Identifying Large Charge Operators

Gil Badel, EPFL
w/ Alexander Monin and Riccardo Rattazzi, to appear soon

July 8, 2022

Large Charge aux Diablerets

Introduction

CFTs in the large charge Q regime are described semiclassically by a superfluid EFT

[Hellerman,Orlando,Reffert,Watanabe,'15]
[Monin,Pirtskhalava,Rattazzi,Seibold,'16]

In cases where there is a small control parameter like $\varepsilon, \frac{1}{N}$, that semiclassical expansion can be performed in a perturbative way in a double scaling limit

$$\varepsilon \to 0, \quad Q \to \infty, \quad \varepsilon Q = \text{fixed}$$

[Badel,Cuomo,Monin,Rattazzi,'19], [Antipin,Bersini,Sannino,Wang,Zhang,'20]
[Alvarez-Gaumé,Orlando,Reffert,'19]

A key ingredient is mapping the theory to the cylinder using the state-operator correspondence

State-operator correspondence

Consider the scalar $U(1)$ model in $3-\varepsilon$ dimensions at the WF fixed point

$$\mathcal{L} = \partial\bar{\phi}\partial\phi + \frac{\lambda^2}{36}(\bar{\phi}\phi)^3$$

Mapping to the cylinder

$$x^\mu = e^\tau n^\mu, \quad \tau = \ln|x|, \quad |n| = 1$$

$$\phi(x) = e^{-\Delta_\phi \tau}\hat{\phi}(\tau,\vec{n}), \quad \bar{\phi}(x) = e^{-\Delta_\phi \tau}\hat{\bar{\phi}}(\tau,\vec{n})$$

The anomalous dimension of ϕ^n is given by energy of the lowest charge-n state

$$\langle\bar{\phi}^n(x_f)\phi^n(x_i)\rangle \propto \frac{1}{|x_i - x_f|^{2\Delta_{\phi^n}}} \xrightarrow{(\tau_f-\tau_i)\to\infty} \mathcal{N} e^{-(\tau_f-\tau_i)\Delta_{\phi^n}}$$

$$\langle\psi_n|e^{-H(\tau_f-\tau_i)}|\psi_n\rangle \xrightarrow{(\tau_f-\tau_i)\to\infty} \tilde{\mathcal{N}} e^{-(\tau_f-\tau_i)\Delta_{\phi^n}}$$

Semiclassical expansion

The latter can be represented as a path integral on the cylinder

$$\hat{\phi} = \frac{\rho}{\sqrt{2}} e^{i\chi}, \quad \hat{\bar{\phi}} = \frac{\rho}{\sqrt{2}} e^{-i\chi}$$

$$S_{\text{cyl}}[\rho, \chi] = \int \mathrm{d}\tau \mathrm{d}\Omega_{d-1} \left(\frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\rho^2(\partial\chi)^2 + \frac{1}{2}m^2\rho^2 + \frac{\lambda^2}{288}\rho^6 \right) - \frac{in(\chi_f - \chi_i)}{\Omega_{d-1}}$$

One expands around the non-trivial saddle as

$$\rho(\tau, \vec{n}) = f + r(\tau, \vec{n}), \quad \chi(\tau, \vec{n}) = -i\mu\tau + \frac{\pi(\tau, \vec{n})}{f\sqrt{2}}$$

$$f^2 = \frac{n}{\mu\Omega_{d-1}}, \quad \mu^2(\mu^2 - m^2) = \frac{\lambda^2 n^2}{48\Omega_{d-1}^2}, \quad m = \frac{d-2}{2}$$

The leading order is given by evaluating the action on the saddle

$$\Delta_{\phi^n}^{(\mathrm{LO})} = \frac{\Delta_{-1}(\lambda_* n)}{\lambda_*} = \frac{1}{\tau_f - \tau_i} S[f, -i\mu\tau] = \frac{n}{3}\left(2\mu + \frac{m^2}{\mu}\right)$$

