Long Range Model at Large Charge and Large N

Elizabeth Helfenberger

Princeton University

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Introduction

Long-range $O(N)$ model in $\mathbb{R}^d$

$$S = \frac{C}{2} \int d^d x d^d y \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}}, \qquad C = \frac{2^s \Gamma(\frac{d+s}{2})}{\pi^{d/2}\Gamma(-\frac{s}{2})}$$

Fifty years old

[Fisher, Ma, Nickel ‘72, Sak ‘73, Sak ‘77]

and has been discussed in many recent papers

[Paulos et. al. 1509.00008, Behan et. al. 1703.03430+1703.05325, Chai et. al. 2107.08052,...]

Short-range: spins have nearest-neighbor interactions

- Kinetic term $(\partial_\mu \phi^I)^2$

Long-range: spin interactions behave as $1/r^{d+s}$

Introduction

Long-range $O(N)$ model in $\mathbb{R}^d$ with quartic interaction

$$S = \frac{C}{2}\int d^dx d^dy \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}} + \frac{g}{4}\int d^dx (\phi^I\phi^I(x))^2, \qquad C = \frac{2^s\Gamma(\frac{d+s}{2})}{\pi^{d/2}\Gamma(-\frac{s}{2})}$$

The scaling dimension of the fundamental fields is $\Delta_\phi = \frac{d-s}{2}$

- $s < d/2$: Gaussian fixed point
  - $\rightarrow$ No anomalous dimension
- $d/2 < s < s_*$: Nontrivial long-range fixed points
  - $\rightarrow$ Δ_ϕ still gets no anomalous dimension
- $s > s_* = 2 - 2\gamma_\phi^{\mathsf{SR}}$, model is described by usual short-range $O(N)$ fixed point

The setup

Long-range $O(N)$ model in $\mathbb{R}^d$ with quartic interaction

$$S = \frac{C}{2}\int d^dx d^dy \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}} + \frac{g}{4}\int d^dx (\phi^I\phi^I(x))^2$$

We study operators with charge j under the global $O(N)$ symmetry

$$\mathcal{O}_j \equiv (u^I\phi^I)^j$$

where u^I is a null auxiliary complex vector

Look at scaling dimensions Δ_j of $\mathcal{O}_j$:

$$\langle \mathcal{O}_j(x_1)\mathcal{O}_j(x_2)\rangle = (u_1^I u_2^I)^j \frac{C_j}{x_{12}^{2\Delta_j}}$$

$$\Delta_j = -\frac{1}{2}|x_{12}|\frac{\partial}{\partial|x_{12}|}\log\langle \mathcal{O}_j(x_1)\mathcal{O}_j(x_2)\rangle$$

The setup

Double-scaling limit

- Take $j \to \infty$, $N \to \infty$, keep $\hat{j} \equiv j/N$ finite
- Small $\hat{j}$: ordinary $1/N$ perturbation theory
- j same order as N: new semiclassical saddle point emerges

Summary of results

In the limit of large $\hat{j}$, we find a very different result for scaling dimensions in long-range $O(N)$ model compared to short-range:

Δ_j	Short-Range $O(N)$	Long-Range $O(N)$
Small $\hat{j}$ limit	$N\left(\frac{d-2}{2}\hat{j} + \mathcal{O}(\hat{j}^2)\right)$	$N\left(\frac{d-s}{2}\hat{j} + \mathcal{O}(\hat{j}^2)\right)$
Large $\hat{j}$ limit	$N\left(\hat{j}^{\frac{d}{d-1}}A_1(d) + \hat{j}^{\frac{1}{d-1}}A_2(d)\right)$	$N\left(\frac{d+s}{2}\hat{j} + A(d,s)\hat{j}^{\frac{s}{d+s}} + \dots\right)$

where the short-range scaling dimensions were computed in [Alvarez-Gaume, Orlando, Reffert 1909.02571; Giombi, Hyman 2011.11622]

Saddle point approximation

Apply usual large N procedure of introducing a Hubbard-Stratonovich auxiliary field σ and dropping the σ^2 term in the action:

$$S = \frac{C}{2} \int d^d x d^d y \frac{\phi^I(x)\phi^I(y)}{|x-y|^{d+s}} + \frac{1}{2} \int d^d x \sigma(x) \phi^I \phi^I(x)$$

