

Feynman integrals in dimensional regularisation and extension of Calabi-Yau motives

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based on [1]=[arXiv:1912.06201v2](https://arxiv.org/abs/1912.06201v2), [2]=[arXiv:2008.10574v1](https://arxiv.org/abs/2008.10574v1) and
[3]=[arXiv:2108.05310](https://arxiv.org/abs/2108.05310)



Introduction: Emerging relation between:

Feynman integrals \Leftrightarrow Algebraic/Arithmetic Geometry

Planar Feynman graph	Geometry	Max Cut Integrals
1-loop	Pts in Fano 1-fold	rational functions
2-loop	elliptic curve in Fano 2-fold	elliptic functions
3-loop	K3 surface in Fano 3-fold	K3-periods
4-loop	CY-3-fold in Fano 4-fold	CY 3-fold periods
⋮	⋮	⋮

I. Gel'fand, S. Bloch, P. Vanhove, M.Kerr, C. Duran, S. Weinzierl, F. Brown, O. Schnetz, J. Bourjaily, A. McLeod, M.

Hippel, M. Wilhelm, J. Broedel, L. Trancredi, S. Müller-Stach, . . . + 248 cits. in [3]

Definition: A Calabi -Yau n -fold M is a Kähler manifold of complex dimension n with the following additional equivalent properties

- a.) The Ricci curvature **vanishes** $R_{i\bar{j}} = 0$
- b.) The canonical class is **trivial** $K_M = c_1(T_M) = 0$
- c.) It has a no-where vanishing **holomorphic** $(n, 0)$ -form Ω (and a type $(1, 1)$ Kähler form ω)
- d.) It has $SU(n)$ holonomy
- e.) It has two covariant constants spinors ...

Constructions: Trick make $c_1(M) = 0$ by **adjunction formula**: Take any Fano variety F , i.e. a Kähler variety with $c_1(F) > 0$ and

- a). take a section of K_F (compact)
- b). take the complement of a section of K_F in F (non-compact)
- c) take the total space of the anti-canonical line bundle $\mathcal{O}(-K_F) \rightarrow F$ (non-compact)

Examples:

- a.) $F = \mathbb{P}^2$, $K_F = 3H$, i.e. a section of K_F is a homogenous cubic, often written in Weierstrass form

$$wy^2 = 4x^3 - g_2(z)xw^2 - g_3(z)w^3$$

Using the Weierstrass $\mathcal{P}(\tau(z), \zeta)$ -function this is identified with a 2-torus with complex structure $\tau(z) \sim_{loc} z$, which is (Ricci) flat. Note $\Omega = dx/y$

- b.) $F = \mathbb{P}^1 \sim S^2$, $K_F = 2H$. The section $x^2 + y^2 = 0$ defines two Pts. A sphere minus 2 Pts is 2-cylinder

which is (Ricci) flat

- c.) $F = \mathbb{P}^1$: $\mathcal{O}(-2) \rightarrow \mathbb{P}^1$ is the local neighbourhood of a two sphere in $K3$, which is flat by the adjunction formula

Main properties:

- Their complex moduli spaces \mathcal{M}_{cs} are **unobstructed**. This leads to **families** of CY n-folds over \mathcal{M}_{cs} .
- Periods $\Pi(z) = \int_{\Gamma} \Omega(z)$ parametrize locally faithfully the \mathcal{M}_{cs} (local) **Torelli Theorem**

- Periods are completely characterized by **homogenous linear differential equations** called **Gauss-Manin– or Picard-Fuchs system**.
- CY manifolds come generically in **mirror pairs** (M, W) for which the complex deformations and the complexified Kähler deformations are **exchanged**
- Periods fulfill **Griffiths transversality** conditions that leads to a **special geometric structure** on \mathcal{M}_{cs}

For the Banana diagrams: this dictionary is best worked out [1,2]. The aim of this talk is to explain the applications of the dictionary relating Feynman integrals to families of Calabi-Yau motives and to extend them to include the dimensional regularization parameter [3] ϵ and comment on the extensions to other graphs.

Consider **l-loop Feynman integrals** in general dimensions $D \in \mathbb{R}_+$ of the form

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^l \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^p \frac{1}{D_j^{\nu_j}} \quad (1)$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \dots, p$ are the propagators, q_j is the j^{th} momenta through D_j , $m_j^2 \in \mathbb{R}_+$ are masses, $i \cdot 0$ indicates the choice of contour/branchcut in \mathbb{C} .

Subject to momentum conservation the p_j are linear in the external momenta p_1, \dots, p_E , $\sum_{i=j}^E p_j = 0$ and the loop momenta k_r .

$$\epsilon := \frac{D_0 - D}{2}$$

describes the deviation from a critical dimension D_0 , which depends on the graph.

The Feynman integral depends besides D on dot

products of p_i and the masses m_j^2 , written compactly in a vector $\underline{x} = (x_1, \dots, x_N) = (p_{i_1} \cdot p_{i_2}, m_j^2)$.

Actually, dimensional analysis of $I_{\underline{\nu}}$ shows that it depends only on the ratios of two parameters x_i . For example, we can choose

$$z_k := x_k / x_N \quad \text{for } 1 \leq k < N$$

and label now the parameters of $I_{\underline{\nu}}$ by these \underline{z} .

The propagator exponents $\underline{\nu} \in \mathbb{Z}^p$ span a lattice. There is a finite set of integrals $I_{\underline{\nu}}(\underline{x}, D)$ so called **master integrals** which generate all integrals in this lattice.