The NLO is given by the casimir energy of fluctuations around the saddle. Use the quadratic action

$$S^{(2)}[r,\pi] = \int \mathrm{d}\tau \mathrm{d}\Omega_{d-1} \left(\frac{1}{2}(\partial r)^2 + \frac{1}{2}(\partial \pi)^2 - 2i\mu r \partial_\tau \pi + 2(\mu^2 - m^2) r^2\right)$$

to get the spectrum of said fluctuations

$$\omega_A^2(\ell) = J_\ell^2 + 2(2\mu^2 - m^2) - 2\sqrt{J_\ell^2 \mu^2 + (2\mu^2 - m^2)^2}$$

$$\omega_B^2(\ell) = J_\ell^2 + 2(2\mu^2 - m^2) + 2\sqrt{J_\ell^2 \mu^2 + (2\mu^2 - m^2)^2}$$

Definition of the superfluid excitations

$$\begin{pmatrix} r \\ \pi \end{pmatrix} = \begin{pmatrix} \frac{2\mu}{\omega_B^2(0)} p_\pi \\ \hat{\pi} - i p_\pi \tau \left(1 - \frac{4\mu^2}{\omega_B^2(0)}\right) \end{pmatrix} Y_{00}$$

$$+ \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left(V_A(\ell) A_{\ell m} e^{-\omega_A(\ell)} Y_{\ell m} + h.c. \right)$$

$$+ \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(V_B(\ell) B_{\ell m} e^{-\omega_B(\ell)} Y_{\ell m} + h.c. \right)$$

where $V_{A,B}(\ell)$ are vectors fixed by the quadratic EOM.

$$[A_{\ell m}, A^\dagger_{\ell' m'}] = \delta_{\ell\ell'}\delta_{mm'}, \quad [B_{\ell m}, B^\dagger_{\ell' m'}] = \delta_{\ell\ell'}\delta_{mm'}, \quad [\hat{\pi}, p_\pi] = i$$

$$\Delta_{\phi^n}^{(\mathrm{NLO})} = \frac{\Delta_{-1}(\lambda_* n)}{\lambda_*} + \Delta_0(\lambda_* n)$$

$$= \frac{n}{3}\left(2\mu + \frac{m^2}{\mu}\right) + \frac{1}{2}\sum_{\ell=0}^{\infty} n_\ell \left[\omega_A(\ell) + \omega_B(\ell) - 2\omega_0(\ell)\right]$$

with $\omega_0(\ell) = \ell + \frac{d-2}{2}$ and $n_\ell = \frac{(2l+d-2)\Gamma(\ell+d-2)}{\Gamma(\ell+1)\Gamma(d-1)}$.

Spectrum of fluctuations

This computation directly yields the spectrum of charge-n states made of a few excitations around the saddle, at NLO (only leading order in the splittings).

Say these excitations are created by acting on the lowest-energy charge-n state $|n\rangle$ with creation operators $A^\dagger_{\ell,m}$, $B^\dagger_{\ell,m}$

$$\prod_i A^\dagger_{\ell_i,m_i} \prod_j B^\dagger_{\tilde{\ell}_j,\tilde{m}_j} |n\rangle$$

that state has energy

$$\Delta_n(\{\ell_i\},\{\tilde{\ell}_j\}) = \Delta^{(\mathrm{NLO})}_{\phi^n} + \sum_i \omega_A(\ell_i) + \sum_j \omega_B(\tilde{\ell}_j)$$

Note : $\omega_A(0) = 0$ and $\omega_A(1) = 1 + O(\varepsilon)$.

