OPE coefficients in strongly coupled SCFTs

Raffaele Savelli
University of Rome "Tor Vergata" & INFN

Les Diablerets
July 07, 2022

Based on work with
A. Bissi, F. Fucito, A. Manenti, F. Morales

arXiv: 2112.11899

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Not gauge theories at strong coupling,
but isolated, non-Lagrangian theories!

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Setting up the problem

- CFTs are ubiquitous in physics: Invaluable frameworks to shed light on key properties of realistic QFTs

- Conformal invariance puts strong constraints on observables:

$$G_{ab}(x) = \langle \mathcal{O}_a(x)\mathcal{O}_b(0)\rangle = \frac{c_a\,\delta_{ab}}{|x|^{2\Delta_a}}$$

primary of dim. Δ_a

$$\mathcal{O}_a(x)\mathcal{O}_b(0) = \sum_c \frac{\lambda_{abc}\,\mathcal{O}_c(0)}{|x|^{\Delta_a+\Delta_b-\Delta_c}} + \ldots$$

subleading for |x|->0

- Restrict to 4D $\mathcal{N}=2$ SCFTs :
  - Vacua & interactions preserve $SU(2)_R \times U(1)_R$ R-symmetry
  - Consider chiral primaries $\mathcal{O}_i$ (focus on Coulomb branch)

Setting up the problem

- 2-pt functions non-trivial only between chiral & anti-chiral:

$$G_{ij}(x) = \langle \mathcal{O}_i(x) \bar{\mathcal{O}}_j(0) \rangle \; = \; \frac{c_i \, \delta_{ij}}{|x|^{R_i}} \longrightarrow \text{R-charge} = 2\Delta_i$$

- OPE provides a ring structure:

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k \lambda_{ijk} \, \mathcal{O}_k(0) \; + \; \text{reg.} \qquad \lambda_{ijk} \neq 0 \quad \text{iff} \quad R_k = R_i + R_j$$

- Orthonormal set of operators:

$$\hat{\mathcal{O}}_i \equiv \frac{\mathcal{O}_i}{G_{ii}} \quad \Longrightarrow \quad \left\{ \begin{array}{rcl} \hat{G}_{ij} & = & \delta_{ij} \\ \hat{\lambda}_{i,j,i+j} & = & \sqrt{\frac{G_{i+j,i+j}}{G_{ii}G_{jj}}} \end{array} \right.$$

invariant under homogeneous rescalings $\mathcal{O}_i \to v^i \, \mathcal{O}_i$

SUSY Localization

- Often in SQFT's integrals on moduli spaces give low-energy approximation of path integrals
- Approximation becomes exact if the theory is suitably deformed
- Compute G_{ij} on S^4 : [Pestun '07]
  - Ω-background $\varepsilon_1 = \varepsilon_2 = \varepsilon = R^{-1}$ [Nekrasov '02]

$$C_{ij} = \langle \mathcal{O}_i(N)\bar{\mathcal{O}}_j(S)\rangle_{S^4} = \frac{\int_{\mathrm{i}\mathbb{R}^r} \mathrm{d}^r a\, u_i(a,\epsilon,q)\bar{u}_j(\bar{a},\epsilon,\bar{q})|Z_{\mathbb{R}^4_\Omega}(a,\epsilon,q)|^2}{\int_{\mathrm{i}\mathbb{R}^r} \mathrm{d}^r a\, |Z_{\mathbb{R}^4_\Omega}(a,\epsilon,q)|^2}$$

  - $u_i(a,\epsilon,q) = \langle \mathcal{O}_i \rangle_{\mathbb{R}^4_\Omega}$ (quantum) CB coord. (~TrΦ^i in gauge th.)
  - $Z_{\mathbb{R}^4_\Omega}(a,\epsilon,q) = e^{\sum_{g=0}^{\infty} \epsilon^{2g-2}\mathcal{F}_g(a,q)}$ Nekrasov p.f. ($F_0 = F_{SW}$)