A set of master integrals can be found by using **integration by parts (IBP) identities**

$$\int \prod_{r=1}^l \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^\mu} \left(q_l^\mu \prod_{j=1}^p \frac{1}{D_j^{\nu_j}} \right) = 0 .$$

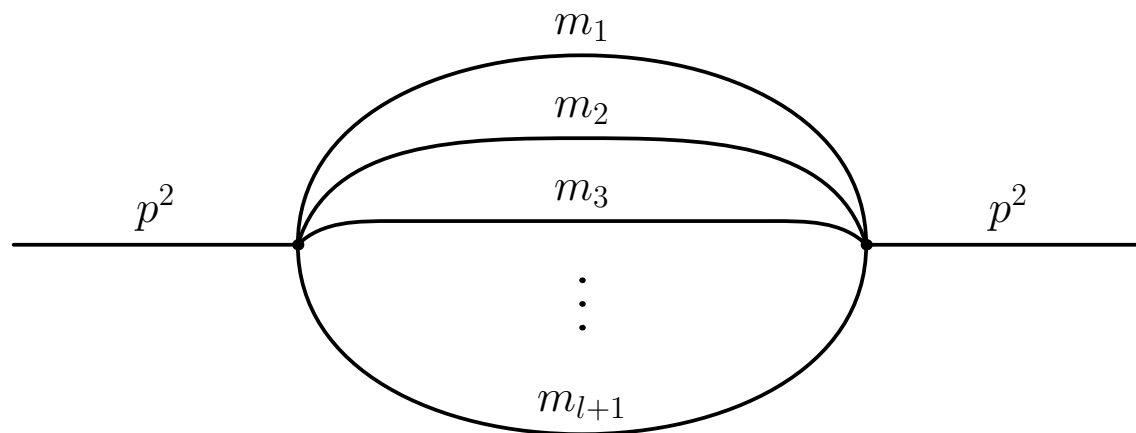
These give relations between different exponents $\underline{\nu}$.

Among the elements in the lattice \mathbb{Z}^p and, in particular, for the master integrals one can define **sectors** and a **semi-ordering** on the latter by defining a map

$$\underline{\nu} \mapsto \underline{\vartheta}(\underline{\nu}) =: (\theta(\nu_j))_{1 \leq j \leq p} .$$

where θ is the Heaviside step function. The semi-ordering is then defined by $\underline{\mathcal{V}}(\underline{\nu}) \leq \underline{\mathcal{V}}(\underline{\tilde{\nu}})$, iff $\theta(\nu_j) \leq \theta(\tilde{\nu}_j)$, $\forall j$. This defines an inclusive order on subgraphs with less propagators and therefore simpler topology.

Main Example: A very simple series of such Feynman amplitudes with loop order l are the **banana diagrams** in critical dimension $D_0 = 2$:



$$D_j = k_j^2 - m_j^2, \quad 1 \leq j \leq l,$$

$$D_{l+1} = (k_1 + \dots + k_l - p)^2 - m_{l+1}^2,$$

$$\underline{z} = (m_1^2/p^2, \dots, m_{l+1}^2/p^2).$$

Master Integrals: The banana graph has $2^{l+1} - 1$ master integrals in $l + 2$ sectors:

$l + 1$ sectors correspond to $\vartheta(\underline{\nu}) = (1, \dots, 1, 0, 1 \dots 1)$. These sectors correspond all to l -loop tadpole integrals

$$J_{l,i}(\underline{z}; \epsilon) = \frac{(-1)^{l+1} (p^2)^{l\epsilon} \epsilon^l}{\Gamma(1 + l\epsilon)} I_{1..1,0,1..1}(\underline{x}; D) = -\frac{\Gamma(1 + \epsilon)^l}{\Gamma(1 + l\epsilon)} \prod_{\substack{j=1 \\ j \neq i}}^{l+1} z_j^{-\epsilon}.$$

$2^{l+1} - l - 2$ master integrals come from the sector $\vartheta(\underline{\nu}) = (1, \dots, 1)$, $\underline{k} \in \{0, 1\}^{l+1}$, $1 \leq |\underline{k}| \leq l - 1$,

$$J_{l,\underline{0}}(\underline{z}; \epsilon) = \frac{(-1)^{l+1}}{\Gamma(1 + l\epsilon)} (p^2)^{1+l\epsilon} I_{1,\dots,1}(\underline{x}; 2 - 2\epsilon),$$

$$J_{l,\underline{k}}(\underline{z}; \epsilon) = (1 + 2\epsilon) \cdots (1 + |\underline{k}|\epsilon) \partial_{\underline{z}}^{\underline{k}} J_{l,\underline{0}}(\underline{z}; \epsilon).$$

Here $|\underline{k}| = \sum_{j=1}^{l+1} k_j$ and $\partial_{\underline{z}}^{\underline{k}} =: \prod_{i=1}^{l+1} \partial_{z_i}^{k_i}$.

The number of master integrals changes **discontinuously**, when \underline{x}, ϵ changes:

- E.g. if $m_i^2 = m^2$, the **equal-mass case** yields only $l + 1$ master integrals

$$J_{l,0}(z; \epsilon) = \frac{(-1)^{l+1}}{\Gamma(1+l\epsilon)} (m^2)^{l\epsilon} \epsilon^l I_{1,\dots,1,0}(p^2, m^2; 2-2\epsilon) = -\frac{\Gamma(1+\epsilon)^l}{\Gamma(1+l\epsilon)},$$

$$J_{l,1}(z; \epsilon) = \frac{(-1)^{l+1}}{\Gamma(1+l\epsilon)} (m^2)^{1+l\epsilon} I_{1,\dots,1}(p^2, m^2; 2-2\epsilon),$$

$$J_{l,k}(z; \epsilon) = (1+2\epsilon) \cdots (1+k\epsilon) \partial_z^{k-1} J_{l,1}(z; \epsilon), \quad \text{for } 2 \leq k \leq l,$$

- In the **critical dimension**, i.e. $\epsilon = 0$, we have only $2^{l+1} - \binom{l+2}{\lfloor \frac{l+2}{2} \rfloor} + 1$ independent master integrals. This was shown in [2] by analyzing the horizontal cohomology of a complete intersection Calabi-Yau geometry M_{l-1}^{Cl} , i.e. the derivatives of the holomorphic Ω_{l-1} form, modulo the Griffith reduction relations or the vertical cohomology of its mirror W_{l-1}^{Cl}

$$h_{\text{hor}}^{l-1-k,k}(M_{l-1}^{\text{Cl}}) = \begin{cases} \binom{l+1}{k} & \text{if } k \leq \lfloor \frac{l}{2} \rfloor - 1 \\ \binom{l+1}{l-1-k} & \text{otherwise} \end{cases}$$

First order IBP and Gauss-Manin connection: Physics experience shows that one can recast the IBP relations as

$$d\underline{I}(\underline{x}; \epsilon) = \mathbf{A}(\underline{x}; \epsilon) \underline{I}(\underline{x}; \epsilon) ,$$

where $d = \sum_{k=1}^N dx_k \partial_{x_k}$ and $\mathbf{A}(\underline{x}; \epsilon)$ is a matrix of rational one-forms. In this first order form one can identify the master integrals $d\underline{I}(\underline{x}; \epsilon)$ as the Hodge bundle over \underline{x}, ϵ and the first order differential form as its flat **Gauss-Manin connection**.

