamma p}}(\eta + 1), \mu_F \right) \frac{d\hat{\sigma}_{i\gamma}(\eta, z)}{dz}}, \]

where \( z = \frac{P_p}{P_q} = \frac{p^-}{q^-} \), \( \eta = \frac{\hat{s}}{M^2} - 1 \) with \( \hat{s} = S_{\gamma p} x \).

Plots of the integrand:
Asymptotics $\hat{\sigma}_{\text{NLO}}(\eta \to \infty)$

[Kraemer, 1995]:

$$\hat{\sigma}_{i\gamma} \propto c_{i\gamma}^{(0)}(\eta) + 4\pi\alpha_s \left[ c_{i\gamma}^{(1)}(\eta) + \bar{c}_{i\gamma}^{(1)}(\eta) \ln \frac{\mu_F^2}{m_c^2} + \frac{\beta_0}{8\pi^2} c_{i\gamma}^{(0)}(\eta) \right]$$

Numerical NLO result

(FDC code, Yu Feng)

Numerical NLO result (FDC code, Yu Feng)

$c^{(1)}$ and $\bar{c}^{(1)}$ at $\eta \to \infty$ from HEF as function of $p_{T_{\text{min}}}^{J/\psi}$
\( \gamma(q) + p(P) \rightarrow J/\psi(p) + X \) in HEF

HEF resummed partonic cross section:

\[
\frac{d\hat{\sigma}_{i\gamma}^{\text{HEF}}(\eta, z)}{dz} = \frac{1}{2zM^2} \int_{\frac{1}{z}}^{1+\eta} dy \int_{0}^{\infty} d\mathbf{q}_T^2 \ C_{gi} \left( \frac{y}{\eta + 1}, \mathbf{q}_T^2, \mu_F, \mu_R \right) \mathcal{H}(y, \mathbf{q}_T^2, z),
\]

actually resums \( \ln \frac{1}{z^+} = \ln \frac{\eta + 1}{y} \). Is resummation of only \( \ln(1 + \eta) \) possible? Yes.

The \( \mathcal{H} \) is the integral of the HEF coefficient function (\( H \)):

\[
R_+ (\mathbf{q}_{T1}, q_1^+) + \gamma(q) \rightarrow c\bar{c} \left[ ^3 S_1^{(1)} \right] (p) + g(k),
\]

over the PS of the gluon (\( y = q_1^+ q^- / M^2 \)):

\[
\mathcal{H}(y, \mathbf{q}_{T1}^2, z) = \int_{0}^{\infty} \frac{dk^+ dk^-}{2(2\pi)^2} \int d^2 \mathbf{p}_T \ H(\hat{s}, \hat{t}, \hat{u}, (\mathbf{q}_{T1} \cdot \mathbf{p}_T), \mathbf{q}_{T1}^2) \\
\times \delta(q^- (1 - z) - k^-) \delta(q_1^+ - \frac{M_T^2}{q^- z} - k^+) \delta(k^+ k^- - (\mathbf{q}_{T1} - \mathbf{p}_T)^2),
\]
InEW matching results

CT18NLO, $\sqrt{S_{yp}} = 700.0$ GeV

CT18NLO, $z_{\text{max}} = 0.9$
Analytic results for $\eta \to \infty$ asymptotics of $d\hat{\sigma}/dp_T$

Can be derived via expansion of HEF formula up to NLO and applying **IBP-reduction** to $q_T$-integrals.

\[
\frac{dc_1(z, \rho, \eta \to \infty)}{dzd\rho} = c_1^{(R)}(z, \rho) \\
\quad + c_1^{(1)}(z, \rho) \ln \left[ \frac{z^2(1-z)^2}{(\rho + (1-z)^2)^2} \right] + c_1^{(2)}(z, \rho) \ln \left[ \frac{(\rho + 1 - z)^2}{(1-z)(\rho + 2 - z)} \right] \\
\quad + \tilde{c}_1^{(3)}(z, \rho) \frac{\sqrt{(1+\rho)((2-3z)^2 + (2-z)^2\rho)}}{(1+\rho)((2-3z)^2 + (2-z)^2\rho)} \\
\times \ln \left[ \frac{\rho(2-z) - (3-2z)z + 2 - \sqrt{(\rho + 1)(\rho(z-2)^2 + (2-3z)^2)}}{\rho(2-z) - (3-2z)z + 2 + \sqrt{(\rho + 1)(\rho(z-2)^2 + (2-3z)^2)}} \right],
\]

where $\rho = p_T^2/M^2$ and

\[
c_1^{(1)}(z, \rho) = \frac{-z^3}{(\rho + 1)^2(\rho + (z-1)^2)^2(\rho - 2z + 1)^4} \\
\times \left\{ 5(\rho + 1)^4 + 4(2\rho + 1)z^6 - (\rho + 1)(23\rho + 31)z^5 + (\rho + 1)(\rho(12\rho + 77) + 89)z^4 \\
- 2(\rho + 1)(\rho + 3)(\rho(\rho + 18) + 21)z^3 + 2(\rho + 1)^2(\rho(3\rho + 32) + 47)z^2 \\
- (\rho + 1)^3(11\rho + 35)z \right\},
\]

and so on...
Analytic results for $\eta \to \infty$, $\rho \gg 1$ asymptotics

was mentioned in my “pheno” talk on Tuesday:

$$
\rho \to \infty : \quad \frac{d c_1(z, \rho, \eta \to \infty)}{dz d\rho} = -\frac{2(z - 1)z}{\rho^3(z - 2)^2} \\
\times \left\{ z^4 - 2z^3 - z^2 \ln \left[ \frac{(1 - z)z^2}{\rho(2 - z)^2} \right] - 4z + 4 \right\} + O(\rho^{-4}),
$$

$$
\frac{d \bar{c}_1(z, \rho, \eta \to \infty)}{d\rho dz} = \frac{2z(1-z)}{\rho^4} (1 - z(1 - z))^2 + O(\rho^{-5}),
$$

$$
\hat{\sigma}_{i\gamma} \propto c_{i\gamma}^{(0)}(\eta) + 4\pi\alpha_s \left[ c_{i\gamma}^{(1)}(\eta) + \bar{c}_{i\gamma}^{(1)}(\eta) \ln \frac{\mu_F^2}{m_c^2} + \frac{\beta_0}{8\pi^2} c_{i\gamma}^{(0)}(\eta) \right]
$$
Conclusions and outlook

- **Message 1:** Quarkonium production at $p_T \sim M \ll \sqrt{S}$ is the unique part of collider phenomenology where BFKL-type resummation is not just desirable, but unavoidable.

- **Message 2:** NLO corrections with $\hat{s} \sim M^2$ are as numerically important as those with $\hat{s} \gg M^2$ at any $\sqrt{S}$. Matching between HEF and NLO CF is always required!

- The high-energy instability of the NLO cross section is related with lack of the $\alpha_s^n \ln^{n-1} \frac{\hat{s}}{M^2}$ corrections in $\hat{\sigma}$ at $\hat{s} \ll M^2$.

- The HEF at DLA is the formalism to solve this problem if the standard fixed-order PDFs are to be used.

- Matching between NLO CF and HEF has to be performed.

- NLO CF+NLL HEF calculation is in progress.

- Future plans:
  - $\chi_{cJ}$ production in DLA+NLO CF
  - rapidity distributions in DLA+NLO CF
  - $p_T$-distribution of $J/\psi$ in DLA+NLO CF
  - next-to-DLA corrections...

Thank you for your attention!