Reviewing spinning correlators in CFTs in a sector of large global charge

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Outline

1. The O(2) sector at large charge
   - Classical treatment
   - Canonical quantization
2. Path integral methods
   - Two-point functions
     - $\langle Q | Q \rangle$ correlator
     - $\left\langle {}^{Q}_{\ell_2 m_2} \middle| {}^{Q}_{\ell_1 m_1} \right\rangle$ correlators
3. Correlators with current insertions
4. Conclusions

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• Consider a conformal field theory (CFT) in D-dimensional flat space with an $O(2)$ internal symmetry. Generically, it can be a subgroup of a larger global symmetry.

• Since flat space is conformally equivalent to the cylinder $\mathbb{R} \times S^{D-1}$ we will work in the cylinder frame.

• Consider the state $|Q\rangle$ generated by the scalar primary $\mathscr{O}^Q$ with $O(2)$ charge Q.

• We are interested in correlators of such primaries at long distances, the easiest of which can be expressed on the cylinder as

$$\langle Q, \infty | Q, -\infty \rangle = \lim_{\beta \longrightarrow \infty} \langle Q | e^{-\beta H_{\text{cyl}}} | Q \rangle \, .$$

• There is strong indication that as Q becomes very large, this correlator on the cylinder has a description in terms of a weakly coupled effective field theory (EFT):

$$\frac{SO(D+1,1)\times U(1)_Q}{SO(D)\times U(1)_{D+\mu Q}}\,,\quad \text{valid for energy scales} \qquad \frac{1}{R}\ll E\ll \mu\sim\frac{Q^{1/(D-1)}}{R},$$

where R is the cylinder radius.

Arxiv: 1505.01537, 2008.03308, 1611.02912

• The parameter $\mu(Q)$ can be interpreted as the chemical potential dual to the quantum number Q which is the fixed control parameter. The symmetry-breaking pattern is known as the *conformal superfluid phase*.

- The corresponding EFT in Euclidean spacetime has been computed in terms of a Goldstone field $\chi = -i\mu\tau + \pi(\tau, \boldsymbol{n})$.
- $\pi(\tau, \boldsymbol{n})$ are the fluctuations over the fixed-charge ground state $\chi = -i\mu\tau$.

Arxiv: 1505.01537, 1611.02912

- The action of the EFT is

$$S = -c_1 \int_{\mathbb{R}\times S^{D-1}} \mathrm{d}\tau \mathrm{d}S \, (-\,\partial_\mu \chi\, \partial^\mu \chi)^{D/2} + \text{curvature couplings},$$

where c_1 an unknown Wilsonian coefficient which depends on the ultraviolet (UV) theory (*i.e.* the starting CFT_D) and $\mathrm{d}S = R^{D-1}\mathrm{d}\Omega$.

• This is to be interpreted as an action for the fluctuation $\pi(\tau, \boldsymbol{n})$ with cutoff $\Lambda \sim \mu$, so that a hierarchy is generated, and it is controlled by the dimensionless ratio $(R\mu) \gg 1$.

• Every observable in the EFT is expressed as an expansion in inverse powers of μ. In particular, the ground-state action takes the form

$$S^{\circledast} = \left(\frac{\tau_2 - \tau_1}{R}\right) \sum_{r=0}^{\infty} \alpha_r (R\mu)^{D-2r},$$

where the coefficients α_r depend on c_1 and all other Wilsonian coefficients associated to curvature terms.

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• Neglecting curvature couplings and expanding to quadratic order in $\pi(\tau, \boldsymbol{n})$, the EFT Lagrangian reads

$$\mathscr{L} = -c_1\mu^D - ic_1\mu^{D-1}D\dot{\pi} + c_1\mu^{D-2}\frac{D(D-1)}{2}\left(\dot{\pi}^2 + \frac{1}{D-1}(\partial_i\pi)^2\right) + \mathcal{O}(\mu^{D-3}).$$

• The conjugate momentum to π is defined in the usual manner from the quadratic Lagrangian

$$\Pi = i\frac{\delta\mathscr{L}}{\delta\dot{\pi}}\bigg|_{\text{lin}} = c_1 D\mu^{D-1} + ic_1 D(D-1)\mu^{D-2}\dot{\pi}.$$

• At leading order, this gives rise to the usual canonical Poisson brackets.

• The fields π and Π can be decomposed into a complete set of solutions of the equations of motion (EOM) :

$$\begin{aligned}\pi(\tau,\boldsymbol{n}) &= \pi_0 - \frac{i\Pi_0\tau}{c_1\Omega_D R^{D-1}D(D-1)\mu^{D-2}} \\ &+ \frac{1}{\sqrt{c_1 R^{D-1}D(D-1)\mu^{D-2}}}\sum_{\ell\geq 1,m}\left(\frac{a_{\ell m}}{\sqrt{2\omega_\ell}}e^{-\omega_\ell\tau}Y_{\ell m}(\boldsymbol{n}) + \frac{a^*_{\ell m}}{\sqrt{2\omega_\ell}}e^{\omega_\ell\tau}Y^*_{\ell m}(\boldsymbol{n})\right), \\ \Pi(\tau,\boldsymbol{n}) &= c_1 D\mu^{D-1} + \frac{\Pi_0}{\Omega_D R^{D-1}} \\ &+ i\sqrt{\frac{c_1 D(D-1)\mu^{D-2}}{R^{D-1}}}\sum_{\ell,m}\left(-a_{\ell m}\sqrt{\frac{\omega_\ell}{2}}e^{-\omega_\ell\tau}Y_{\ell m}(\boldsymbol{n}) + a^*_{\ell m}\sqrt{\frac{\omega_\ell}{2}}e^{\omega_\ell\tau}Y^*_{\ell m}(\boldsymbol{n})\right),\end{aligned}$$

Arxiv: 1610.04495

• π_0 and Π_0 are constant zero modes of the fields, $\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ is the volume of the $D-1$-sphere and the $Y_{\ell m}$ are hyperspherical harmonics.

- The dispersion relation for the oscillator modes reads

$$R\omega_\ell = \sqrt{\frac{\ell(\ell + D - 2)}{(D-1)}} \,.$$

- Adding higher-curvature terms in the EFT will add subleading corrections in $1/Q$ to this expression.
- The complex Fourier coefficients $a_{\ell m}$ can be extracted as follows:

$$a_{\ell m} = \sqrt{\frac{c_1 D(D-1)\mu^{D-2}}{2\omega_\ell\, R^{D-1}}} \int \mathrm{d}S \left[\pi(\tau, \boldsymbol{n})\, \partial_\tau \left(Y^*_{\ell m}(\boldsymbol{n}) e^{\omega_\ell \tau}\right) - \partial_\tau \pi(\tau, \boldsymbol{n}) Y^*_{\ell m}(\boldsymbol{n}) e^{\omega_\ell \tau}\right].$$

- The canonical Poisson bracket between π and Π corresponds to the Fourier mode brackets $\{a_{\ell m}, a^\dagger_{\ell' m'}\} = \delta_{\ell\ell'}\delta_{mm'}$.

• The classical $O(2)$ current and conserved charge are

$$J^{\mu} = \frac{\delta \mathscr{L}}{\delta\, \partial_{\mu}\chi}\,, \qquad Q = \int \mathrm{d}S\, J^{\tau} = c_1 D \Omega_D (R\mu)^{D-1} + \Pi_0 .$$

• The leading contribution to the charge comes from the homogeneous term corresponding to the ground state.
• This relates the EFT scale μ to the ground state charge Q_0 as

$$\mu = \left[\frac{Q_0}{c_1 D R^{D-1} \Omega_D}\right]^{1/(D-1)} .$$

• At leading order in the fluctuations, the charge Q of a generic solution of the EOM depends only additively on the zero mode Π_0,

$$Q = Q_0 + \Pi_0 \ .$$

• Using the state-operator correspondence, we can compute the scaling dimension of the operator $\mathscr{O}^Q$ from the cylinder Hamiltonian.

• A generic solution of the EOM corresponds to an operator with scaling dimension

$$\Delta = RE_{\text{cyl}} = \Delta_0 + \frac{\partial \Delta_0}{\partial Q_0}\Pi_0 + \frac{1}{2}\frac{\partial^2 \Delta_0}{\partial Q_0\,\partial Q_0}\Pi_0^2 + R\sum_{\ell\geq 1,m}\omega_\ell a^*_{\ell m}a_{\ell m},$$

• Where

$$\Delta_0 = c_1(D-1)\Omega_D(\mu R)^D + \mathcal{O}((R\mu)^{D-2}), \quad \frac{\partial \Delta_0}{\partial Q_0} = R\mu, \quad \frac{\partial^2 \Delta_0}{\partial Q_0\,\partial Q_0} = \frac{1}{c_1 D(D-1)\Omega_D(R\mu)^{D-2}}.$$

• The quantity Δ_0 corresponds to the leading "classical" contribution to the action.