To provide an iterative solution scheme one searches for a new basis $\underline{I}(\underline{x}; \epsilon) = \mathbf{M}(\underline{x}; \epsilon)\underline{J}(\underline{z}; \epsilon)$

$$d\underline{J}(\underline{z}; \epsilon) = \tilde{\mathbf{A}}(\underline{z}; \epsilon)\underline{J}(\underline{z}; \epsilon) ,$$

$$\tilde{\mathbf{A}}(\underline{z}; \epsilon) = \mathbf{M}(\underline{x}; \epsilon)^{-1} [\mathbf{A}(\underline{x}; \epsilon)\mathbf{M}(\underline{x}; \epsilon) - d\mathbf{M}(\underline{x}; \epsilon)] ,$$

so that

- $J_{i,0}(\underline{z}; \epsilon) = \lim_{\epsilon \rightarrow 0} J_i(\underline{z}; \epsilon)$ are finite and non-zero
- $\mathbf{A}_0(\underline{z}) := \lim_{\epsilon \rightarrow 0} \tilde{\mathbf{A}}(\underline{z}; \epsilon)$ regular

and one composes the master integrals into its sectors

$$\underline{J}(\underline{z}; \epsilon) = (\underline{J}_1(\underline{z}; \epsilon)^T, \dots, \underline{J}_s(\underline{z}; \epsilon)^T)^T,$$

so that $\tilde{\mathbf{A}}(\underline{z}; \epsilon)$ becomes block-diagonal and the master integrals in each sector satisfy an inhomogeneous differential equation

$$d\underline{J}_r(\underline{z}; \epsilon) = \mathbf{B}_r(\underline{z}; \epsilon) \underline{J}_r(\underline{z}; \epsilon) + \underline{N}_r(\underline{z}; \epsilon), \quad 1 \leq r \leq s.$$

where the inhomogeneity $\underline{N}_r(\underline{z}; \epsilon) = W_l(z)^{-1} \tilde{N}_l(\underline{z}; \epsilon)$ contains integrals from **lower sectors**, which should have been characterized **analytically** in previous steps of the iterative scheme.

The special role of the banana integrals in this program:

- The lower sectors are all tadpoles yielding already analytic expressions.
- Banana integrals do occur in the iterative procedure within each more complicated Feynman diagram.
- The homogenous solutions or maximal cuts correspond in the critical dimension or said differently in leading order in $\epsilon \rightarrow 0$ the period integrals of families of Calabi-Yau $(n = l - 1)$ -folds.

More general definitions and expectations:

- **Period integrals** for geometric families of Calabi-Yau n -folds M_n

$$\Pi_{ij}(\underline{z}) = \int_{\Gamma_i} \gamma^j(\underline{z}),$$

are pairings

$$\Pi : H_n(M_n, \mathbb{Z}) \times H^n(M_n, \mathbb{C}) \rightarrow \mathbb{C}.$$

- Here Γ_i is a fixed basis of homology $H_n(M_n, \mathbb{Z})$ and $\gamma^j(\underline{z})$ a basis of cohomology $H^n(M_n, \mathbb{C})$ varying with the complex structure \underline{z} .

- One can made a fix choice $\tilde{\gamma}^j \in H^n(M_n, \mathbb{Z})$ so that $\int_{\Gamma_i} \tilde{\gamma}^j = \delta_i^j$ and $\int_{M_n} \tilde{\gamma}^i \wedge \tilde{\gamma}^j = \Sigma^{ij}$ yields the intersection form Σ .
- Σ is an **even** lattice form or an **odd** integer symplectic form if n is **even** or **odd**, respectively.
- The periods are solution to a homogenous Gauss-Manin-System and correspond for fixed Γ_i to the **maximal cut** integrals.
- Maximal cut integrals are characterized either by being homogeneous solutions of the GM-System or that the

corresponding contours enclose all propagator poles.

- For families of Calabi-Yau manifolds (motives) the first order Gauss-Manin system is equivalent to the Picard-Fuchs differential ideal (PFI).
- For $l = 1$ the periods are **rational** — for $l = 2$ **elliptic functions**.
- For higher l they generalise to **periods of** (weight or) dimension $n = l - 1$ **families of Calabi-Yau manifold (motives)**.

- Calabi-Yau motives are characterized by their Picard-Fuchs differential ideal, the intersection form Σ and monodromies in $\mathcal{O}(\Sigma, \mathbb{Z})$.
- The inhomogenous solutions for $\epsilon \rightarrow 0$ correspond to the extension of these Calabi-Yau motives by chain integrals.
- Both structures can be analytically solved everywhere in the parameter space using the PFI, the $\hat{\Gamma}$ class and its extension.
- Connecting graphs of different topologies using the

symmetry preserving ϵ regularisation resembles strongly the unification of topologies using the α' regularisation and string/QFT correspondence principle.

Status of this program for the banana integrals:

- For the banana graphs the PFI is a Gel'fand-Kapranov-Zelevinskĭ ideal and the program for $\epsilon \rightarrow 0$ has been completed [1,2].
- In this work [3] we further generalize that to include the general ϵ dependence.

