The analytic structure of the large charge expansion

Matías Torres
In collaboration with Oleg Antipin, Jahmall Bersini and Francesco Sannino

Introduction

Conformal Field Theory

Theories with conformal invariance play a big role in theoretical and experimental physics. Some examples are:

- Critical Phenomena
- Renormalization group
- AdS/CFT
- String theory

A CFT is determined by it is conformal data $\{\mathcal{O}_i, \{\Delta_i, C_{ijk}\}\}$.

The standard method to deal with interacting QFTs is perturbation theory. Though it is a powerful tool, also has limitations. For example:

- Perturbative expansions have a zero radius of convergence.
- Cannot capture non-perturbative effects.
- Multileg amplitudes (multiparticle production).

Resurgence

It is a well known fact that the large order behavior of the perturbative expansion contains non-trivial information that is lost in the perturbative regime.

Given a perturbative expansion

$$\mathcal{O}(x) = \sum_n c_n x^n, \tag{1}$$

we have:

- If the series convergence, c_n contains information about the singularities of $\mathcal{O}(x)$.
- If the series diverge factorially, c_n contains information about the non-perturbative sectors of the theory.

Darboux's theorem

Given a function which has a convergent perturbative series, the Darboux's theorem relates the large order behaviour of the coefficients with the singularities of the function. Concretely if $\mathcal{O}(x) = \sum_n c_n x^n$ has a branch cut around at x_0

$$\mathcal{O}(x) = f(x)\left(1 - \frac{x}{x_0}\right)^{-p} + \text{ analytic}\,, \quad x \to x_0, \tag{2}$$

For large n we have

$$c_n \sim \frac{1}{x_0^n}\left[f(x_0)\begin{pmatrix} n+p-1 \\ n \end{pmatrix} - x_0 f'(x_0)\begin{pmatrix} n+p-2 \\ n \end{pmatrix} - \dots\right], \tag{3}$$

where x_0 corresponds to the nearest singularity of $\mathcal{O}(x)$ around the origin.

$$p = 1 + \lim_{n\to\infty} n\left(x_0 \frac{c_n}{c_{n-1}} - 1\right), \tag{4}$$

$$f(x_0) = \lim_{n\to\infty} \frac{c_n}{\left(\frac{1}{x_0}\right)^n \begin{pmatrix} n+p-1 \\ n \end{pmatrix}}. \tag{5}$$

Dondi *et al* [1], studied the spectrum of charged operators in the $O(2N)$ model in $d = 3$ in the double scaling limit

$$N \to \infty, \quad Q \to \infty, \quad \hat{q} = \frac{Q}{2N} = \text{fixed}, \tag{6}$$

In this limit the scaling dimension $\Delta = \Delta(\hat{q})$ has different behaviours for small and large $\hat{q}$. In particular, for small $\hat{q}$ the perturbative series convergent while for large $\hat{q}$ is $(2n)!$ divergent. The central object to compute non-operturbative corrections was

$$\mathrm{Tr}\, e^{\Delta_{S^{d-1}} t}. \tag{7}$$

Here it was found that the non-perturbative contribution have a geometrical origin given by the wordline instantons.

[1] Resurgence of the large-charge expansion. Dondi, Kalogerakis, Orlando, Reffert 2001.

Goal

Our goal was to understand the analytic properties of the semiclassical expansion for charged operators in different models.

- $O(N)$ model in $d = 4 - \epsilon$ and $d = 3 - \epsilon$ in the double scaling limit

$$Q \to \infty, \quad g \to 0, \quad gQ = \text{fixed}, \tag{8}$$

  where g is the interacting coupling constant. Here the scaling dimension takes the form

$$\Delta_Q = \frac{1}{g}\Delta_{-1}(gQ) + \Delta_0(gQ) + \ldots \tag{9}$$

We studied the small and large gQ expansion for Δ_{-1} and Δ_0.