• Canonical quantization in the cylinder frame is obtained by τ-slicing, associating a Hilbert space $\mathscr{H}_Q$ to each fixed τ.
• This poses no conceptual problems since the cylinder is a direct product of the time direction and a curved manifold.
• The mode coefficients in the decompositions are promoted to field operators with non-vanishing commutators,

$$[\pi_0, \Pi_0] = i, \qquad \left[a_{\ell m}, a^{\dagger}_{\ell' m'}\right] = \delta_{\ell\ell'}\delta_{mm'}.$$

• These are equivalent to the canonical equal-τ commutator $[\pi(\tau, \boldsymbol{n}), \Pi(\tau, \boldsymbol{n}')] = i\delta_{S^{D-1}}(\boldsymbol{n}, \boldsymbol{n}')$, where $\delta_{S^{D-1}}(\boldsymbol{n}, \boldsymbol{n}')$ is the invariant delta function on S^{D-1}.

• To build a representation of the Heisenberg algebra we start with a vacuum $|Q\rangle$ which satisfies

$$a_{\ell m}\,|Q\rangle = \Pi_0\,|Q\rangle = 0.$$

• As we are in finite volume, the $O(2)$ charge is a well-defined operator acting on $\mathscr{H}_Q$ as

$$\mathscr{Q} = \int \mathrm{d}S\,\Pi(\tau, \boldsymbol{n}) = Q_0 \mathbb{1} + \Pi_0\ , \qquad \mathscr{Q}\,|Q\rangle = Q_0\,|Q\rangle\,.$$

• The non-zero charge of the vacuum can be increased by acting with the mode π_0, which is the only one carrying non-zero charge,

$$[\mathscr{Q}, \pi_0] = -i\ , \qquad [\mathscr{Q}, a_{\ell m}] = \left[\mathscr{Q}, a^\dagger_{\ell m}\right] = 0.$$

• This does not lead to a degeneracy in the spectrum since these states live at the cut-off.

• The quantized quadratic Hamiltonian corresponding to the classical expression of the scaling dimension can be written as the sum of a normal-ordered operator $:H:$ and a vacuum contribution

$$D = R :H: + \Delta_1 \mathbb{1}, \qquad \text{where} \qquad \Delta_1 := \frac{1}{2} \sum_{\ell \geq 1, m} (R\omega_\ell).$$

• The vacuum contribution needs regulation and has physical consequences.

• This first appeared in $D = 3$ in
Arxiv: 1611.02912
• And $D = 4, 5, 6$ in
Arxiv: 2010.00407

• From the point of view of the large-charge expansion, the one-loop correction comes at order $\mathcal{O}\left(Q^0\{\log Q\}\right)$.
• We need to keep track in the tree-level computation also of all the curvature terms up to this order.
• In $D = 3$ we know that

$$\Delta_0 = d_{3/2} Q^{3/2} + d_{1/2} Q^{1/2} + \mathcal{O}\left(Q^{-1/2}\right),$$

• In general there will be $\lceil (D+1)/2 \rceil$ terms with positive Q-scaling, each controlled by a Wilsonian coefficient
Arxiv: 2008.03308

• The commutators between D and the various modes show which ones generate excited states when acting on the vacuum:

$$[D, a_{\ell m}] = -R\omega_\ell a_{\ell m}, \qquad \left[D, a^\dagger_{\ell m}\right] = R\omega_\ell a^\dagger_{\ell m},$$

$$[D, \pi_0] = -i\frac{\partial \Delta_0}{\partial Q_0} - i\frac{\partial^2 \Delta_0}{\partial Q_0^2}\Pi_0, \qquad [D, \Pi_0] = 0.$$

• The Hilbert space $\mathscr{H}_Q$ of the theory is described as the Fock space generated by states of the form

$$a^\dagger_{\ell_1 m_1} \dots a^\dagger_{\ell_k m_k} |Q\rangle$$

with charge Q_0 and scaling dimension

$$\Delta = \Delta_0 + \Delta_1 + \sum_{i=1}^{k} (R\omega_{\ell_k}).$$

• These states are also known as *superfluid phonon* states in the literature.

- From the CFT_D perspective, these states correspond to primary operators with different quantum numbers than $\mathscr{O}^Q$ but same $O(2)$ charge.
- The only exception are states including at least one $a^\dagger_{1m}$ which are descendants since their energy is $R\omega_1 = 1$.
- Not all phonon states can be described within the EFT. When the ℓ-quantum number becomes too large, their contribution $R\omega_\ell$ can compete with the leading Δ_0 term, breaking the large-Q expansion.
- Phonon states with comparable energy ω_ℓ should be excluded from the EFT. This sets a cutoff for the ℓ-quantum number as

$$\ell_{\text{cutoff}} \sim Q^{1/(D-1)}.$$

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• The structure of the spectrum and the existence of the above-mentioned charged spinning primaries is a direct prediction of the superfluid hypothesis for generic a $O(2) - \mathrm{CFT}_D$.

• Canonical quantization is the appropriate framework for this discussion. One will expect corrections to scaling dimensions and the spectrum structure coming from interactions corresponding to subleading terms in large Q.

• These are best discussed within a path integral formulation.

Outline

• An equivalent basis of the fixed-τ Hilbert space $\mathscr{H}_Q$ is given by the field/momentum eigenstates

$$\chi(\boldsymbol{n})\,|\chi\rangle = \chi(\boldsymbol{n})\,|\chi\rangle\,, \qquad \Pi(\boldsymbol{n})\,|\Pi\rangle = \Pi(\boldsymbol{n})\,|\Pi\rangle\,.$$

• Their bracket is fixed by the canonical commutation relations,

$$\langle\chi|\Pi\rangle = e^{i\int \mathrm{d}S\chi\Pi}.$$

• Generically, the vacuum $|Q\rangle$ is a superposition of momentum eigenstates without the Π_0-component

$$|Q\rangle = \mathscr{N}_Q \int \mathcal{D}\Pi\,\delta(\Pi_0)\Psi_Q(\Pi)\,|\Pi\rangle\,,$$

where $\mathscr{N}_Q$ is a normalization factor.

- In the limit of large separation, $\tau \to \infty$, correlators will not depend on the specifics of the vacuum wave function Ψ_Q. Without loss of generality, we can take $\Psi_Q = 1$.
- The overlap of $|Q\rangle$ with field eigenstates is then given by

$$\langle\chi|Q\rangle = \begin{cases} \mathcal{N}_Q \exp\left\{\frac{iQ}{\Omega_D R^{D-1}} \int \mathrm{d}S\, \chi\right\} & \text{if } \chi \text{ is constant,} \\ 0 & \text{otherwise} \end{cases}$$

- Generically, on a τ-slice, the zero-modes of any field configuration can be separated by integrating on the sphere

$$\chi_0 = \int \mathrm{d}S\, \chi.$$

- This bracket sets the correct boundary conditions for any correlators in the path integral representation of the form $\langle Q| \ldots |Q\rangle$.

• The vacuum correlator with cylinder times $\tau_2 > \tau_1$ can be written as follows:

$$\langle Q|e^{-\frac{(\tau_2-\tau_1)}{R}D}|Q\rangle = |\mathscr{N}_Q|^2 \int \mathcal{D}\chi \exp\left[-S[\chi] - \frac{iQ}{\Omega_D R^{D-1}} \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int \mathrm{d}S\, \dot{\chi}\right] :- \mathscr{A}(\tau_1, \tau_2)\,,$$

• We introduced the notation $\mathscr{A}(\tau_1, \tau_2)$ for future convenience.
• The path integral can be computed as a saddle-point expansion around a field configuration $\chi^{\text{saddle}}(\tau, \boldsymbol{n})$ which is a solution to the minimization problem

$$\delta S[\chi] = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \mathrm{d}S \left(-\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \chi)}\right)\delta\chi + \int \mathrm{d}S \left(\frac{\partial \mathscr{L}}{\partial(\partial_\tau \chi)} + \frac{iQ}{\Omega_D R^{D-1}}\right)\delta\chi\Big|_{\tau_1}^{\tau_2}.$$

• The bulk EOM requires the (Euclidean) $O(2)$ conserved current:

$$\frac{\partial \mathscr{L}}{\partial(\partial^\mu \chi)} = c_1 D(-\partial_\mu \chi\, \partial_\mu \chi)^{D/2-1}\, \partial_\mu \chi = J_\mu$$

to be divergence-free.

• The general solution compatible with the boundary conditions is the homogeneous configuration $\chi^{\circledast}(\tau, \boldsymbol{n}) = -i\mu\tau + \pi_0$, with π_0 constant and the parameter μ fixed by the boundary condition to:

$$c_1 D \mu^{D-1} = \frac{Q}{\Omega_D R^{D-1}},$$

• The action expansion for this ground-state fluctuation $\chi(\tau, \boldsymbol{n}) = \chi^{\circledast}(\tau, \boldsymbol{n}) + \pi(\tau, \boldsymbol{n})$ is, to quadratic order,

$$S = \Delta_0 \frac{\tau_2 - \tau_1}{R} + c_1 \mu^{D-2} \frac{D(D-1)}{2} \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int \mathrm{d}S \left(\dot{\pi}^2 + \frac{1}{(D-1)R^2} (\partial_i \pi)^2 \right) + \mathscr{O}(\mu^{D-3}).$$

• The boundary term eliminates the linear term and correspondingly the zero-mode terms as expected.

