Gauge and Yukawa interactions at large charge

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Motivation

- Calculate correlation functions without the use of Feynman diagrams
- Obtain information for the spectrum of the QFT in strongly coupled regions

Purpose

Calculate contributions to the anomalous dimensions the large charge Q scalar operators ϕ^Q from fermion fields and gauge interactions

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle = \frac{1}{|x_{fi}|^{2\Delta_Q}}$$

General approach

- Start with a QFT with a global symmetry and obtain its Wilson-Fisher fixed point
- Map the theory on the cylinder $\mathbb{R} \times S^{D-1}$

Operator/State Correspondence

Every operator of a CFT corresponds to state on Hilbert space

$$\mathcal{O}_{\Delta_Q} \leftrightarrow |Q\rangle, \qquad \Delta_Q \leftrightarrow E_Q/R$$

- Consider a **large charge state** $|Q\rangle$ and fix the charge
- Calculate the 2pt function

$$\langle Q|e^{-HT}|Q\rangle = \frac{1}{Z}\int \mathcal{D}\Phi e^{-S_{\text{eff}}+\text{charge fixing}}$$

- Obtain the anomalous dimensions for κ_I couplings is then

$$E_Q R = \Delta_Q = \sum_{j=-1} \frac{1}{Q^j}\Delta_j\left(Q\kappa_I^*, \{q_i\}\right)$$

Models under study

- Gauge Invariant models (Scalar Q.E.D)
- Nambu-Jona-Lasinio-Yukawa (NJLY)
- Asymptotic safe model (Litim-Sannino model)

In what follows we are focuced in Gauge Invariant models...

Consider the action

$$S = \int d^4x \left(\frac{1}{4} F_{\mu\nu}^2 + (D_\mu \phi)^\dagger D_\mu \phi + \frac{\lambda}{24} (\bar{\phi}\phi)^2 \right)$$

where $D_\mu = \partial_\mu + ieA_\mu$ and e, λ couplings constants.

Symmetries

$$\phi \to e^{i\alpha(x)}\phi, \qquad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$$

Equations of Motion

$$-D^\mu D_\mu \phi + m^2 \bar{\phi} + \frac{\lambda_0}{12} (\bar{\phi}\phi)\bar{\phi} = 0\,, \qquad \partial_\mu F^{\mu\nu} = J^\nu$$

Fixed Point

$$\lambda_* = \frac{3}{20} \left(19\epsilon \pm i\sqrt{719}\epsilon \right)\,, \qquad e_*^2 = 24\pi^2 \epsilon$$

- Use conformal coupling of scalars ($\mathcal{R}$ scalar curvature)

$$\int d^D x |\partial\phi|^2 \to \int d^D x \sqrt{-g}\Big(|\partial\phi|^2 + \mathcal{R}|\phi|^2\Big)$$

while ***the other fields are unaffected***

- Express $\phi(x) = \frac{\rho(x)}{\sqrt{2}} e^{i\chi(x)}$
- Impose charge fixing condition

The action reads

$$S_{eff} = \int d^D x \sqrt{-g}\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\rho^2(\partial\chi)^2 + \frac{1}{2}m^2\rho^2 + e\rho^2 A_\mu \partial^\mu \chi + \frac{1}{2}e^2\rho^2 A_\mu A^\mu + \frac{\lambda_0}{24}\rho^4 + \frac{i\,Q}{R^{d-1}\Omega_{d-1}}\dot{\chi}\Big)$$

Computation of γ_{ϕ^Q}

- Obtain equations of motion

$$
\begin{aligned}
& -\nabla^2\rho + \rho[(\partial\chi)^2 + m^2] + 2e\rho(\partial_\mu\chi + eA_\mu)A^\mu + \frac{\lambda}{6}\rho^3 = 0\,, \\
& \nabla_\mu(\rho^2 g^{\mu\nu}\partial_\nu\chi + e\,\rho^2 A^\mu) = 0\,, \\
& \rho^2\dot{\chi} = -\frac{iQ}{R^{D-1}\Omega_{D-1}}
\end{aligned}
$$

- Write dynamical fields $\to$ (classical solutions+variations)

$$
\rho(x) = f + r(x), \quad \chi(x) = -i\mu + \frac{1}{\sqrt{f}}\,\pi(x), \quad A_\mu(x) = 0 + A_\mu(x)
$$

where μ is the chemical potential...