Localization & Mixing

- F_g's encode moduli dependence of gravitational couplings
- Known in gauge theory at weak coupling (instanton expansion)
- But they are intrinsic: Correspond to "quantizing" SW geometry

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- F_g's encode moduli dependence of gravitational couplings
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- But they are intrinsic: Correspond to "quantizing" SW geometry
- S^4 is conformally equivalent to $\mathbb{R}^4$ ✓
- BUT S^4 breaks $U(1)_R$ $\Rrightarrow$ Operators of different R-charge mix!
  - Mixing w/ higher charge forbidden by locality
  - Remove mixing: $G_{ij} = C_{ij} - \sum_{m,n} C_{im} C^{mn} C_{nj}$ $\quad \{R_{m,n}\} < \{R_{i,j}\}$
  - Same as Gram-Schmidt orthogonalization

[Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, Pufu '16]; [Rodriguez-Gomez, Russo '16]

The generalized prepotential

- In a CFT F_g's are (local) homogeneous functions of $\{a_1,\ldots,a_r\}$ of degree 2-2g → dim-less variables $x_k = (\varepsilon/a_k)^2$

- E.g. at rank 1 : $\log Z_{\mathbb{R}^4_\Omega}(x,q) = \mathcal{F}_1(x,q) + \sum_{g\neq 1} \mathfrak{f}_g(q)\, x^{g-1}$

  - Asymptotic series, but not a perturbative expansion

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  - ▸ Asymptotic series, but not a perturbative expansion

- With no a-priori reasons, neglect all $F_{g>1}$

  - ▸ Why should the large-R region $x \ll 1$ be dominant ?

  - ▸ Is this truncation even yielding a finite answer ? YES, always !

- Still, results are surprisingly accurate !

The "u-plane" integral

- $\exists$ exact analytic expression for F_1 dependent on SW geometry:

$$\mathcal{F}_1(a) = \frac{1}{2}\log\left[\det\left(\frac{\partial u_i}{\partial a_j}\right)\right] + \frac{1}{12}\log[\Delta(u(a))]$$

SW-curve discriminant

[Moore, Witten '97]
[Nakajima, Yoshioka '03]
[Shapere, Tachikawa '08]

  - Derivable from modular invariance of path integral
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- Switch integration variables $\{a_1, \dots, a_r\} \rightarrow \{u_1, \dots, u_r\}$

$$C_{ij} = \frac{\int_{\mathbb{R}^r_+} \mathrm{d}^r u \; u_i \bar{u}_j \, |\Delta(u)|^{\frac{1}{6}} \, e^{R^2(\mathcal{F}_0(u)+\bar{\mathcal{F}}_0(\bar{u}))}}{\int_{\mathbb{R}^r_+} \mathrm{d}^r u \, |\Delta(u)|^{\frac{1}{6}} \, e^{R^2(\mathcal{F}_0(u)+\bar{\mathcal{F}}_0(\bar{u}))}}$$

✦ u_k : parameters of the classical SW curve

✦ $F_0(u)$: computable from periods $(a, a_D)(u)$

Convergence

- Re $F_0(u)$ is negative semi-definite in the integration domain

$$\mathcal{F}_0 = \frac{1}{2}\sum_i a_i \frac{\partial \mathcal{F}_0}{\partial a_i} = -\pi \mathrm{i} \sum_i a_i a_D^i = -\pi \mathrm{i} \sum_{i,j} \tau^{ij} a_i a_j$$

F_0 degree-2 homogeneous

$\tau = \partial a_D / \partial a$ degree-0 homogeneous

- $0 = \sum_i a_i \frac{\partial \tau^{jk}}{\partial a_i} = \sum_i a_i \frac{\partial^3 \mathcal{F}_0}{\partial a_j \partial a_k \partial a_i} = \sum_i a_i \frac{\partial \tau^{ji}}{\partial a_j}$