• The normalization $\mathscr{N}_Q$ is chosen such that the correlator takes the form

$$\mathscr{A}(\tau_1, \tau_2) = R^{-2(\Delta_0+\Delta_1+\dots)} \exp\left\{-\frac{(\tau_2-\tau_1)}{R}\left[\Delta_0+\Delta_1+\dots\right]\right\},$$

• The correction Δ_1 is the Casimir energy of the fluctuation π around the homogeneous ground state χ.
• This corresponds to the two-point function in $\mathbb{R}^D$ normalized to unity. The Weyl map to $\mathbb{R}^D$ can then be performed as

$$\langle \mathscr{O}^{-Q}(x_2)\mathscr{O}^{Q}(x_1)\rangle_{\mathbb{R}^D} = \left(\frac{|x_1|}{R}\right)^{-\Delta_Q}\left(\frac{|x_2|}{R}\right)^{-\Delta_Q}\langle \mathscr{O}^{-Q}(\tau_2, \boldsymbol{n}_2)\mathscr{O}^{Q}(\tau_1, \boldsymbol{n}_1)\rangle_{\text{cyl}}\,.$$

• In the state-operator correspondence, the reference states $|Q\rangle$ and $\langle Q|$ correspond to insertions of scalar primaries at $\tau = \pm\infty$:

$$|Q\rangle :- \mathscr{O}^{Q}(-\infty)\,|0\rangle\,, \qquad \langle Q| :- \langle 0|\,\mathscr{O}^{Q}(\infty)^{\dagger},$$

• Correlators of one-phonon states are obtained by acting with a single creation operator $a^\dagger_{\ell m}$ on the vacuum $|Q\rangle$:

$$\left|{}^{\,Q}_{\ell m}\right\rangle = a^\dagger_{\ell m}\,|Q\rangle\,, \qquad\qquad \text{where} \qquad\qquad \left|{}^{\,Q}_{00}\right\rangle = |Q\rangle\,.$$

• In canonical quantization, using the commutation relations of the $a_{\ell m}$ and $a^\dagger_{\ell m}$ the two-point function is found to be

$$\begin{aligned}\left\langle {}^{\;Q}_{\ell_2 m_2}\right|\left.{}^{\;Q}_{\ell_1 m_1}\right\rangle &= \langle Q| a_{\ell_2 m_2}\, e^{-(\tau_2-\tau_1)D/R} a^\dagger_{\ell_1 m_1}|Q\rangle = \mathscr{A}(\tau_1,\tau_2)\, e^{-(\tau_2-\tau_1)\omega_\ell}\delta_{\ell_1\ell_2}\delta_{m_1 m_2}\\ &= R^\Delta e^{-\Delta(\tau_2-\tau_1)/R}\delta_{\ell_1\ell_2}\delta_{m_1 m_2}\,,\end{aligned}$$

• Δ is the conformal dimension, consistent with the general structure of a conformal two-point function on the cylinder.

• This is true to quadratic order in the Hamiltonian. We expect loop corrections to shift the spectrum in a complicated way.
• It is convenient to formulate the correlator as a path integral. This can be done in a straightforward manner by expressing $a_{\ell m}$ in terms of the fields, so that one finds

$$\langle {}^{Q}_{\ell_2 m_2} | {}^{Q}_{\ell_1 m_1} \rangle = \frac{c_1 D(D-1)\mu^{D-2}}{2R^{D-1}\sqrt{\omega_{\ell_2}\omega_{\ell_1}}} \int \mathrm{d}S(\boldsymbol{n}_2) \int \mathrm{d}S(\boldsymbol{n}_1)\, Y^*_{\ell_2 m_2}(\boldsymbol{n}_2)\, Y_{\ell_1 m_1}(\boldsymbol{n}_1)$$
$$\mathscr{A}(\tau_1, \tau_2) \lim_{\substack{\tau \to \tau_1 \\ \tau' \to \tau_2}} (\omega_{\ell_2} - \partial_{\tau'})(\omega_{\ell_1} + \partial_\tau) \langle \pi(\tau', \boldsymbol{n}_2)\pi(\tau, \boldsymbol{n}_1)\rangle ,$$

• The two-point function of the Goldstone fluctuations is defined as

$$\langle \pi(\tau_2, \boldsymbol{n}_2)\pi(\tau_1, \boldsymbol{n}_1)\rangle = \frac{1}{\langle Q, \tau_2 | Q, \tau_1 \rangle} \int \mathcal{D}\pi\, \pi(\tau_2, \boldsymbol{n}_2)\pi(\tau_1, \boldsymbol{n}_1)\, e^{-S[\pi]}.$$

- The information about the spectrum is contained in the full π-fluctuation two-point function.
- In this formalism, the result in the canonical quantization is replicated by using the tree-level propagator, which on the cylinder reads

$$\langle \pi(\tau_2, \boldsymbol{n}_2)\pi(\tau_1, \boldsymbol{n}_1)\rangle = \frac{1}{c_1 D(D-1)(\mu R)^{D-2}}\left(\sum_{\ell=1}^{\infty}\sum_{m} e^{-\omega_\ell|\tau_2-\tau_1|}\frac{Y_{\ell m}(\boldsymbol{n}_2)^* Y_{\ell m}(\boldsymbol{n}_1)}{2R\omega_\ell} - \frac{|\tau_2-\tau_1|}{2R\Omega_D}\right).$$

• The computation of $\langle {}^{Q}_{\ell_2 m_2} | {}^{Q}_{\ell_1 m_1} \rangle$ in canonical quantization is generalized to states with more phonon excitations.
• For two phonon excitations

$$\langle {}^{Q}_{(\ell_2 m_2)\otimes(\ell_2' m_2')} | {}^{Q}_{(\ell_1 m_1)\otimes(\ell_1' m_1')} \rangle = \langle Q | a_{\ell_2 m_2} a_{\ell_2' m_2'} \, e^{-(\tau_2-\tau_1)D/R} a^\dagger_{\ell_1' m_1'} a^\dagger_{\ell_1 m_1} | Q \rangle$$

$$= \mathscr{A}(\tau_1, \tau_2) \, e^{-(\tau_2-\tau_1)\left(\omega_{\ell_2}+\omega_{\ell_2'}\right)} \left(\delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \delta_{\ell_1' \ell_2'} \delta_{m_1' m_2'} + \delta_{\ell_1 \ell_2'} \delta_{m_1 m_2'} \delta_{\ell_1' \ell_2} \delta_{m_1' m_2} \right) .$$

• For states with higher numbers of phonon excitations the energy is just corrected accordingly and there is a sum over all possible permutations of Kronecker deltas.

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• Working with an EFT at large charge guarantees that the physics at the fixed point is captured by a free theory.

• Hence, we are able to explicitly compute three- and four-point functions with current insertions between spinful large-charge primaries $\mathscr{O}^{Q}_{\ell m}$ for a strongly coupled system using only the operator algebra.

• Some of these correlators have already appeared in the literature in the scalar case $\ell = 0$.

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- The classical conserved currents in the model are

$$J_\mu = c_1 D(-\partial_\mu\chi\,\partial^\mu\chi)^{D/2-1}\,\partial_\mu\chi,$$
$$T_{\mu\nu} = c_1\left\{D(-\partial_\mu\chi\,\partial^\mu\chi)^{D/2-1}\,\partial_\mu\chi\,\partial_\mu\chi + g_{\mu\nu}(-\partial_\mu\chi\,\partial^\mu\chi)^{D/2}\right\}.$$

- Their integrals are related to the conserved charges of $\mathbb{R}^d$.

- On the cylinder, the currents are expanded in fluctuations around $\chi(\tau, \boldsymbol{n})$ up to quadratic order as

$$
\begin{aligned}
J_\tau &= -i\frac{Q}{\Omega_D R^{D-1}}\left\{1+\frac{i}{\mu}(D-1)\dot{\pi}-\frac{(D-2)(D-1)}{2\mu^2}\left[\dot{\pi}^2+\frac{(\partial_i\pi)^2}{R^2(D-1)}\right]+\mathcal{O}\left(\mu^{-3}\right)\right\},\\
J_i &= \frac{Q}{\Omega_D R^{D-1}}\left\{\frac{1}{\mu R}\,\partial_i\pi+\frac{i}{\mu}\frac{(D-2)}{\mu R}\,\dot{\pi}\,\partial_i\pi+\mathcal{O}\left(\mu^{-3}\right)\right\},\\
T_{\tau\tau} &= -\frac{\Delta_0}{\Omega_D R^D}\left\{1+i\frac{D}{\mu}\dot{\pi}-\frac{D(D-1)}{2\mu^2}\left[\dot{\pi}^2+\frac{(D-3)\,(\partial_i\pi)^2}{R^2(D-1)^2}\right]+\mathcal{O}\left(\mu^{-3}\right)\right\},\\
T_{\tau i} &= -i\frac{\Delta_0}{\Omega_D R^D}\left[\frac{1}{\mu R}\frac{D}{D-1}\partial_i\pi+\frac{i}{\mu}\frac{D}{\mu R}\dot{\pi}\partial_i\pi+\mathcal{O}\left(\mu^{-3}\right)\right]\\
T_{ij} &= \frac{\Delta_0}{\Omega_D R^D}\frac{h_{ij}}{(D-1)}\left\{1+i\frac{D}{\mu}\dot{\pi}-\frac{D(D-1)}{2\mu^2}\left[\dot{\pi}^2+\frac{(\partial_i\pi)^2}{R^2(D-1)}\right]+\mathcal{O}\left(\mu^{-3}\right)\right\}\\
&\quad+\frac{\Delta_0}{\Omega_D R^D}\frac{1}{(\mu R)^2}\frac{D}{(D-1)}\left\{\partial_i\pi\,\partial_j\pi+\mathcal{O}\left(\mu^{-3}\right)\right\},
\end{aligned}
$$

- h_{ij} is the metric on the $D-1$-sphere and homogeneity of χ guarantees that $T_{\tau i} = J_i = 0$ at leading order.