- Use the representation

$$
\langle Q|e^{-H\tau}|Q\rangle = \frac{1}{Z}\int \mathcal{D}\rho\mathcal{D}\chi\mathcal{D}A\, e^{-S_{eff}}
$$

Classical contributions Δ_{-1}

Plugging the classical solutions to S_{eff} we get

$$S_{eff} = \frac{Q}{2}\left(\frac{3}{2}\mu + \frac{1}{2}\frac{m^2}{\mu}\right)$$

Use e$_s$om and solve for the critical point $(*)$

$$\mu(\mu^2 - m^2) = \frac{\lambda_0\, Q}{4R^{D-1}\Omega_{D-1}} \rightarrow \boxed{R\mu_* = \frac{3^{1/3} + \left[9\frac{\lambda_* Q}{(4\pi)^2} - \sqrt{81\frac{(\lambda_* Q)^2}{(4\pi)^4} - 3}\right]^{2/3}}{3^{2/3}\left[9\frac{\lambda_* Q}{(4\pi)^2} - \sqrt{81\frac{(\lambda_* Q)^2}{(4\pi)^4} - 3}\right]^{1/3}}}$$

Solution

$$4\,\Delta_{-1} = \frac{3^{\frac{2}{3}}\left(x + \sqrt{-3 + x^2}\right)^{\frac{1}{3}}}{3^{\frac{1}{3}} + \left(x + \sqrt{-3 + x^2}\right)^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}\left(3^{\frac{1}{3}} + \left(x + \sqrt{-3 + x^2}\right)^{\frac{2}{3}}\right)}{\left(x + \sqrt{-3 + x^2}\right)^{\frac{1}{3}}}$$

1-loop contribution (Δ_0)

The quadratic action of fluctuation is given by

$$S^{(2)} = \int d^D x \sqrt{-g} \Big(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu r)^2 + \frac{1}{2} (\partial_\mu \pi)^2 - \frac{1}{2}\, 2(m^2 - \mu^2) r^2 \\ - 2i\mu r \partial_\tau \pi + ef \partial_\mu \pi A^\mu - 2ie\mu f r A_0 + \frac{1}{2} (ef)^2 A_\mu A^\mu \Big)$$

- ***After Higgs mechanism there is a local residual symmetry (Elitzur's theorem)***

$$\delta r = 0, \quad \delta\pi = f\,\alpha(x), \quad \delta A_\mu = -\frac{1}{e} \partial_\mu \alpha(x)$$

- Gauge Fixing via R_ξ-gauge

$$S^{(2)} \to S^{(2)} + \frac{1}{2} \int d^D x \sqrt{g}\, G^2, \qquad G^2 = \frac{1}{\xi} \left(\nabla_\mu A^\mu + ef\pi \right)^2$$

One-loop evaluation

Path Integral

$$\langle Q|e^{-HT}|Q\rangle = \text{classical} \times \underbrace{\frac{1}{Z}\int \mathcal{D}r\mathcal{D}\pi\mathcal{D}A\, e^{-\left(S^{(2)}+\frac{1}{2}\int d^Dx\sqrt{g}\,G^2\right)}\det\frac{\delta G}{\delta\alpha}}_{\propto\,\Delta_0}$$

The quadratic part of the exponential is written

$$\begin{aligned}\mathcal{L}^{(2)} =& \frac{1}{2}A_\mu\left(-g^{\mu\nu}\nabla^2+\mathcal{R}^{\mu\nu}+\left(1-\frac{1}{\xi}\right)\nabla^\mu\nabla^\nu+(ef)^2g^{\mu\nu}\right)A_\nu \\ &+\frac{1}{2}\begin{pmatrix} r & \pi\end{pmatrix}\begin{pmatrix}-\nabla^2+2(\mu^2-m^2) & -2i\mu\partial_\tau \\ 2i\mu\partial_\tau & -\nabla^2+\frac{1}{\xi}e^2f^2\end{pmatrix}\begin{pmatrix} r \\ \pi\end{pmatrix} \\ &-2if\mu rA^0+ef\left(1-\frac{1}{\xi}\right)A_\mu\partial^\mu\pi\end{aligned}$$

- $\mathcal{R}^{\mu\nu}=\frac{D-2}{R^2}g^{\mu\nu}$ *the Ricci tensor on* S^{D-1} *(obviously* $\mathcal{R}^{00}=0$*)*
- $-\nabla^2=-\partial_\tau^2+\left(-\nabla^2_{S^{D-1}}\right)$