Two Fock spaces

	Vacuum Fock space	**Hydrodynamic Fock space**
Background	Trivial vacuum $\|0\rangle$	Charge-n $\|n\rangle = \frac{1}{\sqrt{n!}}(a^{\dagger}_{00})^n\|0\rangle$
Fields	$\hat{\phi}, \hat{\bar{\phi}}$	r, π
Creation ops.	$a^{\dagger}_{\ell,m}, b^{\dagger}_{\ell,m}$	$A^{\dagger}_{\ell,m}, B^{\dagger}_{\ell,m}$

Can we express the operators corresponding to the latter picture, in terms of the elementary fields $\phi, \bar{\phi}$ and derivatives ?

Identifying Large Charge Operators

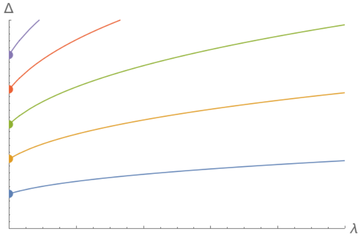
At small charge ($\lambda n \ll 1$) the operators which have definite anomalous dimension can be computed perturbatively. [Brown,1980]

At large charge however ($\lambda n \gg 1$), it is not clear what an operator expression means.

In all cases, operators are organized into conformal multiplets formed by one primary and all its descendents. At fixed large charge, this structure parallels the hydrodynamick Fock space.

Identifying Large Charge Operators

By continuity of the operator spectrum in the coupling, we can consider families of operators $\mathcal{O}_\lambda^{(n,\ell,\alpha)}(x)$ living across all couplings. The idea is then to label each family by its $\lambda = 0$ instance.



Since the conformal multiplet structure is independent of λ, classification of operators can be achieved by classifying charge-n primaries in free theory.

State-operator correspondence

We consider the free CFT in 3 dimensions

$$\mathcal{L} = \partial\hat{\bar{\phi}}\partial\hat{\phi} + \frac{1}{4}\hat{\bar{\phi}}\hat{\phi}$$

The state-operator correspondence is given by

$$a^\dagger_{\ell m}|0\rangle = \frac{\sqrt{2\ell+1}}{\ell!}\int \mathrm{d}\Omega_2\, Y_{\ell m} n^{\mu_1}\cdots n^{\mu_\ell}\partial_{\mu_1}\cdots\partial_{\mu_\ell}\phi(0)|0\rangle$$

$$b^\dagger_{\ell m}|0\rangle = \frac{\sqrt{2\ell+1}}{\ell!}\int \mathrm{d}\Omega_2\, Y_{\ell m} n^{\mu_1}\cdots n^{\mu_\ell}\partial_{\mu_1}\cdots\partial_{\mu_\ell}\bar{\phi}(0)|0\rangle$$

In this picture the correspondence is simple, thus we can express primary operators in terms of creation-annihilation operators.

Charge-n primary states

Expressing the conformal generators in terms of creation-annihilation operators, e.g.

$$K_3 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{(\ell+1)^2 - m^2} \left(a^\dagger_{\ell m} a_{\ell+1,m} + b^\dagger_{\ell m} b_{\ell+1,m} \right)$$

their action is easy to compute by commutation relations. We find directly two sets of primary operators

$$\mathcal{A}^\dagger_{\ell,\ell} = \sum_{k=0}^{\ell} \gamma_{k,\ell} (a^\dagger_{00})^{\ell-k-1} (a^\dagger_{1,1})^k a^\dagger_{\ell-k,\ell-k}, \qquad 2 \le \ell \le n$$

$$\mathcal{B}^\dagger_{\ell,\ell} = \sum_{k=0}^{\ell} \gamma_{k,\ell} (a^\dagger_{00})^{\ell-k+1} (a^\dagger_{1,1})^k b^\dagger_{\ell-k,\ell-k}, \qquad 0 \le \ell \le n$$

Charge-n primary states

These can be lowered using J_-, building the full $SO(3)$ multiplets $\mathcal{A}^\dagger_{\ell,m}, \mathcal{B}^\dagger_{\ell,m}$. These operators have charge equal to spin ℓ.