Green's function $G(x_1, x_2; \sigma)$ of fundamental operators with respect to this action is

$$\delta^{IJ} G(x_1, x_2; \sigma) \equiv \int \mathcal{D}\phi \phi^I(x_1) \phi^J(x_2) e^{-S}$$

Saddle point approximation

Two-point function of large charge operators (after $j!$ Wick contractions) is

$$\begin{aligned}\langle \mathcal{O}_j(x_1)\mathcal{O}_j(x_2)\rangle &= (u_1^I u_2^I)^j j! \int \mathcal{D}\sigma \frac{[G(x_1,x_2;\sigma)]^j}{\int \mathcal{D}\phi e^{-S}} \\ &= (u_1^I u_2^I)^j j! \int \mathcal{D}\sigma e^{-NS_{\text{eff}}} \\ &\approx \mathcal{N} e^{-NS_{\text{eff}}(\sigma_*)}\end{aligned}$$

where the effective action is

$$S_{\text{eff}} = \frac{1}{2}\log\det\left[\frac{C}{|x-y|^{d+s}} + \sigma(x)\delta^d(x-y)\right] - \hat{j}\log(G(x_1,x_2;\sigma))$$

with $\hat{j} \equiv j/N$

Saddle point approximation

Extremizing S_{eff} with respect to σ gives the saddle point equation

$$2\hat{j}G(x_1, x; \sigma_*)G(x_2, x; \sigma_*) = -G(x, x; \sigma_*)G(x_1, x_2; \sigma_*)$$

Strategy to solve the saddle

- Expand $G(x, y; \sigma_*)$ in powers of σ_* and obtain integral recursive relation for Green's function
- Plug in reasonable ansatz for the profile of $\sigma_*(x)$ at the saddle point
- Use technology of fishnet integrals to turn a complicated iterative series of integrals into a single integral and single sum

Green's function

The Green's function can be defined as

$$\int d^d x' \left(\frac{C}{|x - x'|^{d+s}} + \sigma_*(x)\delta^d(x - x') \right) G(x', y; \sigma_*) = \delta^d(x - y)$$

Expand $G(x, y, \sigma_*)$ in powers of σ^*

$$G = G^{(0)} + G^{(1)} + G^{(2)} + \ldots$$

Plug into definition

$$\int d^d x' \frac{C}{|x - x'|^{d+s}} G^{(0)}(x', y) = \delta^d(x - y)$$

$$\int d^d x' \frac{C}{|x - x'|^{d+s}} G^{(L+1)}(x', y; \sigma^*) = -\sigma_*(x) G^{(L)}(x, y; \sigma^*)$$

Green's function

Leading order result is the usual two-point function without any large charge operators

$$G^{(0)}(x,y) = \frac{C_\phi}{|x-y|^{d-s}}, \qquad C_\phi = \frac{\Gamma\left(\frac{d-s}{2}\right)}{2^s \pi^{\frac{d}{2}} \Gamma\left(\frac{s}{2}\right)}$$

Get the result for order L Green's function by iteratively applying the above result

$$\begin{aligned} G^L(x,y,\sigma_*) &= (-1)^L \left(\prod_{k=1}^{L} \int d^d z_k \sigma^*(z_k) G^0(z_k, z_{k+1}) \right) G^0(x, z_1) \\ &= (-1)^L \left(\prod_{k=1}^{L} \int d^d z_k \sigma^*(z_k) \right) \left(\prod_{j=0}^{L} \frac{C_\phi}{|z_{j+1} - z_j|^{d-s}} \right) \end{aligned}$$

where $z_0 = y$, $z_{L+1} = x$

Green's function

To proceed, use ansatz for profile of $\sigma(x)$ at the saddle point:

$$\begin{aligned}\sigma_*(x;x_1,x_2) &= \lim_{N\to\infty}\frac{\int \mathcal{D}\sigma\ \sigma(x)e^{-NS_{\text{eff}}}}{\int \mathcal{D}\sigma e^{-NS_{\text{eff}}}} \\ &= \lim_{N\to\infty}\frac{\langle \mathcal{O}_j(x_1,u_1)\mathcal{O}_j(x_2,u_2)\sigma(x)\rangle}{\langle \mathcal{O}_j(x_1,u_1)\mathcal{O}_j(x_2,u_2)\rangle}\end{aligned}$$