• We will discuss correlators of these currents in the canonically quantized setting which is sufficient for leading-order results.

• Integrating J_τ or $T_{\tau\tau}$ over spatial slices gives rise to the charge operators D and $\mathscr{Q}$.

• When inserted at time τ they measure the scaling dimension and the $O(2)$-charge of any operator insertion contained in the half-cylinder $(\tau, -\infty) \times S^{D-1}$. This fact is expressed by the Ward identities

$$\langle \mathscr{Q}(\tau) \prod_i \mathscr{O}_i(\tau_i, \boldsymbol{n}_i) \rangle = \sum_{\tau_i < \tau} Q_i \, \langle \prod_i \mathscr{O}_i(\tau_i, \boldsymbol{n}_i) \rangle \, ,$$

$$\langle D(\tau) \prod_i \mathscr{O}_i(\tau_i, \boldsymbol{n}_i) \rangle = \sum_{\tau_i < \tau} \Delta_i \, \langle \prod_i \mathscr{O}_i(\tau_i, \boldsymbol{n}_i) \rangle \, .$$

• These identities hold order by order in a loop expansion and can be used to constrain correlators with insertions of the currents.

- We first compute the correlator of two spinning primaries $\mathscr{O}^{Q}_{\ell m}|0\rangle = a^{\dagger}_{\ell m}|Q\rangle$ inserted at times τ_1, τ_2 with an insertion of $J_\mu(\tau, x)$ at time $\tau_1 < \tau < \tau_2$. To leading order one finds

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_\tau(\tau, \boldsymbol{n}) \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -i \frac{Q}{\Omega_D R^{D-1}} \Bigg\{ \mathscr{A}_{\Delta_Q + R\omega_{\ell_2}}(\tau_1, \tau_2)\, \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$$
$$+ \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau)\,(D-1)(D-2)\Omega_D \frac{R\sqrt{\omega_{\ell_2}\omega_{\ell_1}}}{2D\Delta_0} \left[Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}) - \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{R^2(D-1)\omega_{\ell_2}\omega_{\ell_1}} \right] \Bigg\},$$

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_i(\tau, \boldsymbol{n}) \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = i \frac{Q(D-2)}{2\Delta_0 R^{D-1} D}\, \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \left[\sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}}\, Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n}) - (1 \leftrightarrow 2)^* \right].$$

- For later convenience we have introduced

$$\mathscr{A}_{\Delta_Q + R\omega_{\ell_2}}(\tau_1, \tau_2) \coloneqq \mathscr{A}(\tau_1, \tau_2)\, e^{-(\tau_2 - \tau_1)\,\omega_{\ell_2}}, \qquad \mathscr{A}^{\Delta_2}_{\Delta_1}(\tau_1, \tau_2 \mid \tau) \coloneqq e^{-\Delta_2(\tau_2 - \tau)/R - \Delta_1(\tau - \tau_1)/R}.$$

• This generalizes $\mathscr{A}(\tau_1, \tau_2) = \mathscr{A}_{\Delta}(\tau_1, \tau_2)$ defined by the two-point function $\langle Q|Q\rangle$.
• If we integrate J_τ over the sphere, we obtain the conserved charge, so the integral over the three-point function with a J_τ is fixed by the Ward identity, giving us a consistency check:

$$\int \mathrm{d}S(\boldsymbol{n})\, \langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_\tau(\tau, \boldsymbol{n}) \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -iQ\, \mathscr{A}_{\Delta_Q + R\omega_{\ell_2}}(\tau_1, \tau_2)\, \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}.$$

• The current is a sum of a classical piece and quantum corrections. The classical part is homogeneous and, by charge conservation, time-independent. It follows that the classical contribution to the three-point function must be proportional to the two-point function

$$\left\langle {}^{Q}_{\ell_2 m_2} \middle| {}^{Q}_{\ell_1 m_1} \right\rangle = \mathscr{A}_{\Delta_Q + R\omega_{\ell_2}}(\tau_1, \tau_2)\, \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}.$$

• The quantum piece will in general give a contribution that has the same tensor structure as the left-hand side (LHS) and since it is not homogeneous, can be decomposed into spherical harmonics. Moreover, by charge conservation its integral must vanish.
• In the same way, the classical piece of J_i is zero so we only have the inhomogeneous quantum contribution in the J_i correlator.

• The separation into a homogeneous classical part and a space-dependent quantum contribution applies to any physical observable.
• The special case of $\ell_i = 0$ corresponds to the scalar ground state $|Q\rangle$ and Ward identities guarantee that $\langle Q|J_i|Q\rangle = 0$ to all orders.
• From the above relations one can extract the corresponding operator product expansion (OPE) coefficient:

$$C_{\mathcal{O}^{Q}_{\ell m} J_\tau}^{\mathcal{O}^{Q}_{\ell m}} = \frac{\langle \mathcal{O}^{-Q}_{\ell m} J_\tau(\tau, \boldsymbol{n}) \mathcal{O}^{Q}_{\ell m} \rangle}{\langle \mathcal{O}^{-Q}_{\ell m} \mathcal{O}^{Q}_{\ell m} \rangle} = -i \frac{Q}{\Omega_D R^{D-1}}.$$

• These correlators can also be computed for higher-phonon states.

• We next compute the correlators with two insertions of J_μ at cylinder times $\tau < \tau'$ between insertions of $\mathscr{O}^{Q}_{\ell m}$ at τ_1 and τ_2 such that $\tau_2 > \tau > \tau' > \tau_1$. Two insertions of J_τ result in

$$\begin{aligned}\langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_\tau(\tau,\boldsymbol{n}) J_\tau(\tau',\boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1}\rangle &= -\,\mathscr{A}_{\Delta_Q+R\omega_{\ell_2}}(\tau_1,\tau_2)\,\frac{Q^2}{\Omega_D^2 R^{2D-2}}\,\delta_{\ell_1\ell_2}\delta_{m_1 m_2}\\ &\times\left\{1+\frac{(D-1)^2}{2D\,\Delta_1}\sum_\ell e^{-|\tau-\tau'|\omega_\ell}R\omega_\ell\,\frac{(D+2\ell-2)}{(D-2)}\,C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\right\}\\ +\Bigg\{\mathscr{A}^{\Delta_Q+R\omega_{\ell_2}}_{\Delta_Q+R\omega_{\ell_1}}(\tau_1,\tau_2\mid\tau)\,&\frac{Q^2(D-1)^2}{2\Omega_D R^{2D-2}D}\,\frac{R\sqrt{\omega_{\ell_1}\omega_{\ell_2}}}{\Delta_0}\Bigg(-\frac{Y^*_{\ell_2 m_2}(\boldsymbol{n})Y_{\ell_1 m_1}(\boldsymbol{n}')}{e^{-(\tau-\tau')\omega_{\ell_1}}}\\ +\frac{(D-2)}{(D-1)}\Bigg[\frac{\partial_i Y_{\ell_1 m_1}(\boldsymbol{n})\,\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})}{(D-1)\,R^2\omega_{\ell_1}\omega_{\ell_2}} &- Y_{\ell_1 m_1}(\boldsymbol{n})Y^*_{\ell_2 m_2}(\boldsymbol{n})\Bigg]\Bigg)+\Big((\tau,\boldsymbol{n})\leftrightarrow(\tau',\boldsymbol{n}')\Big)\Bigg\},\end{aligned}$$

where we have introduced the Gegenbauer polynomials $C_\ell^{D/2-1}(\boldsymbol{n}\cdot\boldsymbol{n}') = \frac{(D-2)\Omega_D}{D+2\ell-2}\sum_m Y^*_{\ell m}(\boldsymbol{n})Y_{\ell m}(\boldsymbol{n}')$.