The Lagrangian for the $O(N)$ model with a sixth interaction is

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i + \frac{g^2}{8 \times 3!} \left(\phi_i \phi_i\right)^3 . \tag{10}$$

In $d = 3 - \epsilon$ the IR fixed point has the form

$$\frac{g^2}{(4\pi)^2} = \frac{2\epsilon}{22 + 3N}, \tag{11}$$

Here the beta function is zero at one-loop for $d = 3$, therefore the theory is one-loop invariant at $d = 3$. The leading and next to leading scaling dimension are [2][2]

$$\Delta_{-1}(gQ) = gQF_{-1}\left(\frac{g^2 Q^2}{2\pi^2}\right), \quad F_{-1}(x) = \frac{1 + \sqrt{1+x} + \frac{x}{3}}{\sqrt{2}(1 + \sqrt{1+x})^{\frac{3}{2}}}, \quad x = \frac{g^2 Q^2}{2\pi^2}, \tag{12}$$

$$\Delta_0(gQ) = \Delta_0^{(a)}(gQ) + \left(\frac{N}{2} - 1\right) \Delta_0^{(b)}(gQ), \tag{13}$$

[2]Feynman diagrams and the large charge expansion in 3ϵ dimensions, Rattazzi *et al.* 2019.

where

$$\Delta_0^{(a)}(gQ) = \frac{1}{2}\sum_{\ell=0}^{\infty} n_\ell \left[\omega_+(\ell) + \omega_-(\ell)\right] \tag{14}$$

$$\Delta_0^{(b)}(gQ) = \sum_{\ell=0}^{\infty} n_\ell \omega_*(\ell) \tag{15}$$

Here the dispersion relations are

$$\omega_\pm^2(\ell) = J_\ell^2 + 2\left(2\mu^2 - \frac{(d-2)^2}{4}\right) \pm 2\sqrt{J_\ell^2\mu^2 + \left(2\mu^2 - \frac{(d-2)^2}{4}\right)^2}, \tag{16}$$

$$\omega_*(\ell) = \sqrt{J_\ell^2 + \mu^2}. \tag{17}$$

The spectrum contains:

- A gapless mode ω_- with velocity $v = \frac{1}{\sqrt{2}}$.
- A gapped mode with mass $\omega_+(0) = 2\sqrt{2\mu^2 - \frac{(d-2)^2}{4}}$.
- (N-2) spectator modes with mass $\omega_*(0) = \mu$.

Here μ is related to the charge Q via $\mu = \frac{1}{2\sqrt{2}}\sqrt{1 + \sqrt{1 + \frac{g^2Q^2}{2\pi^2}}}$.

Small and large expansion

Considering these definitions we can study the expansion for small and large values $x = \frac{g^2 Q^2}{2\pi^2}$. Numerically, using the Darboux's theorem we found:

- Small x expansion:

$$\Delta_{-1}(x) = f_{-1}(x)(1+x)^{3/2} + \text{analytic}, \tag{18}$$

$$\Delta_0(x) = f_0(x)(1+x)^{1/4} + g_0(x, N)(1+x)^{1/2} + \text{ analytic }. \tag{19}$$

- Large x expansion:

$$\Delta_{-1}(x) = f_{-1}(x)(1+x)^{3/2} + \text{analytic}. \tag{20}$$

It is interesting to notice that at $x = -1$ radial mode ω_- becomes massless.

On the other hand, the story it is completely different for $\Delta_0(x)$.

We will only focus on $\Delta_0^{(b)}(x)$

$$\Delta_0^{(b)} = \sum_{l=0}^{\infty}(2\ell+1)\sqrt{\mu^2+\ell(\ell+1)}, \tag{21}$$

$$= \frac{1}{\mu}\sum_{k=0} a_k \mu^{-2k}, \tag{22}$$

where

$$a_k = \sum_{m=1}^{k+2} \frac{(-1)^{k+1}B_{2m}\Gamma\left(k+\frac{1}{2}\right)}{4\sqrt{\pi}m\Gamma(k+2)}\left[2\begin{pmatrix} k+1 \\ -k+2m-3 \end{pmatrix} + \begin{pmatrix} k+1 \\ -k+2m-2 \end{pmatrix}\right]. \tag{23}$$

the large order behaviour is given

$$a_k \approx -\pi^{-2k-5}\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{5}{2}\right), \tag{24}$$

which implies that the perturbative series has a double-factorially divergence.