Noting $\Delta_\sigma = s + \mathcal{O}(1/N)$, conformal symmetry requires

$$\sigma_*(x;x_1,x_2) = c_\sigma \frac{|x_1-x_2|^s}{|x_1-x|^s|x_2-x|^s}$$

Green's function

Plug ansatz into expression for $G^{(L)}$

$$G^L(x, y, \sigma_*) = (-1)^L \left(\prod_{k=1}^{L} \int d^d z_k \sigma^*(z_k) \right) \left(\prod_{j=0}^{L} \frac{C_\phi}{|z_{j+1} - z_j|^{d-s}} \right)$$

$$= C_\phi(-C_\phi c_\sigma |x_1 - x_2|^s)^L$$

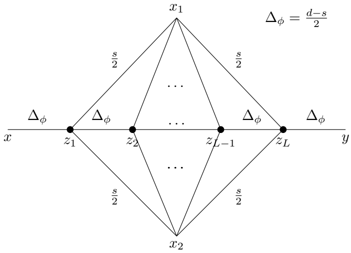
$$\times \left(\prod_{k=1}^{L} \int \frac{d^d z_k}{|z_k - x_1|^s |z_k - x_2|^s} \right) \left(\prod_{j=0}^{L} \frac{1}{|z_{j+1} - z_j|^{d-s}} \right)$$

Green's function

The integral

$$I = \left(\prod_{k=1}^{L} \int \frac{d^d z_k}{|z_k - x_1|^s |z_k - x_2|^s} \right) \left(\prod_{j=0}^{L} \frac{1}{|z_{j+1} - z_j|^{d-s}} \right)$$

was computed in [Derkachov, Ferrando, Olivucci 2108.12620] in the very different context of fishnet Feynman integrals



Green's function

The result can be expressed in terms of Gegenbauer polynomials $C_l^{(\frac{d-2}{2})}(x)$

$$\xi = \frac{x - x_1}{|x - x_1|^2} - \frac{x_2 - x_1}{|x_2 - x_1|^2}$$
$$\eta = \frac{y - x_1}{|y - x_1|^2} - \frac{x_2 - x_1}{|x_2 - x_1|^2}$$

$$I = \frac{\Gamma\left(\frac{d-2}{2}\right)}{(|\xi||\eta|)^{\frac{d-s}{2}}\pi^{\frac{-dL}{2}}} \sum_{l=0}^{\infty} \left(l + \frac{d-2}{2}\right) \times C_l^{(\frac{d-2}{2})}\left(\frac{\xi \cdot \eta}{|\xi||\eta|}\right) \int \frac{du}{2\pi} \left(\frac{\xi^2}{\eta^2}\right)^{iu} (Q_l(u))^{L+1}$$

where $Q_l(u)$ is a ratio of Gamma functions

$$Q_l(u) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{d-s+2l}{4} - iu\right)\Gamma\left(\frac{d-s+2l}{4} + iu\right)}{\Gamma\left(\frac{d-s}{2}\right)\Gamma\left(\frac{d+s+2l}{4} + iu\right)\Gamma\left(\frac{d+s+2l}{4} - iu\right)}$$

Green's function

The result for the Lth Green's function is then

$$G^L = \frac{(-C_\phi c_\sigma)^L}{|x-x_1|^{d-s}|y-x_1|^{d-s}} \frac{\Gamma\left(\frac{d-2}{2}\right)}{(|\xi||\eta|)^{\frac{d-s}{2}}\pi^{\frac{-dL}{2}}} \sum_{l=0}^{\infty}\left(l+\frac{d-2}{2}\right) \times C_l^{\left(\frac{d-2}{2}\right)}\left(\frac{\xi\cdot\eta}{|\xi||\eta|}\right)\int\frac{du}{2\pi}\left(\frac{\xi^2}{\eta^2}\right)^{iu}(Q_l(u))^{L+1}$$