Laplacian Eigenvalues on S^{D-1}

- Represent determinant via *ghosts* :

$$\det \frac{\delta G}{\delta \alpha} = \bar{c}\left(-\nabla^2 + (ef)^2\right) c$$

- The A_0 is a scalar field (belongs to the $\mathbb{R}$ part)
- Split the gauge field $A^i = B^i + C^i$ as

$$\nabla_i B^i = 0 \quad \left(\text{kernel of } \nabla^i\right), \qquad C^i = \nabla^i f \quad \left(\text{image of } \nabla^i\right)$$

$-\nabla^2_{S^{D-1}}$	**scalars**	**vectors**
eigenvalues	$\frac{1}{R^2}\ell(\ell + D - 2)$	$\frac{1}{R^2}\left(\ell\left(\ell + D - 2\right) - 1\right)$
degeneracies	n_b	n_A

$$n_b(\ell) = \frac{(2\ell + D - 2)\Gamma(\ell + D - 2)}{\Gamma(D-1)\Gamma(\ell + 1)}$$

$$n_A(\ell) = \frac{\ell(\ell + D - 2)(2\ell + D - 2)\Gamma(\ell + D - 3)}{\Gamma(\ell + 2)\Gamma(D - 2)}$$

$n_{\text{ghost}} = -2n_b$

Evaluation

The path integral takes the form

$$\langle Q|e^{-HT}|Q\rangle = \text{classical} \times \frac{1}{Z} \left(\det B \times \det \Phi_{4\times 4} \times \det C_{\text{ghosts}}\right)^{-1/2}$$

Results

- $$\boxed{\det B = -\partial_\tau^2 - \nabla^2_{S^{D-1}} + \frac{D-2}{R^2} + (ef)^2}$$

- Scalar field matrix (r, π, A_0, B_i)

$$\begin{pmatrix} -\omega^2 + J_\ell^2 + 2(\mu^2 - m^2) & -2i\mu\omega & -2ie\mu f & 0 \\ 2i\mu\omega & -\omega^2 + J_\ell^2 + \frac{1}{\xi}e^2 f^2 & -ef\left(1 - \frac{1}{\xi}\right)\omega & -ief\left(1 - \frac{1}{\xi}\right)|J_\ell| \\ -2ie\mu f & ef\left(1 - \frac{1}{\xi}\right)\omega & -\frac{1}{\xi}\omega^2 + J_\ell^2 + (ef)^2 & i\left(1 - \frac{1}{\xi}\right)\omega|J_\ell| \\ 0 & ief\left(1 - \frac{1}{\xi}\right)|J_\ell| & i\left(1 - \frac{1}{\xi}\right)\omega|J_\ell| & -\omega^2 + \frac{1}{\xi}J^2_{\ell(s)} + (ef)^2 \end{pmatrix}$$

giving

$$\boxed{\xi \det \Phi = (\omega + \omega_+^2)(\omega + \omega_-^2)(\omega + \omega_1^2)^2}$$

Field	ω_ℓ	ℓ_0
B_i	$\sqrt{J^2_{\ell(v)} + (d-2) + e^2 f^2}$	1
C_i	$\sqrt{J^2_\ell + e^2 f^2}$	1
$(c, \bar{c})$	$\sqrt{J^2_\ell + e^2 f^2}$	0
A_0	$\sqrt{J^2_\ell + e^2 f^2}$	0
ϕ	$\sqrt{J^2_\ell + 3\mu^2 - m^2 + \frac{1}{2}e^2 f^2 \pm \sqrt{\left(3\mu^2 - m^2 - \frac{1}{2}e^2 f^2\right)^2 + 4J^2_\ell \mu^2}}$	0

Table: The fields and their energies as a function of the chemical potentials with a nonvanishing VEV for $\phi, \bar{\phi}$. Note that J^2_ℓ are the Laplacian scalar eigenvalues and $J^2_{\ell(v)}$ are the vector eigenvalues.