According to resurgence theory we can promote this asymptotic series to a transeries of the form

$$\phi(z) = \phi^{(0)}(z) + \sum_{j\neq 0} \sigma_j e^{-A_j/z^{1/\beta_j}} z^{-b_j/\beta_j} \Phi^{(j)}(z), \quad \Phi^{(j)}(z) \sim \sum_{i=0}^{\infty} a_i^{(j)} z^{i/\beta_j}, \tag{25}$$

where the parameters β_j, A_j and b_j are encoded in the large order behaviour of the a_k coefficients as

$$a_k \sim \sum_j \frac{S_j}{2\pi i} \frac{\beta_j}{A_j^{\beta_j k + b_j}} \sum_{i=0}^{\infty} a_i^{(j)} A_j^i \Gamma\left(\beta_j k + b_j - i\right). \tag{26}$$

We focused only on the leading term

$$\hat{a}_k \equiv a_k^{(m=k+2)}\Big|_{k\to k-2} = -\pi^{-2k-1} \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(k - \frac{3}{2}\right) \zeta(2k). \tag{27}$$

Using the following relation

$$2^{2k} \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(k - \frac{3}{2}\right) = \sqrt{\frac{\pi}{2}} \sum_{i=0}^{\infty} \gamma_i \Gamma\left(2k - \frac{3}{2} - i\right). \tag{28}$$

We found

$$\hat{a}_k = -\frac{1}{4\pi^2} \sum_{j=1} \frac{j^{-3/2}}{(2\pi j)^{2k-3/2}} \sum_{i=0} \gamma_i \Gamma\left(2k - \frac{3}{2} - i\right) \tag{29}$$

Identifying the coefficients of the transeries as

$$\beta_j = 2, \quad b_j = -3/2, \quad A_j = 2\pi j, \quad \frac{S_j}{2\pi i} a_0^{(j)} = -\frac{\gamma_0}{j^{3/2} 8\pi^2}, \quad a_{i>0}^{(j)} = \frac{a_0^{(j)}}{(2\pi j)^i} \frac{\gamma_i}{\gamma_0} \tag{30}$$

the non-perturbative corrections take the form

$$\Delta_0^{(b)} \supset \sum_{j=1} e^{-2\pi j \mu} \mu^{3/2} \sum_{i=0} a_i^{(j)} \mu^{-i} \tag{31}$$

in terms of the charge

$$\Delta_0^{(b)} \supset (gQ)^{5/4} \sum_{j=1} \exp\left(-\frac{\sqrt{\pi}}{2^{3/4}} j \sqrt{gQ}\right) \sum_{i=0} a_i^{(j)} \left(2^{7/4}\sqrt{\pi}\right)^i (gQ)^{-i/2} \tag{32}$$

similarly to [1] the non-perturbative corrections scales as $e^{-\sqrt{Q}}$.

$O(N)$ in $d = 4 - \epsilon$

Now we will focus on the $O(N)$ model in $d = 4 - \epsilon$ with quartic interaction

$$S = \int d^d x \left(\frac{(\partial\phi_i)^2}{2} + \frac{(4\pi)^2 g_0}{4!} (\phi_i\phi_i)^2 \right) \tag{33}$$

This model exhibit a weakly coupled W-F fixed point

$$g^*(\epsilon) = \frac{3\epsilon}{8+N} + O\left(\epsilon^2\right) \tag{34}$$

The leading contribution Δ_{-1} read as

$$\frac{4\Delta_{-1}}{g^* Q} = \frac{3^{\frac{2}{3}}\left(x+\sqrt{-3+x^2}\right)^{\frac{1}{3}}}{3^{\frac{1}{3}}+\left(x+\sqrt{-3+x^2}\right)^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}\left(3^{\frac{1}{3}}+\left(x+\sqrt{-3+x^2}\right)^{\frac{2}{3}}\right)}{\left(x+\sqrt{-3+x^2}\right)^{\frac{1}{3}}}, \tag{35}$$

where $x = 6g^*Q$ while Δ_0 is given by

$$\Delta_0 = \frac{1}{2}\sum_{\ell=0}^{\infty} n_\ell \left[\omega_+(\ell) + \omega_-(\ell) + (N-2)\omega_*(\ell)\right], \tag{36}$$