The full Green's function is just the sum over L:

$$G(x,y,\sigma_*) = \frac{1}{|x-x_1|^{d-s}|y-x_1|^{d-s}} \frac{\Gamma\left(\frac{d-2}{2}\right)}{(|\xi||\eta|)^{\frac{d-s}{2}}} \sum_{l=0}^{\infty}\left(l+\frac{d-2}{2}\right) \times C_l^{\left(\frac{d-2}{2}\right)}\left(\frac{\xi\cdot\eta}{|\xi||\eta|}\right)\int\frac{du}{2\pi}\left(\frac{\xi^2}{\eta^2}\right)^{iu}\frac{Q_l(u)}{1+C_\phi c_\sigma \pi^{d/2} Q_l(u)}$$

Solving the saddle

As promised, we have $G(x, y; \sigma_*)$ in terms of a single integral and single sum

$$G(x, y; \sigma_*) = \sum_{l=0}^{\infty} (\dots) \int du (\dots)$$

Right now σ_* is a just a function of x, x_1, and x_2 taking the form

$$\sigma_*(x; x_1, x_2) = c_\sigma \frac{|x_1 - x_2|^s}{|x_1 - x|^s |x_2 - x|^s}$$

We want to find the c_σ such that the saddle point equation is satisfied.

$$2\hat{j} G(x_1, x; \sigma_*) G(x_2, x; \sigma_*) = -G(x, x; \sigma_*) G(x_1, x_2; \sigma_*)$$

Green's function in various limits

We evaluate the $u-$integral by closing the contour in the upper half plane and performing a sum over residues

$$G(x,y;\sigma_*) = \sum_{l=0}^{\infty}(\dots)\int \frac{du}{2\pi}\left(\frac{\xi^2}{\eta^2}\right)^{iu}\frac{Q_l(u)}{1+C_\phi c_\sigma \pi^{d/2}Q_l(u)}$$

$$= \sum_{l=0}^{\infty}(\dots)i\sum_{\text{poles}}\text{Res}\left[\left(\frac{\xi^2}{\eta^2}\right)^{iu}\left(\frac{1}{1/Q_l(u)+C_\phi c_\sigma \pi^{d/2}}\right)\right]$$

Poles occur when the denominator is zero:

$$\frac{1}{Q_l(u)}+C_\phi c_\sigma \pi^{d/2}=0$$

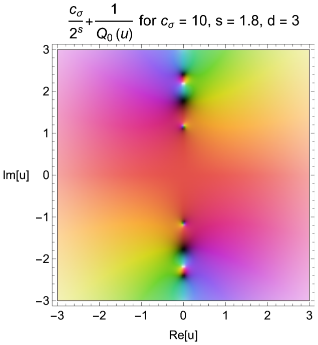
$$\frac{\Gamma\left(\frac{d+s+2l}{4}-iu\right)\Gamma\left(\frac{d+s+2l}{4}+iu\right)}{\Gamma\left(\frac{d-s+2l}{4}+iu\right)\Gamma\left(\frac{d-s+2l}{4}-iu\right)}+\frac{c_\sigma}{2^s}=0$$

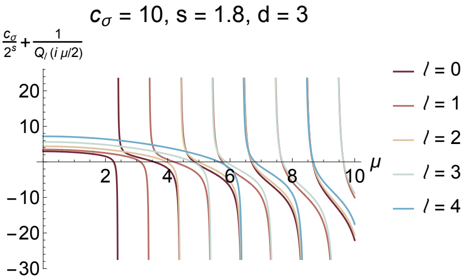
Green's function in various limits

The zeroes of

$$\frac{\Gamma\left(\frac{d+s+2l}{4} - iu\right)\Gamma\left(\frac{d+s+2l}{4} + iu\right)}{\Gamma\left(\frac{d-s+2l}{4} + iu\right)\Gamma\left(\frac{d-s+2l}{4} - iu\right)} + \frac{c_\sigma}{2^s}$$

are on the imaginary axis, and we parametrize them as $u = \frac{i\mu}{2}$ (with $\mu \in \mathbb{R}_{>0}$ for the UHP)