The dictionary between maximal cut integrals and families of Calabi-Yau motives:

	$l = (n + 1)$ -loop banana integrals in $D = 2$ dimensions	Calabi-Yau (CY) geometry
1	Maximal cut integrals in $D = 2$ dimensions	$(n, 0)$ -form periods of CY manifolds or CY motives
2	Dimensionless ratios $z_i = m_i^2/p^2$	Unobstructed compl. moduli of M_n , or equi'ly Kähler moduli of the mirror W_n
3	Integrand-basis for maximal cuts of master integrals in $D = 2$	Middle (hyper) cohomology $H^n(M_n)$ M_n
4	Quadratic relations among maximal cut integrals	Quadratic relations from Griffiths transversality
5	Integration-by-parts (IBP) reduction	Griffiths reduction method

6	Complete set of differential operators annihilating a given maximal cut in $D = 2$ dimensions	Homogeneous Picard-Fuchs differential ideal (PFI) / Gauss-Manin (GM) connection
7	(Non-)maximal cut contours	(Relative) homology of CY geometry $H_n(M_n)$ ($H_{n+1}(F_{n+1}, \partial\sigma_{n+1})$)
8	Contributions from subtopologies to the differential equations	Extensions of the PFI or the GM connection
9	Full banana integrals in $D = 2$ dimensions	Chain integrals in CY geometry or extensions of Calabi-Yau motive
10	Degenerate kinematics (e.g., $m_i^2 = 0$ or $p^2/m_i^2 \rightarrow 0$)	Critical divisors of the moduli space
11	Large-momentum regime $p^2 \gg m_i^2$	Point of maximal unipotent monodromy & $\hat{\Gamma}$ -classes of W_n

12	General logarithmic degenerations	Limiting mixed Hodge structure from monodromy weight filtration
13	Analytic structure and analytic continuation	Monodromy of the CY motive and its extension
14	Special values of the integrals for special values of the z_i	Reducibility of Galois action & L -function values
15	(Generalized?) modularity of Feynman integrals	Global $O(\Sigma, \mathbb{Z})$ -monodromy, integrality of mirror map & instantons expansion

Families of Calabi-Yau motives for the banana integral:

Using the graph polynomials \mathcal{U} and \mathcal{F} we can write the integral generally as:

$$I_{\underline{\nu}}(\underline{p}, \underline{m}, D) = \int_{\sigma_l} \left(\prod_{k=1}^{l+1} x_k^{\delta_k} \right) \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}(p^2, m^2)^\omega} \mu_l.$$

- $\nu_i = 1 + \delta_i$, $\omega := \sum_{i=1}^{l+1} \nu_i - \frac{lD}{2} - 1 + l\epsilon + \sum_i \delta_i$
- $\sigma_{n-1} = \{[x_1 : \dots : x_n] \in \mathbb{P}^{n-1} \mid x_i \in \mathbb{R}_{\geq 0} \forall 1 \leq i \leq n\}$ an open domain,
- $\mu_l = \sum_{k=1}^n (-)^{k+1} x_k dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_n$ measure on \mathbb{P}^l .

The two **Symanzik polynomials** for the banana graph are given by:

$$\mathcal{U} = \left(\prod_{i=1}^{l+1} x_i \right) \left(\sum_{i=1}^{l+1} \frac{1}{x_i} \right) = \sum_{i=1}^{l+1} \prod_{\substack{j=1 \\ j \neq i}}^{l+1} x_j ,$$

$$\mathcal{F}(p^2, \underline{m}^2) = -p^2 \left(\prod_{i=1}^{l+1} x_i \right) + \left(\sum_{i=1}^{l+1} m_i^2 x_i \right) \mathcal{U} .$$

In the critical dimension $D_0 = 2$ one gets **a maximal cut integral**

$$J_{l, \underline{0}}^{\Gamma_T}(z; 0) = \int_{T^l} \frac{\mu_l}{\mathcal{F}(1, \underline{z})} = \int_{T^{l-1}} \oint_{S^1} \frac{\mu_l}{\mathcal{F}(1, \underline{z})} = 2\pi i \int_{\Gamma_T = T^{l-1}} \Omega_{l-1}(z) ,$$

over the **cycle** T^l defined as

$$T^l := \{ [x_1 : \dots : x_{l+1}] \in \mathbb{P}^l \mid |x_i| = 1 \text{ for all } 1 \leq i \leq l+1 \} .$$

Here we used the **Griffiths residue form** for the holomorphic n -form Ω for complete intersections

$$\Omega(\underline{z}) = \frac{1}{(2\pi i)^r} \oint_{S_1^1} \cdots \oint_{S_r^1} \frac{\wedge_{i=1}^m \mu_{n_i}}{P_1 \cdots P_r},$$

where S_k^1 encircles the constraints $P_k = 0$ in the ambient space. The crucial point is that the integral over the S^1 cycle of T^l leads to a closed period integral over $\mathbf{T} = T^{l-1}$ on

$$M_{l-1}^{\text{HS}} = \{ \underline{x} \in \mathbb{P}^l \mid \mathcal{F}(1, \underline{z}; \underline{x}) = 0 \} .$$

Performing all l residua integrals one gets

$$J_{l, \underline{0}}^{\Gamma T}(\underline{z}; \mathbf{0}) = (2\pi i)^l \sum_{n=0}^{\infty} \sum_{|k|=n} \binom{n}{k_1 \dots k_{l+1}}^2 \prod_{i=1}^{l+1} z_i^{k_i},$$

with $|k| = \sum_{i=1}^{l+1} k_i$.

The hypersurface M_{l-1}^{HS} defines a **singular family** of Calabi-Yau motives with $l + 1$ complex parameters. To get a workable smooth model one could deform $F(1, \underline{z}; \underline{x})$ (toric resolution). However, one needs l^2 (complex) moduli to achieve that. This leads to a highly redundant model that is very hard to solve.

A better Calabi-Yau motive: A very elegant way to circumvent this problem was proposed in [2]. Consider the **complete intersection** of two polynomials of degree $(1, \dots, 1)$ in

$$\mathbb{P}_{l+1} := \bigotimes_{i=1}^{l+1} \mathbb{P}_{(i)}^1,$$

i.e., we have after an étale map to mirror coordinates

$$M_{l-1}^{\text{Cl}} = \left\{ \left(w_1^{(i)} : w_2^{(i)} \right) \in \widehat{\mathbb{P}}_{(i)}^1, \forall i \mid \right. \\ \left. P_1 := \sum_{i=1}^{l+1} a^{(i)} w_1^{(i)} + b^{(i)} w_2^{(i)} = \sum_{i=1}^{l+1} c^{(i)} w_1^{(i)} + d^{(i)} w_2^{(i)} =: P_2 = 0 \right\}. \quad (2)$$