- $\{a_1, \ldots, a_r\}$ purely imaginary & $\mathrm{Im}\, \tau > 0$

Argyres-Douglas theories

- Special IR fixed points in the CB of certain asymptotically-free gauge theories [Argyres, Douglas '95]; [Argyres, Plesser, Seiberg, Witten '95]
- Later generalized and classified using geometry (class-*S*, F-theory, Type II geometric engineering …)
- Distinctive features: Intrinsic strong coupling & CB operators of rational dim. $1 < d < 2$
- Conformal bootstrap puts stringent bounds on observables [Beem,Lemos,Liendo,Rastelli,van Rees '14]; [Cornagliotto,Lemos,Liendo '17]; [Gimenez-Grau,Liendo '20];

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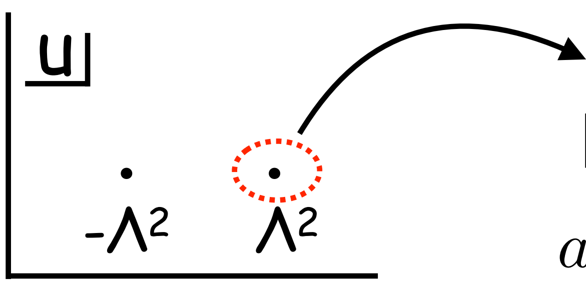
  - We derived more OPE bounds, also for rank-2 theories
  - We found upper bounds on dimension of the lightest neutral unprotected scalar operator in the OPE $u \times \bar{u}$

Argyres-Douglas at rank 1

- Pure SU(2) SYM :

$y^2 = (x^2-u)^2 - \Lambda^4$ singularities (massless d.o.f.) at $\Delta(u)=0$



light magnetic monopole ➟ local physics perturbative in dual e.m. frame

$$a_D \sim (u - \Lambda^2)\,, \qquad a = a_* + \mathrm{i}\, a_D \log(a_D/\Lambda)$$

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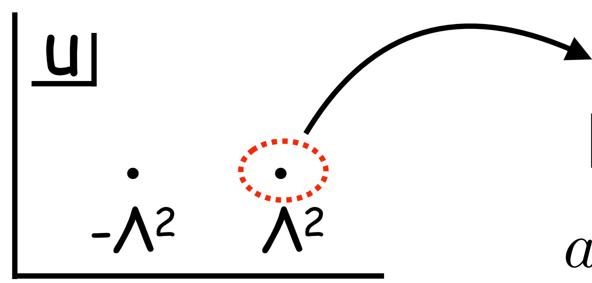

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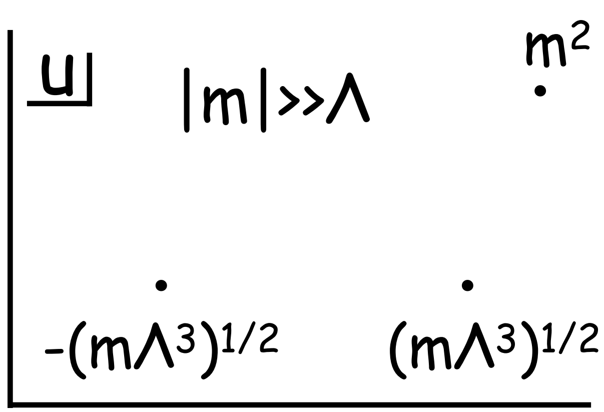
- SU(2) SQCD w/ 1 fundamental flavor :

$y^2 = (x^2-u)^2 - (x-m)\,\Lambda^3$ $\Delta(u,m)=0$ ➠ 3 singularities



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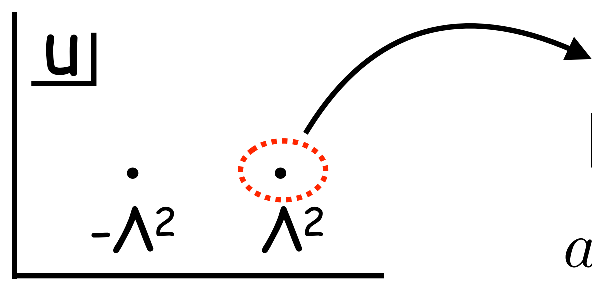