• Integrating this result over the sphere centered at the insertion point $(\tau, \boldsymbol{n})$ gives again the conserved charge as in the Ward identity and must eliminate the τ-dependence

$$\int \mathrm{d}S(\boldsymbol{n}) \, \langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_\tau(\tau, \boldsymbol{n}) J_\tau(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -iQ \, \langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_\tau(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle \, .$$

• The remaining components of the JJ correlators read

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_\tau(\tau, \boldsymbol{n}) J_i(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = 0,$$

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_i(\tau, \boldsymbol{n}) J_j(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \, \frac{Q^2}{2\Omega_D R^{2D-2} \Delta_0 D}$$

$$\Bigg[\partial_i \partial'_j \sum_\ell \frac{e^{-|\tau - \tau'| \omega_\ell}}{R\omega_\ell} \frac{(D + 2\ell - 2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n} \cdot \boldsymbol{n}') \, \delta_{\ell_2 \ell_1} \delta_{m_2 m_1}$$

$$+ \frac{\partial_j Y^*_{\ell_2 m_2}(\boldsymbol{n}') \, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{e^{(\tau - \tau')\omega_{\ell_2}} R \sqrt{\omega_{\ell_1} \omega_{\ell_2}}} + \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n}) \, \partial_j Y_{\ell_1 m_1}(\boldsymbol{n}')}{e^{-(\tau - \tau')\omega_{\ell_1}} R \sqrt{\omega_{\ell_1} \omega_{\ell_2}}} \Bigg] .$$

• For the last correlator the tree-level contribution vanishes, but it is not symmetry protected so that subleading corrections may appear.

• Now we compute correlators with an insertion of the stress-energy tensor T at cylinder time τ with spinning operators $\mathscr{O}^{Q}_{\ell m}$ at τ_1, τ_2 such $\tau_2 > \tau > \tau_1$.
• The insertion of the $T_{\tau\tau}$ component leads to

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau, \boldsymbol{n}) \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -\mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \frac{1}{\Omega_D R^D} \Bigg\{ (\Delta_0 + \Delta_1) \delta_{\ell_2 \ell_1} \delta_{m_2 m_1} + \frac{\Omega_D}{2} R \sqrt{\omega_{\ell_1} \omega_{\ell_2}} \left[(D-1) Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}) - \frac{(D-3)}{(D-1)} \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n}) \, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{R^2 \omega_{\ell_1} \omega_{\ell_2}} \right] \Bigg\}.$$

• For insertions at large separation $\tau_1, \tau_2 \to \pm\infty$ and $\ell_2 = \ell_1$, the three-point function does not depend on the τ-slice of the T insertion.
• Integrating this result over the sphere insertion point $\boldsymbol{n}$ eliminates the τ-dependence according to the Ward identity:

$$\int \mathrm{d}S(\boldsymbol{n}) \, \langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau, \boldsymbol{n}) \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -\mathscr{A}_{\Delta_Q + R\omega_\ell}(\tau_1, \tau_2) \frac{1}{R} (\Delta_0 + \Delta_1 + R\omega_{\ell_2}) \, \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}.$$

• Correlators involving the other components of the stress-energy tensor read

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau i}(\tau, \boldsymbol{n}) \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \, \frac{1}{2R^D} \left[\sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} \, Y^*_{\ell_2 m_2}(\boldsymbol{n}) \, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n}) - (1 \leftrightarrow 2)^* \right],$$

$$\begin{aligned} \langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{ij}(\tau, \boldsymbol{n}) \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \, \frac{1}{(D-1)\Omega_D R^D} \\ \Big\{ h_{ij} \Big[(\Delta_0 + \Delta_1) \, \delta_{\ell_2 \ell_1} \, \delta_{m_2 m_1} \\ + \frac{\Omega_D R \sqrt{\omega_{\ell_1} \omega_{\ell_2}}}{2} \Big((D-1) Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}) - \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n}) \, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{R^2 \omega_{\ell_1} \omega_{\ell_2}} \Big) \Big] \\ + R \sqrt{\omega_{\ell_1} \omega_{\ell_2}} \Omega_D \, \frac{\partial_{(i} Y^*_{\ell_2 m_2}(\boldsymbol{n}) \, \partial_{j)} Y_{\ell_1 m_1}(\boldsymbol{n})}{R^2 \omega_{\ell_1} \omega_{\ell_2}} \Big\}. \end{aligned}$$

• An insertion of the trace of the energy-momentum tensor $T_{\tau\tau} + h^{ij} T_{ij}$ vanishes on any phonon state by conformal invariance.

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} h^{ij} T_{ij}(\tau, \boldsymbol{n}) \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \frac{1}{(D-1)\,\Omega_D R^D} \Bigg\{ (D-1)\,(\Delta_0 + \Delta_1)\,\delta_{\ell_2 \ell_1} \delta_{m_2 m_1} + \frac{\Omega_D R \sqrt{\omega_{\ell_1}\omega_{\ell_2}}}{2} \left((D-1)^2\, Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}) - (D-3)\, \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{R^2 \omega_{\ell_1} \omega_{\ell_2}} \right) \Bigg\},$$

which sums to zero with $\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau, \boldsymbol{n}) \mathscr{O}^{Q}_{\ell_1 m_1} \rangle$.

• Finally, there is a total of six correlators with two insertions of the stress-energy tensor at $\tau > \tau'$ between spinning operators $\mathscr{O}^{Q}_{\ell m}$ at τ_1, τ_2 such that $\tau_2 > \tau > \tau' > \tau_1$ and six correlators involving the various components of one insertion of the stress-energy tensor and one insertion of the $O(2)$-current respectively at times $\tau > \tau'$ between spinning operators $\mathscr{O}^{Q}_{\ell m}$ at τ_1, τ_2 such that we have the ordering $\tau_2 > \tau > \tau' > \tau_1$.

Outline

- We have systematically collected three- and four-point correlators of a CFT with a global $O(2)$ symmetry using the large-charge expansion.
- We have studied in particular correlators with current insertions of J and T sandwiched between either the scalar large-charge ground state or higher phonon states with spin.
- The general structure of our correlators contains contributions with positive Q-scaling coming from the tree-level EFT Lagrangian plus quantum corrections starting at order Q^0 which are independent of the Wilsonian coefficients in odd dimensions.
- A non-trivial position dependence in our correlators must always be due to the quantum corrections since the ground state is homogeneous.

• The results here hold for the $O(2)$ model in D dimensions at large charge or the homogeneous $O(2)$ sector of a CFT with a larger global symmetry group. Once we want to discuss the full non-Abelian structure, things become more complicated.

• The first observation is that the correct quantity to fix is not a set of charges, but a representation.

• The immediate generalization corresponds to fixing the completely symmetric representation.

• Other representations can be obtained in two ways: by exciting type-II Goldstones that are charged under the global symmetry or by starting from an inhomogeneous ground state corresponding to a different saddle point.

• The two approaches must give the same result in the appropriate limit, but have their own technical complications.

• Type-II Goldstones contribute at order $1/\mu$ and for this reason they do not play a role in the computations in the present work, but in order to be studied consistently they require the addition of new subleading terms to the EFT.

• The inhomogeneous saddle is to date only known for the simple case of the $O(4)$ model and an analytic expression is available only in a special limit.
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• Moreover, since it breaks the $SO(D)$ rotational invariance that we have used intensively in our present computations, one expects that computing correlation functions using both the tree-level and the quantum corrections will be more technically challenging.

Thank you!

• Starting from the vacuum $|Q\rangle$ one can obtain a state annihilated by $a_{\ell m}$ with charge $Q_0 + q$ and scaling dimension $\Delta_0(Q_0 + q)$:

$$|Q+q\rangle = e^{i\pi_0 q}\,|Q\rangle = \exp\left[\frac{iq}{\Omega_D R^{D-1}}\int \mathrm{d}S\,\pi(\tau,\boldsymbol{n})\right]|Q\rangle\,.$$

• While these states are all annihilated by the ladder operators, since $[a_{\ell m}, \pi_0] = 0$, they are not zero modes of Π_0.

• They do not represent degenerate vacua, but they have a gap

$$\Delta_0(Q_0+q) - \Delta_0(Q_0) \sim q(R\mu)\,.$$

• π_0 is the only operator on which the $O(2)$-charge acts non-trivially, it has to be compact, $\pi_0 \sim \pi_0 + 2\pi\mathbb{1}$, which implies that $q \in \mathbb{Z}$.

• States with charge $Q_0 + q$ live at the EFT cutoff and will not be discussed any further.

• We are mostly interested in correlators in which the vacuum $|Q\rangle$ is inserted at large separation on the cylinder, namely at $\tau = \pm\infty$. In this case the details on the boundary conditions the vacuum imposes are irrelevant. We can now construct path integrals for the norm of the states which correspond to two-points functions of the corresponding primaries in the CFT_D at large cylinder-time separation.