Any state

$$\left(a^\dagger_{00}\right)^{n-n_{\mathcal{A}}-n_{\mathcal{B}}} \prod_i \mathcal{A}^\dagger_{\ell_i,m_i} \prod_j \mathcal{B}^\dagger_{\tilde{\ell}_j,\tilde{m}_j} |0\rangle$$

is primary and corresponds to an operator of charge n and with $n_{\mathcal{A}} + n_{\mathcal{B}}$ derivatives $\left(n_{\mathcal{A}} = \sum_i \ell_i \text{ and } n_{\mathcal{B}} = \sum_j \tilde{\ell}_j\right)$.

In fact, by a combinatorics argument based on the counting of operators, we are able to show that all free theory primary operators of charge n, and a maximum of n derivatives, are (linear combinations of terms) of the above form.

Equivalent counting in the two Fock spaces

$$\left(a^\dagger_{00}\right)^{n-n_{\mathcal{A}}-n_{\mathcal{B}}} \prod_i \mathcal{A}^\dagger_{\ell_i,m_i} \prod_j \mathcal{B}^\dagger_{\tilde{\ell}_j,\tilde{m}_j}|0\rangle$$

where $2 \le \ell_i \le n$ and $0 \le \tilde{\ell}_j \le n$ is clearly the same number of states as

$$\prod_i A^\dagger_{\ell_i,m_i} \prod_j B^\dagger_{\tilde{\ell}_j,\tilde{m}_j}|n\rangle$$

The conformal multiplet classification of operators parallels the hydrodynamic Fock space structure.

Are they equivalent in the sense that e.g. $(a^\dagger_{00})^{n-\ell}\mathcal{A}^\dagger_{\ell,m}|0\rangle \approx A^\dagger_{\ell,m}|n\rangle$? Is this valid for all $\ell \le n$?

Mapping between the two Fock spaces

Because of the non-linear relation

$$\hat{\phi} = \frac{f + r}{\sqrt{2}} e^{\frac{1}{2}\tau + \frac{i\pi}{f}}$$

the mapping is non-trivial. However, it can be derived order-by order in a n^{-1} expansion, taking into account the scaling

$$a_{00} \sim O(\sqrt{n}), \qquad a_{\ell\neq 0,m} \sim b_{\ell m} \sim O(1)$$

Leading order

$$\exp\left[i\frac{\hat{\pi}}{\sqrt{2n}}\right] = \frac{a_{00}^{\dagger}}{\sqrt{n}}, \qquad p_{\pi} = 0$$

$$B_{\ell m} = \frac{a_{00}\, b_{\ell m}}{\sqrt{n}}, \quad A_{\ell m} = \frac{a_{00}^{\dagger}\, a_{\ell m}}{\sqrt{n}}$$

Next to leading order

$$\exp\left[i\frac{\hat{\pi}}{\sqrt{2n}}\right] = \frac{a_{00}^{\dagger}}{\sqrt{n}}, \quad p_{\pi} = \frac{(Q-n)Y_{00}}{f}$$

The other NLO expressions are complicated and will not be displayed here.

The zero mode $\exp\left[i\frac{\hat{\pi}}{\sqrt{2n}}\right]$ is charged, it is what makes the transition to other fixed-charged sectors. That is why the conjugated momentum is linear in the charge.

Are primaries equal to hydrodynamic modes ?

We can rewrite the conformal generators in terms of the hydrodynamic Fock space operators.

At leading order for the special conformal generators, the expression is simple

$$K_0 = \sqrt{n}A_{1,0}, \quad K_- = -\sqrt{n}A_{1,-1}, \quad K_+ = \sqrt{n}A_{1,1}$$

This seems to indicate that, as expected from the spectrum and backed up by the counting, primary states are precisely those who are created without any $A^\dagger_{1,i}$.

Are primaries always equal to hydrodynamic modes ?