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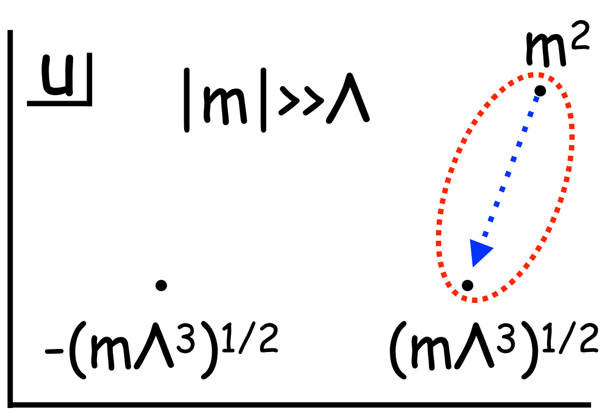

$\Delta(\Delta)(m)=0$ ➟ $m=-3\Lambda/4$

electron & monopole simultaneously massless

local physics: interacting non-Lagrangian CFT !

$$a \sim a_D \sim (u-u_*)^{\frac{1}{d}}$$

Rank-1 CB geometries

- Same story for SU(2) SQCD w/ 1,2,3 fund. flavors :
  - $\mathscr{H}_0$: d=6/5 ; $\mathscr{H}_1$: d=4/3 ; $\mathscr{H}_2$: d=3/2
  - Are large-R saddles of S^4 part. funct. of SQCD [Russo '14,'19]
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- To compute C_{ij} we just need to know the SW data of SCFT
- We can extend computation to other isolated theories!
  - At rank 1, use F-theory on singular elliptically-fibered K3's :

$\mathcal{H}_0: \quad y^2 = x^3 + u$	$E_6: \quad y^2 = x^3 + u^4 \quad d = 3$
$\mathcal{H}_1: \quad y^2 = x^3 + ux$	$E_7: \quad y^2 = x^3 + u^3 x \quad d = 4$
$\mathcal{H}_2: \quad y^2 = x^3 + u^2$	$E_8: \quad y^2 = x^3 + u^5 \quad d = 6$

$\ast$ no bootstrap bounds available yet!

[Minahan, Nemeschansky '96]

Results for rank 1

- At rank 1 all non-trivial info is in d , everything else fixed by dimensional analysis !
- $F_0 = - c^2/2 \ u^{2/d}$, c irrelevant real constant (drops in OPE's)
- $F_1 = \alpha \log u^{1/d}$, $\alpha = 2(a - a_{free}) = 3(d-1)/2$
- Finally: $$\boxed{C_{ij} = \frac{1}{(cR)^{d(i+j)}} \frac{\Gamma[\frac{d}{2}(i+j+3)-1]}{\Gamma[\frac{3d}{2}-1]}}$$
- Comparison to bootstrap:

✳ Same conclusions from matrix model

[Grassi, Komargodski, Tizzano '19]

OPE	method	$\mathcal{H}_0$	$\mathcal{H}_1$	$\mathcal{H}_2$
$\lambda^2_{u\,u\,u^2}$	Boot.	2.167	2.359	2.698
		2.142	2.215	2.298
	Localiz.	2.098	2.241	2.421
$\lambda^2_{u\,u^2\,u^3}$	Boot.	3.637	4.445	
		3.192	3.217	
	Localiz.	3.300	3.674	4.175

Beyond rank 1

- No conceptual obstacle, but technically harder
- Challenge is to compute periods $(a, a_D)^i\ (u_k)$
- E.g. rank-2 Argyres-Douglas $(A_1, A_{4;5})$ from $SU(5;6)$ SYM
  - SW: $y^2 = x^{5,6} - u\,x - v$, $\lambda_{SW} = x\, dy$
- Change variables:
  $(u,v) \longrightarrow (\kappa, v)$, $\kappa = v\, u^{-4/5;-5/6}$
- Choose 1-cycle basis:
  $- i\pi\, a_k\, a_D{}^k (\kappa, v) = F_0 = -\, v^{2/d(v)}\, f(\kappa)$
- C_{ij} : First integrate analytically in v , then numerically in κ