- . These read

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau, \boldsymbol{n}) T_{\tau\tau}(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \frac{\Delta_0}{\Omega_D^2 R^{2D}}$$
$$\Bigg\{ \Bigg[\Delta_0 + 2\Delta_1 + \frac{D}{2} \sum_{\ell} e^{-|\tau-\tau'|\omega_\ell} R\omega_\ell \frac{(D+2\ell-2)}{D-2} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}') \Bigg] \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$$
$$+ \frac{D\Omega_D}{2} R \sqrt{\omega_{\ell_1}\omega_{\ell_2}} \Bigg[Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}') e^{(\tau-\tau')\omega_{\ell_1}} + Y^*_{\ell_2 m_2}(\boldsymbol{n}') Y_{\ell_1 m_1}(\boldsymbol{n}) e^{-(\tau-\tau')\omega_{\ell_2}} \Bigg] \Bigg\}$$
$$+ \Bigg\{ \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \frac{\Omega_D \Delta_0 R \sqrt{\omega_{\ell_1}\omega_{\ell_2}}}{2\Omega_D^2 R^{2D}} \Bigg((D-1)\, Y_{\ell_1 m_1}(\boldsymbol{n}) Y^*_{\ell_2 m_2}(\boldsymbol{n})$$
$$- \frac{(D-3)}{(D-1)} \frac{\partial_i Y_{\ell_1 m_1}(\boldsymbol{n})\, \partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})}{R^2 \omega_{\ell_1}\omega_{\ell_2}} \Bigg) + \Big((\tau, \boldsymbol{n}) \leftrightarrow (\tau', \boldsymbol{n}') \Big) \Bigg\}.$$

This correlator is symmetric under $(\tau, \boldsymbol{n}) \leftrightarrow (\tau', \boldsymbol{n}')$. The $\ell = 0$

special case of this correlator has already appeared .

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{ij}(\tau,\boldsymbol{n}) T_{kn}(\tau',\boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1}\rangle = \mathscr{A}^{\Delta_Q+R\omega_{\ell_2}}_{\Delta_Q+R\omega_{\ell_1}}(\tau_1,\tau_2\mid\tau)\,\frac{\Delta_0}{(D-1)^2\Omega_D^2 R^{2D}}$$
$$\Bigg\{\Bigg[\Delta_0+2\Delta_1+\frac{D}{2}\sum_\ell e^{-|\tau-\tau'|\omega_\ell}R\omega_\ell\,\frac{(D+2\ell-2)}{D-2}\,C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\Bigg]h_{ij}h_{kn}\delta_{\ell_2\ell_1}\delta_{m_2m_1}$$
$$+\frac{D\,\Omega_D}{2}R\sqrt{\omega_{\ell_2}\omega_{\ell_1}}\left(Y^*_{\ell_2m_2}(\boldsymbol{n})Y_{\ell_1m_1}(\boldsymbol{n}')\,e^{(\tau-\tau')\omega_{\ell_1}}+Y^*_{\ell_2m_2}(\boldsymbol{n}')Y_{\ell_1m_1}(\boldsymbol{n})\,e^{-(\tau-\tau')\omega_{\ell_2}}\right)h_{ij}h_{kn}\Bigg\}$$
$$+\Bigg\{\mathscr{A}^{\Delta_Q+R\omega_{\ell_2}}_{\Delta_Q+R\omega_{\ell_1}}(\tau_1,\tau_2\mid\tau)\,\frac{\Omega_D\Delta_0 R\sqrt{\omega_{\ell_1}\omega_{\ell_2}}}{2(D-1)\,\Omega_D^2R^{2D}}\Bigg[2\frac{\partial_{(i}Y^*_{\ell_2m_2}(\boldsymbol{n})\,\partial_{j)}Y_{\ell_1m_1}(\boldsymbol{n})}{R^2(D-1)\,\omega_{\ell_1}\omega_{\ell_2}}+Y^*_{\ell_2m_2}(\boldsymbol{n})Y_{\ell_1m_1}(\boldsymbol{n})h_{ij}$$
$$-\frac{\partial_iY^*_{\ell_2m_2}(\boldsymbol{n})\,\partial_iY_{\ell_1m_1}(\boldsymbol{n})}{R^2(D-1)\,\omega_{\ell_1}\omega_{\ell_2}}h_{ij}\Bigg]h_{kn}+\Big((\tau,\boldsymbol{n},ij)\leftrightarrow(\tau',\boldsymbol{n}',kn)\Big)\Bigg\}.$$

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau i}(\tau,\boldsymbol{n}) T_{\tau j}(\tau',\boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1}\rangle = -\,\mathscr{A}^{\Delta_Q+R\omega_{\ell_2}}_{\Delta_Q+R\omega_{\ell_1}}(\tau_1,\tau_2\mid\tau)\,\frac{\Delta_0 D}{2(D-1)^2\Omega_D R^{2D}}$$
$$\Bigg\{\partial_i\partial'_j\sum_\ell\frac{e^{-|\tau-\tau'|\omega_\ell}}{R\omega_\ell}\,\frac{(D+2\ell-2)}{(D-2)\Omega_D}\,C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\,\delta_{\ell_2\ell_1}\delta_{m_2m_1}+\frac{\partial_iY^*_{\ell_2m_2}(\boldsymbol{n})\,\partial'_jY_{\ell_1m_1}(\boldsymbol{n}')}{R\sqrt{\omega_{\ell_2}\omega_{\ell_1}}e^{-(\tau-\tau')\omega_{\ell_1}}}$$
$$+\frac{\partial'_jY^*_{\ell_2m_2}(\boldsymbol{n}')\,\partial_iY_{\ell_1m_1}(\boldsymbol{n})}{R\sqrt{\omega_{\ell_2}\omega_{\ell_1}}e^{(\tau-\tau')\omega_{\ell_2}}}\Bigg\}.$$

This correlator is symmetric under $(\tau, \boldsymbol{n}, i) \leftrightarrow (\tau', \boldsymbol{n}', j)$.

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau i}(\tau, \boldsymbol{n}) T_{\tau\tau}(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = - \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \frac{\Delta_0 D}{2\Omega_D R^{2D}} \frac{1}{(D-1)}$$
$$\Bigg\{ \partial_i \sum_\ell e^{-|\tau-\tau'|\omega_\ell} \frac{(D+2\ell-2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\, \delta_{\ell_2 \ell_1} \delta_{m_2 m_1} + \sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} \frac{Y^*_{\ell_2 m_2}(\boldsymbol{n}')\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{e^{(\tau-\tau')\omega_{\ell_2}}}$$
$$- \sqrt{\frac{\omega_{\ell_1}}{\omega_{\ell_2}}} \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})\, Y_{\ell_1 m_1}(\boldsymbol{n}')}{e^{-(\tau-\tau')\omega_{\ell_1}}} + \frac{(D-1)}{D} \left[\sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n}) - (1 \leftrightarrow 2)^* \right] \Bigg\}.$$

Since $T_{\tau i}$ vanishes on the ground-state solution, the correlator solely receives a second-order contribution from the linear terms and the quadratic term of $T_{\tau i}$. Moving to the combination $T_{\tau i} T_{jk}$ one finds

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau i}(\tau, \boldsymbol{n}) T_{jk}(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \frac{\Delta_0 D}{2\Omega_D R^{2D}} \frac{h_{jk}}{(D-1)^2}$$
$$\Bigg\{ \partial_i \sum_\ell e^{-|\tau-\tau'|\omega_\ell} \frac{(D+2\ell-2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\, \delta_{\ell_2 \ell_1} \delta_{m_2 m_1} + \sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} \frac{Y^*_{\ell_2 m_2}(\boldsymbol{n}')\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{e^{(\tau-\tau')\omega_{\ell_2}}}$$
$$- \sqrt{\frac{\omega_{\ell_1}}{\omega_{\ell_2}}} \frac{Y_{\ell_1 m_1}(\boldsymbol{n}')\, \partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})}{e^{-(\tau-\tau')\omega_{\ell_1}}} + \frac{(D-1)}{D} \left[\sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n}) - (1 \leftrightarrow 2)^* \right] \Bigg\}.$$

Again, besides the linear terms only the quadratic term of $T_{\tau i}$ contributes at second order. In addition, the correlator $\langle * \rangle\ \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau i}(\tau, \boldsymbol{n})\, h^{jk}(\boldsymbol{n}') T_{jk}(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1}$ differs by a minus sign from the previous correlator with an insertion of $T_{\tau\tau}(\tau', \boldsymbol{n}')$, as imposed by conformal invariance.

$$\begin{aligned}
\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau,\boldsymbol{n}) T_{ij}(\tau',\boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1}\rangle = &- \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1,\tau_2 \mid \tau)\, \frac{\Delta_0}{\Omega_D^2 R^{2D}}\, \frac{h_{ij}}{(D-1)} \\
&\Bigg\{ \Bigg[\Delta_0 + 2\Delta_1 + \frac{D\,\Omega_D}{2} \sum_\ell R\omega_\ell e^{-|\tau-\tau'|\omega_\ell} \frac{(D+2\ell-2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}') \Bigg] \delta_{\ell_2 \ell_1}\delta_{m_2 m_1} \\
&+ \frac{D\,\Omega_D}{2} R\sqrt{\omega_{\ell_2}\omega_{\ell_1}} \Bigg[Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}') e^{(\tau-\tau')\omega_{\ell_1}} + Y^*_{\ell_2 m_2}(\boldsymbol{n}') Y_{\ell_1 m_1}(\boldsymbol{n}) e^{-(\tau-\tau')\omega_{\ell_2}} \Bigg] \Bigg\} \\
&- \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1,\tau_2 \mid \tau)\, \frac{\Delta_0 R\sqrt{\omega_{\ell_1}\omega_{\ell_2}}}{2\Omega_D R^{2D}}\, h_{ij} \left\{ Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}) - \frac{(D-3)}{(D-1)^2} \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{R^2 \omega_{\ell_1}\omega_{\ell_2}} \right\} \\
&- \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1,\tau_2 \mid \tau')\, \frac{\Delta_0 R\sqrt{\omega_{\ell_1}\omega_{\ell_2}}}{2\Omega_D R^{2D}} \Bigg\{ h_{ij} \Bigg[Y^*_{\ell_2 m_2}(\boldsymbol{n}') Y_{\ell_1 m_1}(\boldsymbol{n}') - \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n}')\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n}')}{(D-1)\, R^2 \omega_{\ell_1}\omega_{\ell_2}} \Bigg] \\
&\qquad + 2\, \frac{\partial_{(i} Y^*_{\ell_2 m_2}(\boldsymbol{n}')\, \partial_{j)} Y_{\ell_1 m_1}(\boldsymbol{n}')}{(D-1)\, R^2 \omega_{\ell_1}\omega_{\ell_2}} \Bigg\}.
\end{aligned}$$