Green's function in various limits

In terms of $u = i\mu/2$, the $u-$integral is

$$G(x, y; \sigma_*) = \sum_{l=0}^{\infty}(\dots)i \sum_{\text{poles}} \text{Res}\left[\left(\frac{\xi^2}{\eta^2}\right)^{-\frac{\mu}{2}}\left(\frac{1}{1/Q_l(i\mu/2) + C_\phi c_\sigma \pi^{d/2}}\right)\right]$$

Taking the limit $\xi \to \infty$ or $\eta \to 0$ greatly simplifies the result

- Sum is dominated by smallest μ pole.
- Smallest μ pole corresponds to the $l = 0$ term of the sum $\implies$ neglect $l > 0$ terms

Green's function in various limits

We obtain the following results for the 3 limits:

$$G(x_1, y, \sigma^*) = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}|y - x_1|^{d-s}}\mu'(c_\sigma)\left(\frac{\delta|y - x_2|}{|x_2 - x_1||y - x_1|}\right)^{\mu(c_\sigma)-\frac{d-s}{2}}$$

$$G(x, x_2, \sigma^*) = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}|x - x_2|^{d-s}}\mu'(c_\sigma)\left(\frac{\delta|x - x_1|}{|x_2 - x_1||x - x_2|}\right)^{\mu(c_\sigma)-\frac{d-s}{2}}$$

$$G(x_1, x_2, \sigma^*) = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}|x_2 - x_1|^{d-s}}\mu'(c_\sigma)\left(\frac{\delta^2}{|x_2 - x_1|^2}\right)^{\mu(c_\sigma)-\frac{d-s}{2}}$$

where δ is a regulator denoting $x - x_1$ and/or $y - x_2$

Green's function in coincident point limit

The coincident point limit doesn't simplify as much as the other 3 limits, so we leave it as

$$G(x,x,\sigma^*) = \sum_{l=0}^{\infty} \frac{C_\phi \Gamma\left(\frac{d-2}{2}\right) |x_2 - x_1|^{d-s} (2l+d-2)\Gamma\left(d-2+l\right)}{2\Gamma\left(d-2\right) l! \left(|x-x_1||x-x_2|\right)^{d-s}} \times \int \frac{du}{2\pi} \frac{Q_l(u)}{\left(1 + C_\phi c_\sigma \pi^{d/2} Q_l(u)\right)}$$

$$= \frac{\Gamma\left(\frac{d}{2}\right) F'(c_\sigma)}{\pi^{\frac{d}{2}}} \left(\frac{|x_2 - x_1|}{|x-x_1||x-x_2|}\right)^{d-s}$$

where

$$F(c_\sigma) = \sum_{l=0}^{\infty} \frac{(2l+d-2)\,\Gamma\left(d-2+l\right)}{\Gamma\left(d-1\right) l!} \int \frac{du}{2\pi} \log\left(1 + C_\phi c_\sigma \pi^{d/2} Q_l(u)\right)$$

Green's function in coincident point limit

The notation $F'(c_\sigma)$ anticipates the relationship between $G(x,x,\sigma^*)$ and the functional determinant:

$$\log\det\left[\frac{C}{|x-y|^{d+s}}+\sigma_*(x)\delta^d(x-y)\right]$$
$$=\sum_{L=1}^{\infty}\frac{1}{L}\int d^dx\sigma_*(x)G^{L-1}(x,x,\sigma_*)$$
$$=-2F(c_\sigma)\log\left(\frac{\delta^2}{|x_1-x_2|^2}\right)$$

where δ regulates the divergent x-integral and again

$$F(c_\sigma)=\sum_{l=0}^{\infty}\frac{(2l+d-2)\,\Gamma\,(d-2+l)}{\Gamma\,(d-1)\,l!}\int\frac{du}{2\pi}\log\Big(1+C_\phi c_\sigma\pi^{d/2}Q_l(u)\Big)$$

