Large-order behaviours in CFTs and Resurgence

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AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Based on:

L. Àlvarez Gaumé, D.Orlando, S.Reffert [2008.03308]
N.A.D, I. Kalogerakis, D.Orlando, S.Reffert [2102.12488]

Intro and Motivations

$$\Delta^{\min}_{d=3,\,O(2)}(Q) = c_{\frac{3}{2}}\, Q^{\frac{3}{2}} + c_{\frac{1}{2}}\, Q^{\frac{1}{2}} + \mathcal{C} + \mathcal{O}(Q^{-\frac{1}{2}})$$

Wilson coefficients

Universal constant

[Hellerman *et at.* '15, Cuomo '20 ...]

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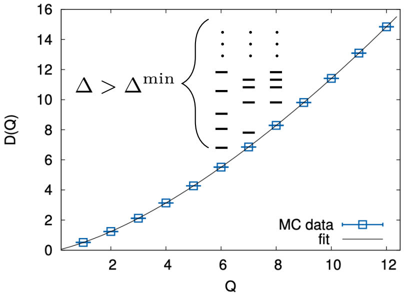

[Banerjee, Chandrasekharan, Orlando '17]

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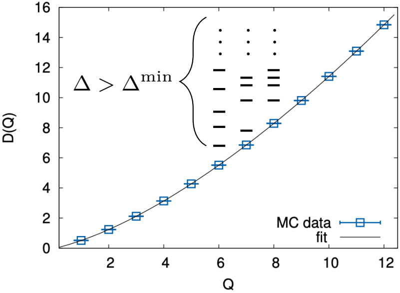
- Extrapolation to small charge operators $Q \sim \mathcal{O}(1)$?
- Exponential corrections $\sim e^{-CQ^{\alpha}}$?

  [Hellerman '21]

  [Grassi, Komargodski, Tizzano '19]



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$$Q := \sum_{i=1}^{N} Q_i, \quad Q, N \to \infty \quad Q/2N = \text{const.}$$

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- The CFT free energy computes a heavy operator scaling dimension:

$$\mathcal{F}(Q) = \Delta(Q)/r_{S^2} \quad \text{for the operator} \quad \phi_{\{i_1} \dots \phi_{i_Q\}} \sim \varphi^Q$$

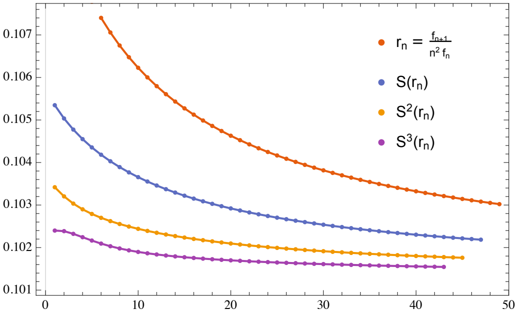
[Giombi, Hyman '20]

Scaling dimension for the operator φ^Q

$$\frac{\Delta(Q)}{2N} = \begin{cases} \overbrace{\frac{1}{2}}^{\text{tree}} \left(\frac{Q}{2N}\right) + \sum_{n=2}^{\infty} b_n \left(\frac{Q}{2N}\right)^n \\ \underbrace{\frac{2}{3}}_{c_{3/2}} \left(\frac{Q}{2N}\right)^{3/2} + \underbrace{\frac{1}{6}}_{c_{1/2}} \left(\frac{Q}{2N}\right)^{1/2} + \sum_{n=0}^{\infty} c_n \left(\frac{Q}{2N}\right)^{-n-\frac{1}{2}} \end{cases}$$

- Two regimes, different saddles contributing.
- The non-trivial $Q/2N \gg 1$ behaviour is absent in the $\mathrm{CFT}_{\mathrm{UV}}$ (free boson).
- Two regimes are connected: information contained in the growth of b_n, c_n.
- On $\mathbb{R} \times S^2$ these have interpretations in term of SSB, gapped GB ecc.

Large-order growth of $\Delta(Q)$



- Regime $Q/2N \gg 1$: growth is $c_n \sim (n!)^2$ (EFT perturbation theory).

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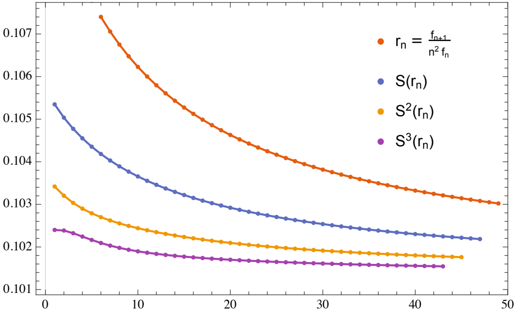

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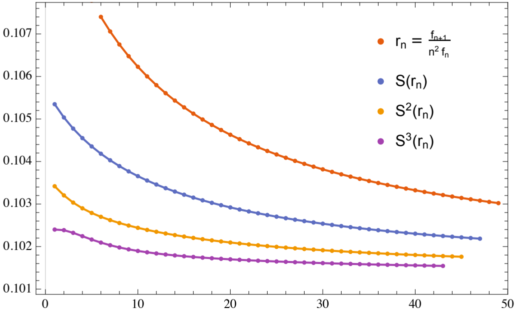

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- Regime $Q/2N \ll 1$: growth is $b_n \sim n^\alpha$ (Large-N perturbation theory).
- Non-perturbative corrections associated to this growth?