This is transversal $dP_1 \wedge dP_2 \neq 0$ when $P_1 = P_2 = 0$ iff

$$\det \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} \neq 0$$

for all $i = 1, \dots, l + 1$. On every $\mathbb{P}_{(i)}^1$ there is a natural $\mathrm{SL}_i(2, \mathbb{C})$ action which allows to make the choices

$$a^{(i)} = -\frac{m_i^2}{p^2} = -z_i, \quad d^{(i)} = x, \quad i = 1, \dots, l + 1,$$

$$b^{(1)} = \frac{x}{w_2^{(1)}}, \quad c^{(1)} = \frac{1}{w_1^{(1)}},$$

$$b^{(i)} = c^{(i)} = 0, \quad i = 2, \dots, l + 1,$$

One can construct a **birational map** from the smooth complete intersection geometry to the singular hypersurface geometry by solving for $P_1 = 0$ such that one gets $x = \sum_{i=1}^{l+1} \frac{m_i^2}{p^2} w_1^{(i)}$. P_2 becomes $P_2 = 1 - x \sum_{i=1}^{l+1} w_2^{(i)}$.

Passing to \mathbb{C}^* coordinates $w_1^{(i)} = W_i$ and $w_2^{(i)} = 1/W_i$, for $i = 1, \dots, l+1$, we find

$$P_2 = p^2 - \left(\sum_{i=1}^{l+1} m_i^2 W_i \right) \left(\sum_{i=1}^{l+1} \frac{1}{W_i} \right).$$

This is the hypersurface written in \mathbb{C}^* coordinates!

As a check we can calculate $J_{l, \underline{0}}^{\Gamma T}(\underline{z}; 0)$ in the complete intersection geometry. With $\mu = 1/2^{l+1} \mu_1^{(1)} \wedge \dots \wedge \mu_1^{(l+1)}$, where $\mu_1^{(i)}$ is the standard measure for \mathbb{P}_i^1 , one gets

$$\begin{aligned}
\frac{1}{(2\pi i)^l} J_{l, \underline{0}}^{\Gamma T}(\underline{z}; 0) &= \frac{1}{(2\pi i)^{l+1}} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mu}{P_1 P_2} \\
&= \frac{1}{(2\pi i)^{l+1}} \oint_{W_1=0} \dots \oint_{W_{l+1}=0} \mu \left(1 - \sum_{i=1}^{l+1} z_i W_i \right)^{-1} \left(1 - \sum_{i=1}^{l+1} \frac{1}{W_i} \right)^{-1} \frac{dW_1}{W_1} \wedge \dots \wedge \frac{dW_{l+1}}{W_{l+1}} \\
&= \frac{1}{(2\pi i)^{l+1}} \oint_{W_1=0} \dots \oint_{W_{l+1}=0} \mu \sum_{\substack{|m|=m \\ |n|=n}} \binom{m}{m_1 \dots m_{l+1}} \binom{n}{n_1 \dots n_{l+1}} \prod_{i=1}^{l+1} (z_i W_i)^{m_i} \left(\frac{1}{W_i} \right)^{n_i} \\
&= \sum_{n=0}^{\infty} \sum_{|k|=n} \binom{n}{k_1 \dots k_{l+1}}^2 \prod_{i=1}^{l+1} z_i^{k_i} , ,
\end{aligned}$$

which is **exactly the same** expression as for the HS motive, i.e., they give rise to the same maximal cuts!

Remarks on Calabi-Yau motives:

- The smooth hypersurface geometry M_n^{HS} and M_n^{Cl} **have not the same topology**. Indeed it is easy to check that

$$\begin{aligned} \chi(M_3^{\text{HS}}) &= 20, & \chi(M_4^{\text{HS}}) &= 540, & \dots, \\ \chi(M_3^{\text{Cl}}) &= -80, & \chi(M_4^{\text{Cl}}) &= 720, & \dots. \end{aligned}$$

- They are related by a **singular transition**. E.g. for $n = 3$ we have

$$X_u^{16,26} \rightarrow X_a^{\text{sing}} \rightarrow \hat{X}^{5,45}. \quad (3)$$

Here $X_u^{16,26}$ is our deformed space M_3^{HS} with $\chi(M_3^{\text{HS}}) =$

20. The superscripts are h_{21} and h_{11} , respectively. X_a^{sing} is the singular five-parameter space in the physical slice written in torus variables, and $\hat{X}^{5,45}$ is a **small resolution of X_a^{sing}** . X_a^{sing} has 30 nodes, where S^3 -spheres are shrinking. Replacing each singular loci by \mathbb{P}^1 resolved them small. Each resolution adds $\chi(\mathbb{P}^1) = 2$ to the Euler number leading the topology $\hat{X}^{5,45}$, which mirror to M_3^{Cl} !

- Indeed, M_n^{Cl} is self mirror in the following sense: Consider the $l + 1$ deformations of (??). By taking derivatives of Ω_n w.r.t. these parameters modulo the

Griffiths partial integration relation

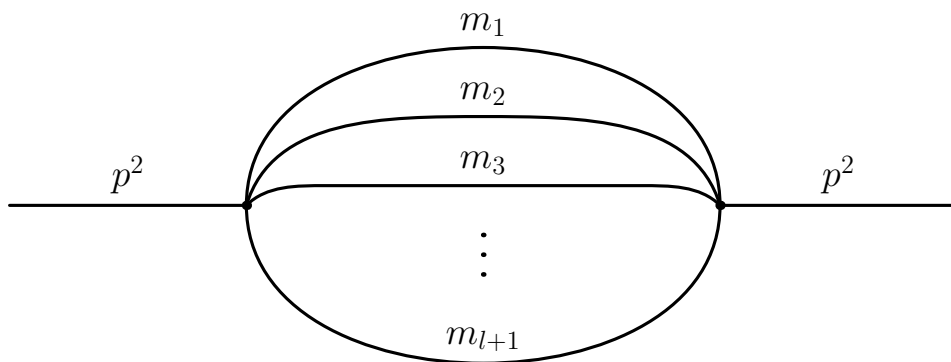
$$\sum_{k \neq j} \frac{m_k}{m_j - 1} \frac{P_j}{P_k} \frac{q \partial_{x_i} P_k}{\prod_{l=1}^r P_l^{m_l}} \mu = \frac{1}{m_j - 1} \frac{P_j \partial_{x_i} q}{\prod_{l=1}^r P_l^{m_l}} \mu - \frac{q \partial_{x_i} P_j}{\prod_{l=1}^r P_l^{m_l}} \mu ,$$

we can define $H_n^{\text{hor}}(M_{l-1}^{\text{Cl}})$. The latter is mirror to the vertical cohomology $H_{\text{vert}}^{k,k}(W_n^{\text{Cl}})$ that is inherited from \mathbb{P}_{l+1} . Similar remarks hold for the associated homology groups. If we restrict ourselves to these vertical- and horizontal subspaces, then the geometries for the banana graphs are **self mirrors**

$$M_{l-1}^{\text{Cl}, \text{res}} = W_{l-1}^{\text{Cl}, \text{res}} .$$

The mirror picture:

The **vertical quantum cohomology** of W_{l-1}^{Cl} relates natural to the banana graph



$$\longleftrightarrow W_{l-1}^{\text{Cl}} = \left(\begin{array}{c|cc} \mathbb{P}_1^1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}_{l+1}^1 & 1 & 1 \end{array} \right) \subset \left(\begin{array}{c|cc} \mathbb{P}_1^1 & 1 & \\ \vdots & \vdots & \\ \mathbb{P}_{l+1}^1 & 1 & \end{array} \right) = F_l.$$

In particular, in the **high energy regime** we get a one-to-one identification of the complexified (large volume) **Kähler parameters** t^k of the $l + 1$ rational curves \mathbb{P}_k^1 with the **physical parameters** m_i^2/p^2

$$t^k \simeq \frac{1}{2\pi i} \int_{\mathbb{P}_k^1} (i\omega - b) + \mathcal{O}(e^{-t^k}) = \frac{\log \left(\frac{m_k^2}{p^2} \right)}{2\pi i} = \frac{\log(z_k)}{2\pi i}$$

for $k = 1, \dots, l + 1$. Away from the limit the mirror symmetry for complete intersections and, in particular, the associated GKZ system provides the exact answer, including the exponentially suppressed $\mathcal{O}(e^{-t^k})$ corrections.

The fibration Structure: $E = \left(\begin{array}{c} \mathbb{P}_1^1 \\ \mathbb{P}_2^1 \\ \mathbb{P}_3^1 \end{array} \parallel \begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array} \right)$ is the elliptic

curve associated to the two-loop graph. The K3 associated to the three-loop graph $K_3 = \left(\begin{array}{c} \mathbb{P}_1^1 \\ \mathbb{P}_2^1 \\ \mathbb{P}_3^1 \\ \mathbb{P}_4^1 \end{array} \parallel \begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array} \right)$ is

fibered in four ways by E over each of its \mathbb{P}_k^1 . The K3 fibres the Calabi-Yau three-fold W_3^{Cl} in five ways and so on.

The $\widehat{\Gamma}$ -classes: A powerful application of the geometric realization W_{l-1}^{Cl} is the $\widehat{\Gamma}$ -class formalism. It relates the Frobenius \mathbb{Q} -basis of solutions at the point of maximal unipotent monodromy (MUM) to an integral \mathbb{Z} -basis of solutions to the PFI.

The latter contains the maximal cut integral that corresponds to the unique period $\Pi_{\mathbf{S}}$ over a $S^{l-1} =: \mathbf{S}$ that vanishes at the nearest conifold and describes the imaginary part of the banana integral above threshold. An extension of the $\widehat{\Gamma}$ -class also yields the full Feynman integral in the critical dimension. Note that $\mathbf{S} \cap \mathbf{T} = 1$

and both cycles play a crucial role in homological mirror symmetry.

Let I_p an index set of order $|I_p| = p$ and define the **Frobenius basis** at the MUM point:

$$S_{(p),k}(\underline{z}) = \frac{1}{(2\pi i)^p p!} \sum_{I_p} \kappa_{(p),k}^{i_1, \dots, i_p} \varpi_0(\underline{z}) \log(z_{i_1}) \cdots \log(z_{i_p}) + \mathcal{O}(\underline{z}^{1+\alpha}).$$

Here $|S_{(p)}(\underline{z})|$ denotes the total number of solutions which are of leading order p in $\log(z_i)$ and $\kappa_{(p),k}^{i_1, \dots, i_p}$ are intersection numbers of the mirror W_{l-1}^{Cl} .

In particular, the Kähler parameters t^k are given by the **mirror map**

$$t^k(\underline{z}) = \frac{S_{(1),k}(\underline{z})}{S_{(0),0}(\underline{z})} = \frac{1}{2\pi i} \left(\log(z_k) + \frac{\Sigma_k(\underline{z})}{\varpi_0(\underline{z})} \right),$$

for $k = 1, \dots, h^{1,1}(W_n) = h^{n-1,1}(M_n)$.

Homological mirror symmetry predicts then

$$\Pi_{\mathbf{S}}(t(\underline{z})) = \int_{W_{l-1}} e^{\omega \cdot t} \widehat{\Gamma}(TW_{l-1}) + \mathcal{O}(e^{-t}) \quad (4)$$

and an extension also yields the full Feynman integral

$$J_{l,\underline{0}}(\underline{z}, 0) = \int_{F_l} e^{\omega \cdot t} \widehat{\Gamma}_{F_l}(TF_l) + \mathcal{O}(e^{-t}) . \quad (5)$$

Here the extended $\widehat{\Gamma}$ -class is given by

$$\widehat{\Gamma}_F(TF) = \frac{\widehat{A}(TF)}{\widehat{\Gamma}^2(TF)} = \frac{\Gamma(1 - c_1)}{\Gamma(1 + c_1)} \cos(\pi c_1) .$$

By comparing the powers of $t^k \sim \log(z_k)$ on both sides of (??),(??) using the mirror map these formulas determine uniquely the exact boundary conditions for the

integrals in terms of topological intersection calculations on W_{l-1}^{Cl} or the Fano variety F_l and the Frobenius basis for the banana graph [2].