Scaling dimensions

Putting all results together, we have

Saddle point equation

$$2\hat{j}G(x_1, x; \sigma_*)G(x_2, x; \sigma_*) = -G(x, x; \sigma_*)G(x_1, x_2; \sigma_*)$$

$$\downarrow$$

$$\boxed{F'(c_\sigma) = -\hat{j}\mu'(c_\sigma)}$$

Scaling dimensions

$$\Delta_j = \frac{N}{2}|x_{12}|\frac{\partial}{\partial|x_{12}|}\left[\frac{1}{2}\log\det\left(\frac{C}{|x-y|^{d+s}} + \sigma(x)\delta^d(x-y)\right) - \hat{j}\log(G(x_1, x_2; \sigma))\right]$$

$$\downarrow$$

$$\boxed{\Delta_j = N\left(F(c_\sigma) + \hat{j}\mu(c_\sigma)\right)}$$

Limiting cases

The scaling dimension is currently stated in terms of 2 complicated functions of c_σ

$$\Delta_j = N\left(F(c_\sigma) + \hat{j}\mu(c_\sigma)\right)$$

$$F(c_\sigma) = \sum_{l=0}^{\infty} \frac{(2l+d-2)\,\Gamma\,(d-2+l)}{\Gamma\,(d-1)\,l!} \int \frac{du}{2\pi} \log\left(1 + C_\phi c_\sigma \pi^{d/2} Q_l(u)\right)$$

$$\mu(c_\sigma) = \text{ smallest positive real root of } 1/Q(i\mu/2) + 2^{-s} c_\sigma$$

For arbitrary c_σ, we can determine Δ_j numerically.

Analytically, we can make progress by taking limits in small and large c_σ (which will correspond to small and large $\hat{j}$)

Small c_σ, $\hat{j}$ limit

When c_σ is small, we have

$$F(c_\sigma) = -\frac{c_\sigma^2 \Gamma\left(\frac{d-s}{2}\right)^4 \Gamma\left(s - \frac{d}{2}\right)}{2^{2s+1}\Gamma(d-s)\Gamma\left(\frac{s}{2}\right)^4 \Gamma\left(\frac{d}{2}\right)}$$

$$\mu(c_\sigma) = \frac{d-s}{2} + \frac{\Gamma\left(\frac{d-s}{2}\right) c_\sigma}{2^{s-1}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{s}{2}\right)} + \mathcal{O}(c_\sigma^2)$$

Then the saddle point equation $F'(c_\sigma) = -\hat{j}\mu'(c_\sigma)$ gives

$$c_\sigma = \frac{\hat{j}2^{s+1}\Gamma(d-s)\Gamma\left(\frac{s}{2}\right)^3}{\Gamma\left(\frac{d-s}{2}\right)^3 \Gamma\left(s - \frac{d}{2}\right)}$$

$$\implies \frac{\Delta_j}{N} = \frac{d-s}{2}\hat{j} + \frac{2\Gamma(d-s)\Gamma\left(\frac{s}{2}\right)^2}{\Gamma\left(\frac{d-s}{2}\right)^2 \Gamma\left(s - \frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}\hat{j}^2 + O(\hat{j}^3)$$

- Matches anomalous dimension computation from standard $1/N$ perturbation theory

Large c_σ, $\hat{j}$ limit

When c_σ is large, we have

$$F(c_\sigma) = \frac{(c_\sigma)^{\frac{d}{s}}\pi}{2^{d-1}\, d\, \Gamma\left(\frac{d}{2}\right)^2 \sin\left(\frac{\pi d}{s}\right)}\left(1 + O\left(\frac{1}{c_\sigma^{2/s}}\right)\right)$$

$$\mu(c_\sigma) = \frac{d+s}{2} + \frac{2^{s+1}\Gamma\left(\frac{d+s}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{-s}{2}\right)c_\sigma} + O\left(\frac{1}{c_\sigma^2}\right)$$

Then the saddle point equation $F'(c_\sigma) = -\hat{j}\mu'(c_\sigma)$ gives

$$c_\sigma = \left(\frac{\hat{j}\; 2^{d+s}\; s\, \Gamma\left(\frac{d+s}{2}\right)\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{\pi d}{s}\right)}{\pi\Gamma\left(-\frac{s}{2}\right)}\right)^{\frac{s}{d+s}}$$

$$\implies \frac{\Delta_j}{N} = \frac{d+s}{2}\hat{j} + A(d,s)\hat{j}^{\frac{d}{d+s}}$$

$$A(d,s) = \frac{2\pi(d+s)}{\Gamma\left(\frac{d}{2}\right)^2 \sin\left(\frac{\pi d}{s}\right) ds}\left(\frac{\Gamma\left(\frac{d+s}{2}\right) s\sin\left(\frac{\pi d}{s}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(-\frac{s}{2}\right)\pi}\right)^{\frac{d}{d+s}}$$

Short-range crossover

Above $s_* = 2 - 2\gamma_\phi^{\mathsf{SR}}$, the low-energy behavior of the model is described by usual short-range $O(N)$ fixed point.