This correlator is not symmetric in $(\tau, \boldsymbol{n}) \leftrightarrow (\tau', \boldsymbol{n}')$, however, by conformal invariance, the correlator $\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau,\boldsymbol{n}) h^{ij}(\boldsymbol{n}) T_{ij}(\tau',\boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1}\rangle$ is symmetric in $(\tau, \boldsymbol{n}) \leftrightarrow (\tau', \boldsymbol{n}')$.

One can check directly that the above correlators satisfy the Ward identity for $T_{\tau\tau}$ insertions in Eq. (**??**). For example, for two insertions of $T_{\tau\tau}$ one finds

$$\int \mathrm{d}S(\boldsymbol{n})\, \langle * \rangle\ \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau,\boldsymbol{n}) T_{\tau\tau}(\tau',\boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} = -\frac{1}{R}(\Delta_0 + \Delta_1 + R\omega_{\ell_2})\ \langle * \rangle\ \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau',\boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1}.$$

In the special case $\ell = 0$, the above correlators simplify as follows:

$$\langle \mathscr{O}^{-Q} T_{\tau\tau}(\tau, \boldsymbol{n}) T_{\tau\tau}(\tau', \boldsymbol{n}') \mathscr{O}^{Q} \rangle = \mathscr{A}(\tau_1, \tau_2) \frac{\Delta_0}{\Omega_D^2 R^{2D}} \Big[\Delta_0 + 2\Delta_1 + \frac{D}{2} \sum_\ell e^{-|\tau-\tau'|\omega_\ell} R\omega_\ell \frac{(D+2\ell-2)}{D-2} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}') \Big],$$

$$\langle \mathscr{O}^{-Q} T_{ij}(\tau, \boldsymbol{n}) T_{kn}(\tau', \boldsymbol{n}') \mathscr{O}^{Q} \rangle = \mathscr{A}(\tau_1, \tau_2) \frac{\Delta_0}{\Omega_D^2 R^{2D}} \frac{h_{ij} h_{kn}}{(D-1)^2} \Big[\Delta_0 + 2\Delta_1 + \frac{D}{2} \sum_\ell \frac{R\omega_\ell}{e^{|\tau-\tau'|\omega_\ell}} \frac{(D+2\ell-2)}{D-2} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}') \Big],$$

$$\langle \mathscr{O}^{-Q} T_{\tau i}(\tau, \boldsymbol{n}) T_{\tau j}(\tau', \boldsymbol{n}') \mathscr{O}^{Q} \rangle = - \frac{\mathscr{A}(\tau_1, \tau_2)\, \Delta_0 D}{2(D-1)^2 \Omega_D^2 R^{2D}} \partial_i \partial'_j \sum_\ell \frac{e^{-|\tau-\tau'|\omega_\ell}}{R\omega_\ell} \frac{(D+2\ell-2)}{D-2} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}'),$$

$$\langle \mathscr{O}^{-Q} T_{\tau i}(\tau, \boldsymbol{n}) T_{\tau\tau}(\tau', \boldsymbol{n}') \mathscr{O}^{Q} \rangle = - \frac{\mathscr{A}(\tau_1, \tau_2)\, \Delta_0 D}{2(D-1) \Omega_D^2 R^{2D}} \partial_i \sum_\ell e^{-|\tau-\tau'|\omega_\ell} \frac{(D+2\ell-2)}{D-2} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}'),$$

$$\langle \mathscr{O}^{-Q} T_{\tau i}(\tau, \boldsymbol{n}) T_{jk}(\tau', \boldsymbol{n}') \mathscr{O}^{Q} \rangle = \frac{\mathscr{A}(\tau_1, \tau_2)\, \Delta_0 D\, h_{jk}}{2(D-1)^2 \Omega_D^2 R^{2D}} \partial_i \sum_\ell e^{-|\tau-\tau'|\omega_\ell} \frac{(D+2\ell-2)}{D-2} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}'),$$

$$\langle \mathscr{O}^{-Q} T_{\tau\tau}(\tau, \boldsymbol{n}) T_{ij}(\tau', \boldsymbol{n}') \mathscr{O}^{Q} \rangle = - \mathscr{A}(\tau_1, \tau_2) \frac{\Delta_0}{\Omega_D^2 R^{2D}} \frac{h_{ij}}{(D-1)} \Big[\Delta_0 + 2\Delta_1 + \frac{D}{2} \sum_\ell R\omega_\ell e^{-|\tau-\tau'|\omega_\ell} \frac{(D+2\ell-2)}{D-2} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}') \Big].$$

The $\ell_1 = \ell_2 = 0$ correlator with insertions of $T_{\tau i} T_{\tau\tau}$ was computed in the macroscopic limit $R \to \infty$. We now consider correlators with There are six correlators involving the various components which can be computed as follows:

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau i}(\tau, \boldsymbol{n}) J_\tau(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -i\, \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau)\, \frac{Q}{2\Omega_D R^{2D-1}}$$
$$\Bigg\{ \partial_i \sum_\ell e^{-|\tau-\tau'|\omega_\ell} \frac{(D+2\ell-2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\, \delta_{\ell_2 \ell_1} \delta_{m_2 m_1} + \sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} \frac{Y^*_{\ell_2 m_2}(\boldsymbol{n}')\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{e^{(\tau-\tau')\omega_{\ell_2}}}$$
$$- \sqrt{\frac{\omega_{\ell_1}}{\omega_{\ell_2}}} \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}')}{e^{-(\tau-\tau')\omega_{\ell_1}}} + \left(\sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} Y^*_{\ell_2 m_2}(\boldsymbol{n})\ \partial_i Y_{\ell_1 m_1}(\boldsymbol{n}) - (1 \leftrightarrow 2)^* \right) \Bigg\}.$$

$T_{\tau i}$ vanishes on the ground state and hence the quadratic contributions only come from the linear terms and the quadratic term of $T_{\tau i}$. The combination $J_i T_{\tau\tau}$ instead leads to

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_i(\tau, \boldsymbol{n}) T_{\tau\tau}(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -i\, \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau)\, \frac{Q}{2\Omega_D R^{2D-1}}$$
$$\Bigg[\delta_{\ell_2 \ell_1} \delta_{m_2 m_1}\, \partial_i \sum_\ell e^{-|\tau-\tau'|\omega_\ell} \frac{(D+2\ell-2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}') + \sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} \frac{Y^*_{\ell_2 m_2}(\boldsymbol{n}')\ \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{e^{(\tau-\tau')\omega_{\ell_2}}}$$
$$- \sqrt{\frac{\omega_{\ell_1}}{\omega_{\ell_2}}} \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})\ Y_{\ell_1 m_1}(\boldsymbol{n}')}{e^{-(\tau-\tau')\omega_{\ell_1}}} + \frac{(D-2)}{D} \left(\sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n}) - (1 \leftrightarrow 2)^* \right) \Bigg].$$

This correlator is related to the previous one. From the expansions in Eq. (**??**) it is clear that this has to be the case.

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau, \boldsymbol{n}) J_\tau(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = i\, \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau)\, \frac{Q(D-1)}{2\Omega_D R^{2D-1}} R\sqrt{\omega_{\ell_2}\omega_{\ell_1}}$$
$$\left\{ \left[\frac{Y^*_{\ell_2 m_2}(\boldsymbol{n}') Y_{\ell_1 m_1}(\boldsymbol{n})}{e^{(\tau-\tau')\omega_{\ell_2}}} + \frac{Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}')}{e^{-(\tau-\tau')\omega_{\ell_1}}} \right] + \sum_\ell \frac{R\omega_\ell\, e^{-|\tau-\tau'|\omega_\ell}}{R\sqrt{\omega_{\ell_2}\omega_{\ell_1}}} \frac{(D+2\ell-2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}') \right.$$
$$+ \frac{(D-2)}{D}\left[Y^*_{\ell_2 m_2}(\boldsymbol{n}') Y_{\ell_1 m_1}(\boldsymbol{n}') - \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n}')\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n}')}{R^2(D-1)\omega_{\ell_2}\omega_{\ell_1}} \right] + \frac{2}{(D-1)} \left[\frac{1}{\Omega_D} \left(\frac{\Delta_0 + \Delta_1}{R\sqrt{\omega_{\ell_1}\omega_{\ell_2}}} \right) \delta_{\ell_2 \ell_1} \delta_{m_2 m_1} \right.$$
$$\left.\left. + \frac{1}{2}\left((D-1) Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}) - \frac{(D-3)}{(D-1)} \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{R^2 \omega_{\ell_1}\omega_{\ell_2}} \right) \right] \right\}.$$

Here, the quadratic term from J_τ vanishes after integration over $\boldsymbol{n}'$, whereas the quadratic term from $T_{\tau\tau}$ remains finite after integration over $\boldsymbol{n}$. This is so because it has to correct the energy by $R\omega_{\ell_2}$, in accordance with the Ward identities.