Small and large x expansion

Using the Darboux's theorem we found that the small large

- Small x expansion:

$$\Delta_{-1} = f_{-1}(x)\left(1+\frac{x}{\sqrt{3}}\right)^{3/2} + \text{ analytic },\tag{37}$$

$$\Delta_0 = f_0(x)\left(1+\frac{x}{\sqrt{3}}\right)^{1/4} + g_0(x,N)\left(1+\frac{x}{\sqrt{3}}\right)^{1/2} + \text{ analytic}\tag{38}$$

- Large x expansion:

$$\Delta_{-1} = f_{-1}(x)\left(1+\frac{x}{\sqrt{3}}\right)^{3/2} + \text{analytic}\tag{39}$$

Again, the point $x=-\sqrt{3}$ corresponds to the value of Q for which the radial mode becomes massless.

Using this we can test a claim made in [3][3], about the relation between different orders in the semiclassical expansion. If $\Delta_j = \sum_n a_{jn}(gQ)^n$ the coefficients should obey

$$\frac{a_{j+1,n-1}}{a_{j,n}} \approx n \tag{40}$$

Solving the previous recursive relation we find

$$a_{j,n} = b_j \left(\frac{1}{-\sqrt{3}}\right)^n \begin{pmatrix} n+j-3/2 \\ n \end{pmatrix} \left[1 + O\left(\frac{1}{n}\right)\right] \tag{41}$$

According to the Darboux's theorem all the Δ_j should be non-analytic around $x = -\sqrt{3}$

$$\Delta_j = f_j(x)\left(1 + \frac{x}{\sqrt{3}}\right)^{1/2-j} + \text{ analytic} \tag{42}$$

Nevertheless this expression does not contains all the information of the singularity.

[3]The Epsilon Expansion Meets Semiclassics , Rattazzi *et al.* 2020

Different from the $d = 3 - \epsilon$ case, the large x expansion of $\Delta_0^{(b)}$ is convergent

$$\Delta_0^{(b)}(gQ) = \sum_{\ell=0}^{\infty}(\ell+1)^2\sqrt{\mu^2 + \ell(\ell+2)} \tag{43}$$

$$= \frac{1}{\Gamma(s)}\int_0^{\infty} dt t^{s-1} e^{-\mu^2 t}\,\mathrm{Tr}\left(e^{\Delta_{S^{3-\epsilon}} t}\right)\Big|_{s=-1/2} \tag{44}$$

$$= \sum_{k=0} a_k \frac{\Gamma\left(-1/2 + k - \frac{3-\epsilon}{2}\right)}{-2\sqrt{\pi}}\mu^{4-\epsilon-2k}, \tag{45}$$

where a_k are the coefficients of the heat kernel expansion $\mathrm{Tr}\left(e^{\Delta_{S^{3-\epsilon}} t}\right) = \sum_{k=0} a_k t^{k+\frac{3-\epsilon}{2}}$. Due to the Gamma function in the numerator, the terms with $k = 0, 1, 2$ diverge in the limit $\epsilon \to 0$ and need to be renormalized.

For example, the $k = 0$ term is given by

$$-a_0 \frac{\Gamma(-2+\epsilon/2)}{2\sqrt{\pi}} \mu^{4-\epsilon} = \left[-\frac{1}{8\epsilon} + \frac{1}{32}\left(4\gamma_E - 5 - 4\log(2)\right) + \frac{1}{8}\log(\mu) + O(\epsilon)\right]\mu^4 \tag{46}$$

By renormalizing the first three coefficients, we obtain a close form for $\Delta_0^{(b)}$

$$\Delta_0^{(b)} = -\frac{5\mu^4}{32} + \frac{\mu^2}{6} - \frac{1}{20} + \frac{1}{8}\left(\mu^2 - 1\right)^2 \left(\log\left(\mu - \frac{1}{\mu}\right) + \gamma_E - \log(2)\right) \tag{47}$$