The anomalous dimension of the fundamental operators is $\mathcal{O}(1/N)$, so in our regime where N is infinite, we should see the crossover happen at $s = 2$.

However, as $s \to 2$, our scaling dimensions at large $\hat{j}$ have a linear dependence on $\hat{j}$:

$$\Delta_j = N\left(\frac{d+s}{2}\hat{j} + A(d,s)\hat{j}^{\frac{s}{d+s}} + \dots\right)$$

as opposed to the short range result, which has scaling dimensions that go as $\hat{j}^{\frac{d}{d-1}}$:

$$\Delta_j = N\left(\hat{j}^{\frac{d}{d-1}}A_1(d) + \hat{j}^{\frac{1}{d-1}}A_2(d)\right)$$

Short-range crossover

What explains this discrepancy?

We can investigate by taking a closer look at $\mu(c_\sigma)$, which was the solution to $1/Q(i\mu/2) + 2^{-s}c_\sigma = 0$

Right at $s = 2$, $\mu(c_\sigma)$ simply has a square root dependence on c_σ at leading order

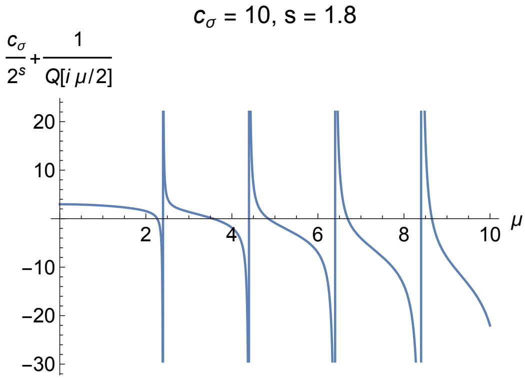
$$\mu(c_\sigma) \underset{s=2}{=} \sqrt{c_\sigma + \left(\frac{d}{2} - 1\right)^2} + \mathcal{O}(1/c_\sigma)$$

Just below $s = 2$, $\mu(c_\sigma)$ at large c_σ saturates to a constant which cannot recover the square root behavior

$$\mu(c_\sigma) = \frac{d+s}{2} + \frac{2^{s+1}\Gamma\left(\frac{d+s}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{-s}{2}\right)c_\sigma} + O\left(\frac{1}{c_\sigma^2}\right)$$

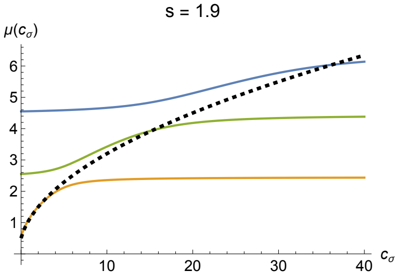
Short-range crossover

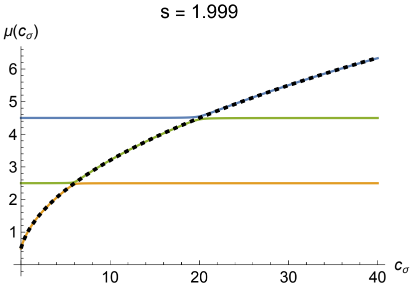
But recall that there was an infinite tower of roots for $1/Q_0(i\mu/2) + 2^{-s}c_\sigma = 0$ and we chose the smallest one.