Comparison with a Barnes integral representations:

Using the identity

$$\frac{1}{(A+B)^\lambda} = \int_{c-i\infty}^{c+i\infty} \frac{d\xi}{2\pi i} A^\xi B^{-\xi-\lambda} \frac{\Gamma(-\xi)\Gamma(\xi+\lambda)}{\Gamma(\lambda)},$$

one can rewrite $\mathcal{F}(p^2, \underline{m}^2)^{-\omega}$ as

$$\begin{aligned} \tilde{I} = & \int \frac{d\xi_0}{2\pi i} \frac{\Gamma(-\xi_0)\Gamma(\xi_0+\omega)}{\Gamma(\omega)} (-p^2)^{\xi_0} \\ & \times \int_{[0,\infty)^l} dx_1 \dots dx_l \left(\prod_{i=1}^{l+1} x_i^{\xi_0+\delta_i} \right) \left(\sum_{i=1}^{l+1} \prod_{\substack{j=1 \\ j \neq i}}^{l+1} x_j \right)^{-\xi_0-\frac{d}{2}} \left(\sum_{i=1}^{l+1} m_i^2 x_i \right)^{-\xi_0-\omega}. \end{aligned} \quad (6)$$

The actual specifications of the contours in this integral is very complicated. But one can correctly close in the large momentum region and here also with $\delta_i = 0$ for all i the contours to find

$$\begin{aligned}
 I_{1,\dots,1}(p^2, \underline{m}^2; 2 - 2\epsilon) &= \frac{1}{\Gamma(1 + l\epsilon)} \left(\frac{1}{-p^2 - i0} \right)^{1+l\epsilon} \sum_{\underline{j} \in \{0,1\}^{l+1}} \frac{\Gamma(-\epsilon)^j \Gamma(\epsilon)^{l+1-j} \Gamma(1 + (j-1)\epsilon)}{\Gamma(-j\epsilon)} \\
 &\times \left[\prod_{i=1}^{l+1} \left(\frac{m_i^2}{-p^2 - i0} \right)^{(j_i-1)\epsilon} \sum_{\underline{n} \in \mathbb{N}_0^{l+1}} \frac{(1+j\epsilon)_n (1+(j-1)\epsilon)_n}{\prod_{i=1}^{l+1} (1 + (-1)^{j_i+1}\epsilon)_{n_i}} \prod_{i=1}^{l+1} \frac{1}{n_i!} \left(\frac{m_i^2}{p^2} \right)^{n_i} \right] \quad (7)
 \end{aligned}$$

and infer the leading asymptotic behavior at large momentum.

Letting $\underline{n} = (0, \dots, 0)$ in eq. (??), we can extract the leading behavior of the banana integrals at large momentum, e.g. for the generic-mass case

$$\begin{aligned}
 I_{1, \dots, 1}(p^2, \underline{m}^2; 2 - 2\epsilon) &= -\frac{1}{\Gamma(1 + l\epsilon)} e^{i\pi l\epsilon} \left(\frac{1}{p^2}\right)^{1+l\epsilon} \\
 &\times \sum_{\underline{j} \in \{0,1\}^{l+1}} e^{i\pi(j-1)\epsilon} \frac{\Gamma(-\epsilon)^j \Gamma(\epsilon)^{l+1-j} \Gamma(1 + (j-1)\epsilon)}{\Gamma(-j\epsilon)} \prod_{i=1}^{l+1} z_i^{(j_i-1)\epsilon} + \mathcal{O}(z_i^2) .
 \end{aligned}
 \tag{8}$$

This gives the **leading asymptotics** of $I_{1, \dots, 1}(p^2, \underline{m}^2; 2 - 2\epsilon)$ and can be used as a **boundary condition** to solve the differential equations for the banana graphs. In particular, in the equal-mass case the

expression further simplifies to

$$J_{l,1}(z; \epsilon) = - \sum_{k=1}^{l+1} \binom{l+1}{k} \frac{\Gamma(-\epsilon)^k \Gamma(\epsilon)^{l+1-k} \Gamma(1 + (k-1)\epsilon)}{\Gamma(-k\epsilon) \Gamma(1+l\epsilon)} e^{(k-1)i\pi\epsilon} z^{1+(k-1)\epsilon} + \mathcal{O}(z^2) .$$

Expanding this around $\epsilon = 0$, one obtains

$$J_{l,1}(z; \epsilon) = \sum_{n=0}^{\infty} J_{l,1}^{(n)}(z) \epsilon^n . \quad (9)$$

The leading order in ϵ , i.e. $J_{l,1}^{(0)}$, **precisely reproduces the logarithmic structure** of the l -loop banana Feynman integral in $D = 2$ spacetime dimensions!

The homogeneous differential operators $\mathcal{L}_{l,\epsilon}$ that annihilate the maximal cuts of the banana integrals in $D = 2 - 2\epsilon$ dimensions:

Loop order l	Differential operator $\mathcal{L}_{l,\epsilon}$
1	$1 + \epsilon - 2z - (1 - 4z)\theta$
2	$(1 + 2\epsilon)(1 + \epsilon - 3z + z\epsilon) + (-2 - 3\epsilon + 10z + 10z\epsilon + 9z^2\epsilon)\theta + (1 - z)(1 - 9z)\theta^2$
3	$(1 + 2\epsilon)(1 + 3\epsilon)(1 + \epsilon - 4z + 2z\epsilon) + (-3 - 12\epsilon + 18z + 60z\epsilon - 11\epsilon^2 + 28z\epsilon^2 + 64z^2\epsilon^2)\theta - 3(-1 + 10z)(1 + 2\epsilon)\theta^2 - (1 - 4z)(1 - 16z)\theta^3$
4	$(1 + 2\epsilon)(1 + 3\epsilon)(1 + 4\epsilon)(1 + \epsilon - 5z + 3z\epsilon) + (-4 - 30\epsilon + 28z + 189z\epsilon + 26z^2\epsilon - 225z^3\epsilon - 70\epsilon^2 + 343z\epsilon^2 - 225z^3\epsilon^2 - 50\epsilon^3 + 84z\epsilon^3 + 414z^2\epsilon^3)\theta + (6 - 63z + 26z^2 - 225z^3 + 30\epsilon - 315z\epsilon - 675z^3\epsilon + 35\epsilon^2 - 343z\epsilon^2 - 363z^2\epsilon^2 - 225z^3\epsilon^2)\theta^2 - 2(2 - 35z + 225z^3 + 5\epsilon - 105z\epsilon + 259z^2\epsilon + 225z^3\epsilon)\theta^3 + (1 - z)(1 - 9z)(1 - 25z)\theta^4$