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau i}(\tau, \boldsymbol{n}) J_j(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -i\, \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau)\, \frac{Q}{2\Omega_D R^{2D-1}} \frac{1}{(D-1)}$$
$$\left\{ \partial_i \partial'_j \sum_\ell \frac{e^{-|\tau-\tau'|\omega_\ell}}{R\omega_\ell} \frac{(D+2\ell-2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\, \delta_{\ell_2\ell_1}\delta_{m_2 m_1} + \frac{\partial'_j Y^*_{\ell_2 m_2}(\boldsymbol{n}') \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{R\sqrt{\omega_{\ell_1}\omega_{\ell_2}}\, e^{(\tau-\tau')\omega_{\ell_2}}} \right.$$
$$\left. + \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n}) \partial'_j Y_{\ell_1 m_1}(\boldsymbol{n}')}{R\sqrt{\omega_{\ell_1}\omega_{\ell_2}}\, e^{-(\tau-\tau')\omega_{\ell_1}}} \right\}.$$

Both $T_{\tau i}$ and J_i vanish on the ground state and hence the only quadratic contribution comes from the two linear terms.

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{ij}(\tau, \boldsymbol{n}) J_\tau(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -i\, \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \frac{Q}{\Omega_D R^{2D-1}}$$
$$\Bigg\{ h_{ij} \Bigg[\frac{(\Delta_0 + \Delta_1)}{\Omega_D (D-1)} \delta_{\ell_2 \ell_1} \delta_{m_2 m_1} + \frac{1}{2} \sum_\ell e^{-|\tau-\tau'|\omega_\ell} R\omega_\ell \frac{(D+2\ell-2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\, \delta_{\ell_2\ell_1}\delta_{m_2 m_1}$$
$$+ \frac{R\sqrt{\omega_{\ell_2}\omega_{\ell_2}}}{2} \Bigg(\frac{Y^*_{\ell_2 m_2}(\boldsymbol{n}') Y_{\ell_1 m_1}(\boldsymbol{n})}{e^{(\tau-\tau')\omega_{\ell_2}}} + \frac{Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n}')}{e^{-(\tau-\tau')\omega_{\ell_1}}} + \Big[1 + \frac{(D-2)}{D}\Big] Y^*_{\ell_2 m_2}(\boldsymbol{n}) Y_{\ell_1 m_1}(\boldsymbol{n})$$
$$- \Big[1 + \frac{(D-2)}{D}\Big] \frac{\partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{(D-1)\, R^2 \omega_{\ell_1}\omega_{\ell_2}} \Bigg) \Bigg] + \frac{R\sqrt{\omega_{\ell_1}\omega_{\ell_2}}}{(D-1)} \frac{\partial_{(i} Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_{j)} Y_{\ell_1 m_1}(\boldsymbol{n})}{R^2 \omega_{\ell_1}\omega_{\ell_2}} \Bigg\}.$$

This correlator is related to the TJ correlator due to the fact that $h^{ij} T_{ij} = -T_{\tau\tau}$, which holds by virtue of conformal invariance.

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_i(\tau, \boldsymbol{n}) T_{jk}(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = i\, \mathscr{A}^{\Delta_Q + R\omega_{\ell_2}}_{\Delta_Q + R\omega_{\ell_1}}(\tau_1, \tau_2 \mid \tau) \frac{Q}{2\Omega_D R^{2D-1}} \frac{h_{jk}}{(D-1)}$$
$$\Bigg\{ \partial_i \sum_\ell e^{-|\tau-\tau'|\omega_\ell} \frac{(D+2\ell-2)}{(D-2)\Omega_D} C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')\, \delta_{\ell_2\ell_1}\delta_{m_2 m_1} + \sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} \frac{Y^*_{\ell_2 m_2}(\boldsymbol{n}')\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n})}{e^{(\tau-\tau')\omega_{\ell_2}}}$$
$$- \sqrt{\frac{\omega_{\ell_1}}{\omega_{\ell_2}}} \frac{Y_{\ell_1 m_1}(\boldsymbol{n}')\, \partial_i Y^*_{\ell_2 m_2}(\boldsymbol{n})}{e^{-(\tau-\tau')\omega_{\ell_1}}} + \frac{(D-2)}{D} \Bigg[\sqrt{\frac{\omega_{\ell_2}}{\omega_{\ell_1}}} Y^*_{\ell_2 m_2}(\boldsymbol{n})\, \partial_i Y_{\ell_1 m_1}(\boldsymbol{n}) - (1 \leftrightarrow 2)^* \Bigg] \Bigg\}.$$

This correlator is proportional to h_{jk} since the quadratic term in the expansion of T_{jk} only appears at cubic order in the correlator. This is no longer the case once one includes higher-order corrections.

In the special case $\ell = 0$ the TJ correlators simplify significantly:

$$\langle \mathscr{O}^{-Q} T_{\tau i}(\tau, \boldsymbol{n}) J_\tau(\tau', \boldsymbol{n}') \mathscr{O}^{Q} \rangle = -i \frac{Q\, \mathscr{A}(\tau_1, \tau_2)}{2\Omega_D^2 R^{2D-1}}\, \partial_i \sum_\ell e^{-(\tau-\tau')\omega_\ell}\, \frac{(D+2\ell-2)}{D-2}\, C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}'),$$

$$\langle \mathscr{O}^{-Q} J_i(\tau, \boldsymbol{n}) T_{\tau\tau}(\tau', \boldsymbol{n}') \mathscr{O}^{Q} \rangle = -i \frac{Q\, \mathscr{A}(\tau_1, \tau_2)}{2\Omega_D^2 R^{2D-1}}\, \partial_i \sum_\ell e^{-(\tau-\tau')\omega_\ell}\, \frac{(D+2\ell-2)}{D-2}\, C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}'),$$

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau\tau}(\tau, \boldsymbol{n}) J_\tau(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = i\, \mathscr{A}(\tau_1, \tau_2)\, \frac{Q(D-1)}{2\Omega_D R^{2D-1}}$$
$$\times \left\{ \sum_\ell R\omega_\ell\, e^{-(\tau-\tau')\omega_\ell}\, \frac{(D+2\ell-2)}{D-2}\, C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}') + \frac{2(\Delta_0+\Delta_1)}{(D-1)\Omega_D}\, \delta_{\ell_2\ell_1}\delta_{m_2 m_1} \right\},$$

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{\tau i}(\tau, \boldsymbol{n}) J_j(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -i \frac{Q\, \mathscr{A}(\tau_1, \tau_2)}{2(D-1)\Omega_D^2 R^{2D-1}}\, \partial_i \partial'_j \sum_\ell \frac{(D+2\ell-2)}{D-2}\, \frac{C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')}{e^{(\tau-\tau')\omega_\ell} R\omega_\ell},$$

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} T_{ij}(\tau, \boldsymbol{n}) J_\tau(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = -i \frac{Q\, \mathscr{A}(\tau_1, \tau_2)}{\Omega_D^2 R^{2D-1}}\, h_{ij}$$
$$\times \left\{ \frac{(\Delta_0+\Delta_1)}{D-1} + \frac{1}{2} \sum_\ell e^{-(\tau-\tau')\omega_\ell} R\omega_\ell\, \frac{(D+2\ell-2)}{D-2}\, C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}') \right\},$$

$$\langle \mathscr{O}^{-Q}_{\ell_2 m_2} J_i(\tau, \boldsymbol{n}) T_{jk}(\tau', \boldsymbol{n}') \mathscr{O}^{Q}_{\ell_1 m_1} \rangle = i \frac{Q\, \mathscr{A}(\tau_1, \tau_2)}{2\Omega_D^2 R^{2D-1}}\, \frac{h_{jk}}{(D-1)}\, \partial_i \sum_\ell \frac{(D+2\ell-2)}{D-2}\, \frac{C_\ell^{\frac{D}{2}-1}(\boldsymbol{n}\cdot\boldsymbol{n}')}{e^{(\tau-\tau')\omega_\ell}}.$$

The correlators $J_i T_{\tau\tau}$ and $T_{\tau i} J_\tau$ in the special case $\ell_1 = \ell_2 = 0$ has appeared in the macroscopic limit $R \to \infty$.