OPE	method	(A_1, A_4)	(A_1, A_5)
$\lambda^2_{u\,u\,u^2}$	Boot. $\{$	2.102	2.231
		2.024	2.055
	Localiz.	1.878	1.929
$\lambda^2_{u\,v\,uv}$	Boot. $\{$	1.125	1.233
		0.981	0.960
	Localiz.	1.043	1.039
$\lambda^2_{v\,v\,v^2}$	Boot. $\{$	2.533	2.709
		2.181	2.195
	Localiz.	2.231	2.203

Large Charge

- For large-charge / large-dimension states, truncating at F_1 is an approximation !

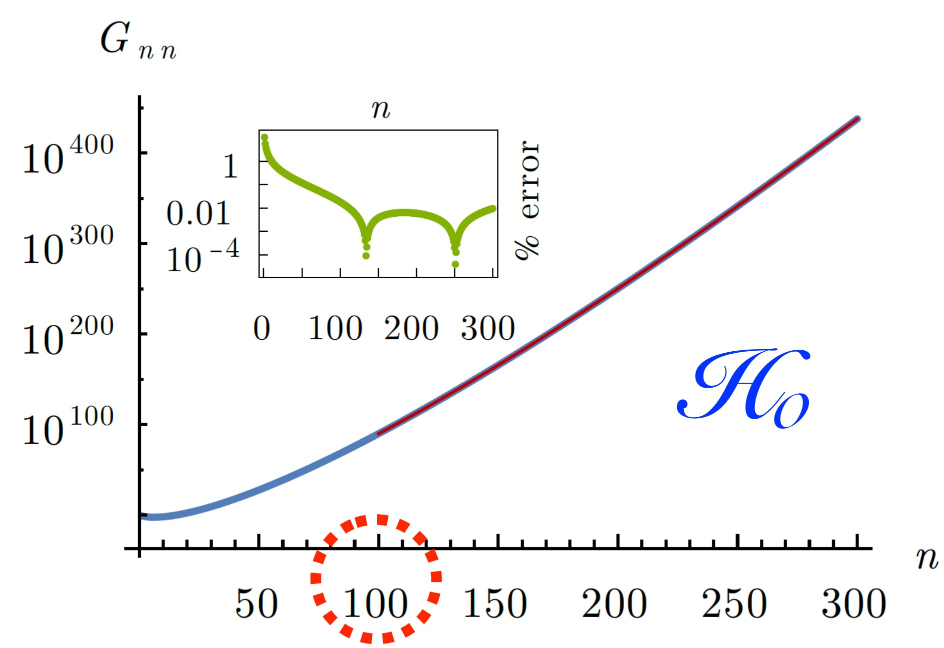
- Recall: $C_{NN} \sim \int_0^\infty \mathrm{d}x \, \exp\left(-\frac{2\mathfrak{f}_0}{x} - d(N+\frac{3}{2})\log x\right)$

- Large-N saddle at $x \approx \frac{2\mathfrak{f}_0}{dN} \implies a \sim \frac{\sqrt{N}}{R}$ All $F_{g>1}$ subleading

- Agreement w/ large-charge asymptotics of rank-1 correlators:

$$G_{NN} \propto \Gamma[dN + \alpha + 1]$$

[Hellerman, Maeda, Orlando, Reffert, Watanabe '18]



Summary & outlook

- ✓ Based on SUSY localization, we found a formula for OPE's between CB operators of isolated 4D $\mathcal{N}=2$ SCFT's of any rank
- ✓ The formula only uses input data from SW geometry
- ✓ Despite truncation of higher gravitational corrections, it gives very accurate results for any value of charge

- ◉ Compute $F_{g>1}$, e.g. using the holomorphic anomaly equation
- ◉ Clarify relationship between large-charge & genus expansion
  - ▸ What's after F_1 ? Is there another saddle point at small radius ?
  - ▸ Estimate F_∞ by comparing to the known $1/J$ expansion ?