It follows that $\Delta_0^{(b)}$ has an esential singularity at $\mu = 0$ and two logarithmic branch cuts at from $\mu = -1$ to $\mu = -\infty$ and from $\mu = 0$ to $\mu = 1$

Monopoles Operators

Now we apply our same analysis to a different model. The action for QED_3 in 3 dimensions is given by

$$S = \int d^3x \left[\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}^i (\partial + iA) \psi^i \right] \tag{48}$$

This theory has associated a topological current

$$J_\mu = \frac{1}{4\pi} \epsilon_{\mu\nu\rho} F^{\nu\rho} \tag{49}$$

For large N_f the theory flows to a conformal theory. Mapping the theory on the sphere we obtain

$$\Delta_Q = E_Q \left[A^Q \right] \equiv -\log Z_{S^2 \times \mathbb{R}} \left[A^Q \right] \tag{50}$$

The leading contribution read as

$$\Delta_{-1} = 4 \sum_{\ell=Q+1}^{\infty} \ell \sqrt{\ell^2 - Q^2} = Q^{3/2} \sum_{k=0}^{\infty} a_k \frac{1}{Q^k} \tag{51}$$

where the coefficients a_k are given by

$$a_k = \frac{2}{\pi^2 k!}(-1)^{k+1} \frac{1}{(4\pi)^k} \Gamma\left(k - \frac{3}{2}\right) \Gamma\left(k + \frac{5}{2}\right) \sin\left(\frac{\pi}{4}(2k+1)\right) \zeta\left(k + \frac{3}{2}\right) \tag{52}$$

this series is asymptotic. The associated Borel transform read as

$$\mathcal{B}\left[\frac{\Delta_{-1}}{Q^{3/2}}\right](t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \tag{53}$$

$$= \sum_{m=1}^{\infty} \frac{(i-1)}{\sqrt{2}\pi m^{3/2}} \left[{}_2F_1\left(-\frac{3}{2}, \frac{5}{2}; 1; -\frac{it}{4m\pi}\right) + i\,{}_2F_1\left(-\frac{3}{2}, \frac{5}{2}; 1; \frac{it}{4m\pi}\right)\right] \tag{54}$$

Here ${}_2F_1(a, b; c; x)$ denotes the Hypergeometric function, which can be analytically continued in the complex plane along any path avoiding the branch points at $x = 1$ and $x = \infty$.

Therefore $\mathcal{B}\left[\frac{\Delta_{-1}}{Q^{3/2}}\right](t)$ features an infinite series of poles $t = 4\pi i m, m \in \mathbb{Z}$. Therefore, both lateral summation coincide. The resumed serie takes the form

$$\Delta_{-1} = Q^{5/2} \int_0^\infty dt e^{-Qt} \mathcal{B}\left[\frac{\Delta_{-1}}{Q^{3/2}}\right](t) \tag{55}$$

$$= \sum \frac{2iQ^2}{\pi m} \left[e^x K_2(x) - e^{-x} K_2(-x)\right], \quad x \equiv 2i\pi m Q \tag{56}$$

where $K_2(x)$ is the modified Bessel function of the second kind.

Conclusions

Conclusions

Considering the semiclassical expansion

$$\Delta_Q = \frac{1}{g}\Delta_{-1}(x) + \Delta_0(x) + \ldots \tag{57}$$

We conclude that:

- The singularities of Δ_{-1} and Δ_0, for small and large x are strickly related to the possitivity of the masses of the spectrum. Even more, their nature is exactly the same in $d = 4 - \epsilon$ and $d = 3 - \epsilon$.
- The form of the of the non-perturbative contributions does not depend of the double scaling. Their origin is merely geometric.
- The behaviour of the large x expansion for Δ_0 is completely different in $d = 3 - \epsilon$ and $d = 4 - \epsilon$. While in the first case Δ_0 acquires non-perturbative contributions from wordline instantons, in $d = 4 - \epsilon$ they do not appear, leading to an analytic expression.

Thank you!