Thermodynamics on $\mathbb{R} \times S^2$

Critical action for the $O(2N)$ theory:

$$S_{\mathbb{R}\times S^2}[\varphi, \sigma] = \int_{\mathbb{R}\times S^2} \mathrm{d}^3x \left\{ |D^{(\mu)}\varphi|^2 + \left(\sigma + \frac{1}{4r_{S^2}} \right) |\varphi|^2 \right\}$$

Auxiliary field $\sigma \sim |\varphi|^2$

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Canonical: (Q, V, T)

$$\mathcal{Z}^{(c)}_{\mathbb{R}\times S^2} = e^{-\beta(\Delta / r_{S^2})}$$

Grand canonical: (μ, V, T)

$$\mathcal{Z}^{(gc)}_{\mathbb{R}\times S^2} = e^{-\beta \Omega}$$

Legendre transform $\mu \leftrightarrow Q = \Omega'(\mu)$

[More in this context: Moser, Orlando, Reffert '21]

Grand potential and BEC phase

- The Large-N result for the gran potential is

$$\frac{\Omega}{2N} = -\left(\mu^2 - \frac{1}{4} - \langle\sigma\rangle\right)\frac{|\langle\varphi\rangle|^2}{2N} + \sum_{\ell=0}^{\infty}(2\ell+1)\sqrt{\ell(\ell+1)+\frac{1}{4}+\langle\sigma\rangle}$$

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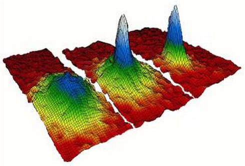
- This is a non-abelian version of Bose-Einstein condensation.

$$\langle\varphi\rangle \sim \sqrt{N} : O(2N) \to O(2N-1)$$

small-μ: symmetry restoration



Non-perturbative contributions - I

- A saddle carrying charge Q has potential and field vev given by

$$\frac{\Omega(\mu)}{2N} = \sum_{\ell=0}^{\infty} (2\ell+1)\sqrt{\ell(\ell+1)+\mu^2}$$

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$$\frac{\Omega(\mu)}{2N} = \frac{1}{\Gamma(s)}\int_0^{\infty} \mathrm{d}t\, t^{s-1}e^{-t\mu^2}\mathrm{Tr}\left\{e^{t\Delta_{S^2}}\right\}\Big|_{s=-1/2}$$

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- The heat trace is the simplest object to run a Resurgence analysis:
  - Grows as $\sim n!$ expanded at $t \to 0^+ \implies$ non-perturbative ambiguities.
  - The Borel transform can be computed exactly.
  - There is an exact integral formula we can find via Borel-Laplace summation.

Non-perturbative contributions - II

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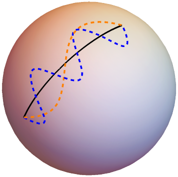
- The large-order analysis is not sensitive to an overall $\mathbb{C}$-coefficients σ_k (Transseries coefficients).
- Determine σ_k (modulo Stokes jumps): $\begin{cases} \text{Selberg-type trace formulas.} \\ \text{Worldline path integral.} \end{cases}$

Worldline path integral: classification of saddles

$$\langle y|e^{-t\Delta_{S^2}}|x\rangle \simeq$$



$$= \int_{x(0)=x}^{y(t)=y} \mathcal{D}x^\mu \, e^{-\frac{1}{4}\int_0^t \mathrm{d}\tau \, g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}$$

[Strassler '92, Schubert '96 (review) ... Bastianelli '05]

Transseries $\leftrightarrow$ saddle-point approximation around (closed) S^2 geodesics.

Transseries for heat trace

$$\mathrm{Tr}\left\{e^{t\Delta_{S^2}}\right\} \xrightarrow[t\to 0]{} \frac{1}{t}\Big\{1+\ldots\Big\} \;\pm i\; \left(\frac{\pi}{t}\right)^{\frac{3}{2}} \sum_{k\neq 0}(-1)^{k+1}|k|\, e^{-\frac{(2\pi k)^2}{4t}} \;\Big\{1+\ldots\Big\}$$

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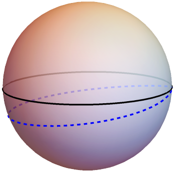

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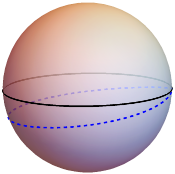

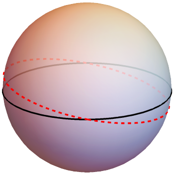

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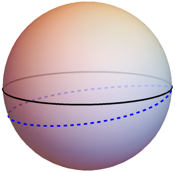
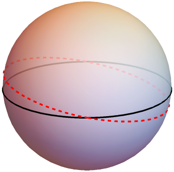
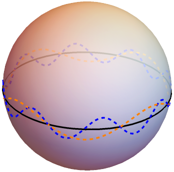
Positive modes







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- Under the assumptions:
  - $\Delta(Q)$ has an asymptotic perturbative expansion for any N.
  - The leading singularity is determined via saddle of a WL integral for a particle with mass $m \sim \mu$ (Gapped goldstone? [Nicolis, Piazza '13...] radial modes? [Grassi, Komargodski, Tizzano '19])
  - There is a scale invariant EFT with cutoff $\Lambda \sim \sqrt{Q}/r_{S^2}$

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- Cooper pairs BEC formation in interacting fermionic large-N CFTs.

Summary and conclusion

In the double-scaling limit of the $O(2N)$, the expansion of $\Delta(Q)$ is:

- asymptotic for $Q/(2N) \gg 1$, with factorial growth $\sim (n!)^2$.
- Contains exponential corrections given by Worldline instantons.
- These are small also for (reasonably) low Q, N. This seems to suggest that the $Q/2N \gg 1$ regime can be extrapolated also to light operators.

Thank you for your attention!