Consider a spin-ℓ primary

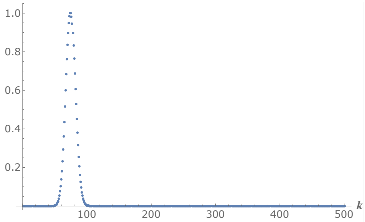
$$|n;\ell,\ell\rangle = (a_{00}^\dagger)^{n-\ell}\mathcal{A}_{\ell,\ell}^\dagger|0\rangle$$
$$= \alpha_0 \sum_{k=0}^{\ell} \gamma_{k,\ell}(a_{00}^\dagger)^{n-k-1}(a_{1,1}^\dagger)^k a_{\ell-k,\ell-k}^\dagger|0\rangle$$

The first term corresponds to the leading-order $A_{\ell\ell}$

$$(a_{00}^\dagger)^{n-1}a_{\ell,\ell}^\dagger|0\rangle \sim \left(a_{00}a_{\ell\ell}^\dagger\right)(a_{00}^\dagger)^n|0\rangle \propto A_{\ell\ell}^\dagger|n\rangle$$

Are primaries always equal to hydrodynamic modes ?

- For $\ell \ll \sqrt{n}$, the norm of each successive terms is suppressed, meaning the hydrodynamic mode description is valid.
- For $\sqrt{n} \leq \ell \ll n$, the norm of the terms peak around $k_{\text{max}} \sim \frac{2\ell^2}{n}$, which means the first term is not dominant; the hydrodynamic description breaks down.



Computation of matrix elements

We can nevertheless compute observables exactly on the primary state as long as $\ell \ll n$. For example at $1 \ll \ell \ll n$:

$$\langle n; \ell, \ell| : \partial_\tau^p \hat{\bar{\phi}}(\tau, \vec{n}) \partial_\tau^p \hat{\phi}(\tau, \vec{n}) : |n; \ell, \ell\rangle$$

$$= \frac{n}{4^{p+1}\pi} \Bigg[\left(1 + \ell^{-1} + \ell^{-2} + \ldots\right) + \frac{\ell + 1 + \ell^{-1} + \ldots}{n} + \frac{\ell^2 + \ell + 1 + \ldots}{n^2} + \ldots$$

$$+ \frac{\ell^{2p-\frac{1}{2}}}{n} \left(\left(1 + \ell^{-1} + \ldots\right) + \frac{\ell + 1 + \ldots}{n} + \ldots \right) \Bigg]$$

The result can reliably be written as a series in ℓ/n.

Summary

- Fluctuations around the large charge saddle form a Fock space generated by operators $A^\dagger_{\ell m}, B^\dagger_{\ell m}$.
- In free theory, the full set of charge-n primary operators with at most n derivatives is generated by known $\mathcal{A}^\dagger_{\ell m}, \mathcal{B}^\dagger_{\ell m}$ (with the exception of $\mathcal{A}^\dagger_{00}$ and $\mathcal{A}^\dagger_{1,i}$).
- In case of spin $\ell < \sqrt{n}$, there is a direct correspondence (at leading order) between A excitations and $\mathcal{A}$ operators, and between B excitations and $\mathcal{B}$ operators. In particular $A^\dagger_{1,i}$ create descendents.
- This direct relation breaks down for $\ell > \sqrt{n}$ (even though the counting is still valid), but the computation of observables is still possible in a series in $\frac{\ell}{n}$.

Thank you !

Semiclassical computation of 4-point functions

The semiclassical method can also be used to compute correlators of the form

$$\langle \bar{\phi}^n (\bar{\phi}\phi)(\bar{\phi}\phi)\phi^n \rangle$$

Using the properties of CFTs, we are able to deduce the operators entering in the OPEs. For instance, the NLO computation reveals that the OPE

$$\phi^n \times (\bar{\phi}\phi)$$

contains operators of spin ℓ and energy $\Delta_{\phi^n} + \omega_{\pm}(\ell)$ for all ℓ.