- *Functional determinant of the ghosts cancel against the contribution stemming from C_i and A_0, leaving a single ghost zero mode ($\ell = 0$) contribution which in turn cancels one of the zero modes of the scalar.*

Following the stadard steps of infinity cancelations we obtain the 1-loop correction

$$\Delta_0 = \frac{1}{16}\left(-15\mu^4 - 6\mu^2 + 8\sqrt{6\mu^2-2} + 5\right) + \frac{1}{2}\sum_{l=1}\sigma(\ell) - \frac{3e^2\left(\mu^2-1\right)\left(3e^2\left(7\mu^2+5\right)+16\pi^2 g\left(5-9\mu^2\right)\right)}{2048\pi^4 g^2}$$

where

$$\sigma(\ell) = \left(\sqrt{\frac{3e^2\left(\mu^2-1\right)}{16\pi^2 g} + 3\mu^2 + \ell(\ell+2) - 1 - \sqrt{\left(\frac{3e^2\left(\mu^2-1\right)}{16\pi^2 g} - 3\mu^2 + 1\right)^2 + 4\ell(\ell+2)\mu^2}} + \sqrt{\frac{3e^2\left(\mu^2-1\right)}{16\pi^2 g} + 3\mu^2 + \ell(\ell+2) - 1 + \sqrt{\left(\frac{3e^2\left(\mu^2-1\right)}{16\pi^2 g} - 3\mu^2 + 1\right)^2 + 4\ell(\ell+2)\mu^2}}\right)(\ell+1)^2 + 2\ell(\ell+2)\sqrt{\frac{3e^2\left(\mu^2-1\right)}{8\pi^2 g} + \ell(\ell+2) + 1} - \frac{-5\mu^4 + 10\mu^2 + 16l^2(l+1)(l+2) + 8l(l+1)\mu^2 - 5}{4l} + \frac{9e^2\left(\mu^2-1\right)\left(3e^2\left(\mu^2-1\right) - 16\pi^2 g\left(\mu^2 + 2l(l+1) - 1\right)\right)}{512\pi^4 g^2 l}$$

3-loop results

$$\Delta_0 = Q\left(-\frac{9e^4}{128\pi^4\lambda}+\frac{3e^2}{16\pi^2}-2\lambda\right)+Q^2\left(\frac{e^4}{256\pi^4}-\frac{e^2\lambda}{12\pi^2}+\frac{2\lambda^2}{9}\right)$$
$$+Q^3\left(\frac{e^6(9\zeta(3)-1)}{1024\pi^6}-\frac{e^4\lambda(3\zeta(3)+1)}{96\pi^4}+\frac{e^2\lambda^2(3-2\zeta(3))}{12\pi^2}+\frac{2}{27}\lambda^3(16\zeta(3)-17)\right)$$

- ***Check***
  Combining the above with the small charge expansion of Δ_{-1} one reproduces the 1-loop scaling dimension of ϕ at the fixed point: $\Delta_{Q=1} = -\frac{9}{2}\epsilon$.
- Making use of the known 2-loop scaling dimension of ϕ we can fix the NNLO term in the charge at order ϵ^2 and obtain the Δ_Q to order ϵ^2.

Since the fixed point is complex we give the result in terms of the running couplings as

$$\Delta_Q^{\text{2-loop}} = Q\left(\frac{d-2}{2}\right)-3\alpha Q+\frac{1}{3}\lambda(Q-1)Q$$
$$+\alpha^2\left(Q^2+\frac{7Q}{3}\right)-\frac{4}{3}\alpha\lambda Q(Q-1)+\frac{\lambda^2}{9}\left(-2Q^3+2Q^2+Q\right)$$

Scalar QCD

The action reads

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{D}_\mu \bar{\phi}\, D^\mu \phi + m^2\, \bar{\phi}\phi + \frac{\lambda}{24} (\bar{\phi}\phi)^2 \right)$$

and is invariant under $\delta\phi^\alpha = i\theta^a(x)(t^a)^\alpha{}_\beta \phi^\beta$ where

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha + ig(t^a)^\alpha{}_\beta A^a_\mu \phi^\beta, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu$$

Comments

- In QCD there are gauge boson self-interactions appearing as third and fourth order in the action
- Calculating Δ_0 these terms are neglected...

NJLY model

$$\mathcal{L}_{\rm NJLY} = \partial_\mu \bar{\phi} \partial^\mu \phi + \bar{\psi}_j \partial\!\!\!/ \psi^j + g\, \bar{\psi}_{Rj} \bar{\phi} \psi_L^j + g\, \bar{\psi}_{Lj} \phi \psi_R^j + \frac{\lambda}{24} \left(\bar{\phi}\phi\right)^2$$

where $j = 1, \ldots, N_f$ is the number of Dirac fermions and ϕ is the complex scalar field. The model enjoys $U(1)$ and chiral symmetry