Properties of Calabi-Yau motives and their significances for Feynman integrals:

- Griffiths transversality: Let $\underline{\Pi}(\underline{z}) = \left(\int_{\Gamma_1} \Omega, \dots, \int_{\Gamma_r} \Omega \right)^T$ the period vector. Then one gets as a generalization of the observations of Bryant and Griffiths for Calabi-Yau n -folds:

$$\underline{\Pi}(\underline{z})^T \Sigma \partial_{\underline{z}}^k \underline{\Pi}(\underline{z}) = \int_{M_n} \Omega \wedge \partial_{\underline{z}}^k \Omega = \begin{cases} 0 & \text{for } 0 \leq r < n \\ C_{\underline{k}}(\underline{z}) & \text{for } |k| = n \end{cases},$$

where the $C_{\underline{k}}(\underline{z})$ are rational functions in the complex structure parameters. For the first equality, expand

Ω in an integer symplectic basis of cohomology. The second equality follows from Griffiths transversality and consideration of the Hodge type. Note an arbitrary local basis $\underline{\tilde{\Pi}}(\underline{z})$ corresponding to an (implicit) choice of a basis of cycles $\tilde{\Gamma}^i \in H_n(M_n, \mathbb{C})$, obtained as independent local solutions of the Picard-Fuchs differential ideal, one can find a $\tilde{\Sigma}$ and write down the corresponding relations $\underline{\tilde{\Pi}}(\underline{z})^T \tilde{\Sigma} \partial_{\underline{z}}^k \underline{\tilde{\Pi}}(\underline{z})$ among the solutions very explicitly. It implies that there are **quadratic relations among the maximal cut integrals**. For the banana graphs we checked explicitly that these are the only ones.

In particular $W_l(z)^{-1} = \Sigma W_l(z) C(z)^{-1}$.

- Self-adjointness: Let the Picard-Fuchs differential ideal be generated by a single (normalized) differential operator (as it is the case for one-parameter families), i.e.

$$\mathcal{L}^{(n+1)} = \partial_z^{n+1} + \sum_{i=0}^n a_i(z) \partial_z^i.$$

Then the **Yukawa coupling** C_n fulfills the differential equation

$$\frac{\partial_z C_n(z)}{C_n(z)} = \frac{2}{n+1} a_n(z). \quad (10)$$

One can define the **adjoint differential operator**

$$\mathcal{L}^{*(n+1)} = \sum_{i=0}^{n+1} (-\partial_z)^i a_i(z).$$

An operator is called **essentially self-adjoint** if

$$\mathcal{L}^{*(n+1)} A(z) = (-1)^{n+1} A(z) \mathcal{L}^{(n+1)},$$

where $A(z)$ satisfies the differential relation $\frac{\partial_z A(z)}{A(z)} = \frac{2}{n+1} a_n(z)$. Note that $A(z)$ is up to a multiplicative constant given by the Yukawa coupling $C_n(z)$.

A one-parameter maximal cut Feynman integral has to be annihilated by a self-adjoint linear differential operator if it comes from a CY geometry!

- Landman's theorem: It states that all possible monodromy matrices of an algebraic n -fold have to obey

$$(\mathbf{T}^k - \mathbb{1})^{n+1} = 0 . \quad (11)$$

Here $k \in \mathbb{N}_0$, implying that the **indicial** α has to be a **rational number**. A monodromy matrix \mathbf{T} can be unipotent of lower order $m < n$, i.e., $(\mathbf{T}^k - \mathbb{1})^{m+1} = 0$. It is clear that m is the size of the biggest Jordan block in \mathbf{T} . The maximal n that can appear is $n = \dim(M)$. It is not too hard to see that the unipotency of order $m \leq n$ implies that a period on an n -fold cannot degenerate worse than with a logarithmic singularity of

type $\log(\Delta)^n$. This has an important consequence for Feynman integrals. Assume that we have a maximal cut of a Feynman integral in integer dimensions that degenerates in a dimensionless physical parameter Δ (or, more generally, some polynomial combination thereof) as $\log(\Delta)^m$. Then it follows from Landman's theorem that the geometry associated to this maximal cut integral cannot be an algebraic manifold of dimension less than m , or a Calabi-Yau motive of weight less than m !

- The $SL(2, \mathbb{C})$ theorem: This uses the **limiting mixed Hodge Structure** to restrict the structure of the Jordan Blocks further. For example, for a Calabi-Yau three-fold the classification of one-parameter operators (known under the name AESZ list) uses the fact that the only possible degenerations are of the following types:
 - The **generic point** F is characterized by generic local exponents.
 - The **conifold point** C has local exponents (a, b, b, c) and a single 2×2 Jordan block.

(conifolds), and $z = \infty$ is a K -point.

$$\begin{aligned}
 & \mathcal{P}_2 \left\{ \begin{array}{cccc} 0 & \frac{1}{9} & 1 & \infty \\ \hline 1 + \epsilon & -2\epsilon & -2\epsilon & 0 \\ 1 + 2\epsilon & 0 & 0 & \epsilon \end{array} \right\}, \quad \mathcal{P}_3 \left\{ \begin{array}{cccc} 0 & \frac{1}{16} & \frac{1}{4} & \infty \\ \hline 1 + \epsilon & 0 & 0 & -\epsilon \\ 1 + 2\epsilon & \frac{1}{2} - 3\epsilon & \frac{1}{2} - 3\epsilon & 0 \\ 1 + 3\epsilon & 1 & 1 & \epsilon \end{array} \right\}, \\
 & \mathcal{P}_4 \left\{ \begin{array}{ccccc} 0 & \frac{1}{25} & \frac{1}{9} & 1 & \infty \\ \hline 1 + \epsilon & 0 & 0 & 0 & 0 \\ 1 + 2\epsilon & 1 - 4\epsilon & 1 - 4\epsilon & 1 - 4\epsilon & \epsilon \\ 1 + 3\epsilon & 1 & 1 & 1 & 1 \\ 1 + 4\epsilon & 2 & 2 & 2 & 1 + \epsilon \end{array} \right\}.
 \end{aligned}
 \tag{12}$$