Short-range crossover

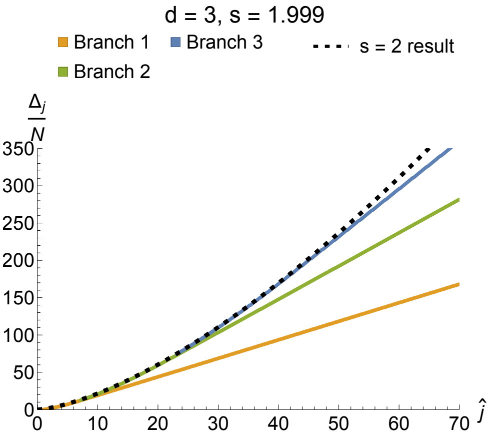
It turns out that if we look numerically at higher solutions of $1/Q_0(i\mu/2) + 2^{-s}c_\sigma = 0$ and "glue" them together, it recovers the square root behavior as we take $s \to 2$





Short-range crossover

This "gluing" in turn recovers the short-range scaling dimension behavior



Short-range crossover

To understand more fully how the transition occurs, we need to compute subleading corrections to the scaling dimensions by including the determinant of fluctuations around the saddle point

Note that when N isn't strictly infinite, the crossover will happen at $s_* = 2 - O(1/N) < 2$

Cylinder calculation

As a check of the flat space calculation, we can compute scaling dimensions by mapping the problem to the cylinder using a Weyl transformation

We start in flat space and, following the procedure in [Paulos, Rychkov, van Rees, Zan 1509.00008], write the long-range action in $D = d + 2 - s$ dimensions with a defect interaction localized in the d-dimensional subspace:

$$S = \frac{\Gamma\left(\frac{s}{2}\right)}{(4\pi)^{1-\frac{s}{2}}} \int d^d x \; d^{2-s} w \frac{1}{2} (\partial_\mu \Phi^I)^2 + \frac{g}{4} \int d^d x (\phi^I \phi^I)^2$$

where Φ^I is a D-dimensional extension of ϕ^I:

$$\Phi^I(x, w = 0) = \phi^I(x)$$

Cylinder calculation

We then perform a Weyl transformation to the cylinder $R \times S^{D-1}$ and perform a similar analysis as [Cuomo, Mezei, Raviv-Moshe 2108.06579], with coordinates on S^{D-1} given by

$$ds^2 = d\theta^2 + \sin^2\theta d\Omega_{d-1} + \cos^2\theta d\Omega_{1-s}$$

Look at fixed large charge states by introducing fixed chemical potential μ

$$S \to S + \frac{\Gamma\left(\frac{s}{2}\right)}{(4\pi)^{1-\frac{s}{2}}} \int_{R\times S^{D-1}} \left[i\mu \left(\dot{\Phi}^1\Phi^2 - \dot{\Phi}^2\Phi^1\right) - \frac{\mu^2}{2}\left((\Phi^1)^2 + (\Phi^2)^2\right)\right]$$

where we introduced a background gauge field in the time direction:
$\partial_0\Phi^1 \to D_0\Phi^1 = \partial_0\Phi^1 + i\mu\Phi^2$, $\partial_0\Phi^2 \to D_0\Phi^2 = \partial_0\Phi^2 - i\mu\Phi^1$

Cylinder calculation

Expand field about the ansatz

$$\Phi^1 + i\Phi^2 = \sqrt{2} f(\theta), \qquad \Phi^3 = \Phi^4 = \cdots = \Phi^N = 0$$

Plug ansatz into action and extremize the following

$$\Delta_j = \left[\frac{S_{\text{cl}}}{T} + \mu j \right]_{\mu=\mu^*}$$

We find that the scaling dimensions match our flat space results for the 2 regimes we checked

- ϵ expansion about lower critical dimension $s = \frac{d+\epsilon}{2}$
- Large N expansion

Conclusion and future directions

- We studied the spectrum of large charge operators in the double-scaling limit of large j and large N in the long-range $O(N)$ model
- We found scaling dimensions interpolate between $\Delta_j \sim \frac{(d-s)}{2} j$ at small $\hat{j}$ and $\Delta_j \sim \frac{(d+s)}{2} j$ at large $\hat{j}$, which is a qualitatively different behavior from the one found in the short range version of the $O(N)$ model
- In future work, would be interesting to look at subleading corrections to scaling dimensions to see in detail how the local behavior is recovered from the $s \to 2$ limit

Thank you!