$$\Delta_0^{(f)}(\lambda^* \bar{Q}, g) = -6 - \frac{3g\left(\mu^2-1\right)\left(g\left(9\mu^2+3\right)+8\pi^2\lambda\left(13-3\mu^2\right)\right)}{512\pi^4\lambda^2} + \frac{1}{2}\sum_{\ell=1}^{\infty}\sigma^{(f)}(\ell) + \frac{\sqrt{\frac{3g\left(\mu^2-1\right)}{\lambda}+2\pi^2(\mu-3)^2}+\sqrt{\frac{3g\left(\mu^2-1\right)}{\lambda}+2\pi^2(\mu+3)^2}}{\sqrt{2}\pi}$$

where

$$\sigma^{(b)}(\ell) = (1+\ell)^2\left[\omega_+(\ell)+\omega_-(\ell)\right] - 2\ell^3 - 6\ell^2 - 2\mu^2 - 2\left(\mu^2+2\right)\ell + \frac{5\left(\mu^2-1\right)^2}{4\ell}$$

and

$$\sigma^{(f)}(\ell) = 2(1+\ell)(2+\ell)[\omega_{f+}(\ell)+\omega_{f-}(\ell)] + \frac{9g^2\left(\mu^2-1\right)^2}{128\pi^4\lambda^2\ \ell} - \frac{3g\left(\mu^2-1\right)\left(4\ell^2+\mu^2+6\ell-1\right)}{16\pi^2\lambda\ \ell} - 2(\ell+1)(\ell+2)(2\ell+3)$$

Asymptotic Safe Model

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) + \mathrm{Tr}(\bar{\Psi} i \not{D} \Psi) + y\ \mathrm{Tr}(\bar{\Psi}_L \Phi \Psi_R + \bar{\Psi}_R \Phi^\dagger \Psi_L)$$
$$+ \mathrm{Tr}(\partial_\mu \Phi^\dagger \partial^\mu \Phi) - u\left[\mathrm{Tr}(\Phi^\dagger \Phi)\right]^2 - v\ \mathrm{Tr}(\Phi\Phi^\dagger\Phi\Phi^\dagger)$$

1-loop

$$\Delta_0^{(f)} = -\frac{\left(4\mu^2-1\right)\alpha_y N_f^2\left(\left(12\mu^2+1\right)\alpha_y N_f^2 - 2\left(12\mu^2-13\right)N_c\left(N_f\alpha_h+2s\alpha_v\right)\right)}{32N_c^2\left(N_f\alpha_h+2s\alpha_v\right)^2}$$
$$+\sqrt{\frac{2\left(4\mu^2-1\right)\alpha_y N_f^2}{N_c\left(N_f\alpha_h+2s\alpha_v\right)}+(2\mu+3)^2}+\sqrt{\frac{2\left(4\mu^2-1\right)\alpha_y N_f^2}{N_c\left(N_f\alpha_h+2s\alpha_v\right)}+(3-2\mu)^2}$$
$$-6+\frac{1}{2}\sum_{l=1}\sigma^{(f)}(\ell)$$

where

$$\sigma^{(f)}(\ell) = \frac{\left(1-4\mu^2\right)^2 N_f^4\alpha_y^2}{8\ell N_c^2\left(N_f\alpha_h+2s\alpha_v\right)^2} - \frac{\left(4\mu^2-1\right)N_f^2\left(4\ell^2+4\mu^2+6\ell-1\right)\alpha_y}{4\ell N_c\left(N_f\alpha_h+2s\alpha_v\right)}$$
$$+(\ell+1)(\ell+2)\left(\sqrt{\frac{2\left(4\mu^2-1\right)N_f^2\alpha_y}{N_c\left(N_f\alpha_h+2s\alpha_v\right)}+4\mu\left(\mu-2\ell+3\right)+4l(l+3)+9}\right.$$
$$\left.+\sqrt{\frac{2\left(4\mu^2-1\right)N_f^2\alpha_y}{N_c\left(N_f\alpha_h+2s\alpha_v\right)}+4\mu\left(\mu+2\ell+3\right)+4\ell(\ell+3)+9}-4\ell-6\right)$$

Future directions

- Perform the Standard Model analog
- Generalize to AdS/CFT in calculating anomalous dimensions of large charge operators in $\mathcal{N} = 4$ and in general in super-conformal field theories and β-deformations
- Check from the point of view of correlation functions dualities in QFT
- Apply in integrable deformations of WZW-models (Araujo, Celikbas, Orlando, Reffert) compare *large Q* *vs* *large k*

Thanks for your attetion!