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Mixed QCD-EW corrections to NC DY : Two-loop virtual amplitudes

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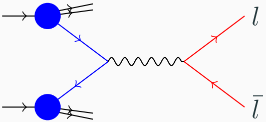
Drell-Yan



- ✓ **One of the standard candle processes**
  - Large cross section and clean experimental signature - important for detector calibration and constraining parton distribution functions
- ✓ **Precise predictions for electroweak parameter**
  - W boson mass (m_W), Weak mixing angle ($\sin^2\theta_{\mathrm{W}}$) ...
- ✓ **New physics potential**
  - Many BSM scenarios with same final states

Perturbative expansion

Parton model

$$\sigma_{tot}(z) = \sum_{i,j \in q,\bar{q},g,\gamma} \int \mathrm{d}x_1 \mathrm{d}x_2 \; f_i(x_1,\mu_F) f_j(x_2,\mu_F) \sigma_{ij}(z,\varepsilon,\mu_F)$$

In the full QCD-EW SM, we have a double series expansion of the partonic cross sections in the electromagnetic and strong coupling constants, α and α_s, respectively:

$$\begin{aligned}
\sigma_{ij}(z) &= \sigma_{ij}^{(0)} \sum_{m,n=0}^{\infty} \alpha_s^m \alpha^n \; \sigma_{ij}^{(m,n)}(z) \\
&= \sigma_{ij}^{(0)} \Big[\sigma_{ij}^{(0,0)}(z) \\
&\quad + \alpha_s \sigma_{ij}^{(1,0)}(z) + \alpha \sigma_{ij}^{(0,1)}(z) \\
&\quad + \alpha_s^2 \sigma_{ij}^{(2,0)}(z) + \alpha \alpha_s \sigma_{ij}^{(1,1)}(z) + \alpha^2 \sigma_{ij}^{(0,2)}(z) \\
&\quad + \alpha_s^3 \sigma_{ij}^{(3,0)}(z) + \alpha \alpha_s^2 \sigma_{ij}^{(2,1)}(z) + \alpha^2 \alpha_s \sigma_{ij}^{(1,2)}(z) + \cdots \Big]
\end{aligned}$$

Why $\sigma_{ij}^{(1,1)}(z)$ is important?

$\alpha_s(m_Z) \simeq 0.118$	$\alpha(m_Z) \simeq 0.0078$	$\frac{\alpha_s(m_Z)}{\alpha(m_Z)} \simeq 15.1$	$\frac{\alpha_s^2(m_Z)}{\alpha(m_Z)} \simeq 1.8$

1. From naive argument of coupling strength, N^3LO QCD $\sim$ mixed NNLO QCD$\otimes$EW.
2. However, in specific phase-space points, fixed order EW corrections can become very large because of logarithmic (weak and QED Sudakov type) enhancement. These effects are large for W mass measurements. On the other hand, these corrections suffer from large uncertainties coming from unphysical scales.
3. N^3LO QCD corrections control the uncertainties arising from the unphysical scales, but they lack the large EW effects.
4. The appearance of photon induced processes $\Rightarrow$ photon PDFs.

The NNLO mixed QCD-EW corrections

- have similar magnitude as N^3LO QCD,
- contain the large EW effects,
- reduce the theoretical uncertainties.

NNLO QCD$\otimes$EW corrections extremely important for high ($\mathcal{O}(10^{-4})$) precision pheno.

Another motivation : Electroweak scheme dependence

The Lagrangian has 3 inputs (g, g', v). More observables (like $G_\mu, \alpha, m_W, m_Z, \sin\theta_W$) are experimentally measured and can be considered as input parameters in different schemes. Such two schemes are

1. G_μ-scheme : where (G_μ, m_W, m_Z) are considered as input
2. $\alpha(0)$-scheme : where (α, m_W, m_Z) are considered as input

The relation between G_μ and α gets EW and mixed QCD⊗EW corrections.

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi\alpha}{2\sin^2\theta_W \cos^2\theta_W m_Z^2}(1+\Delta r)$$

At LO, $\alpha(G_\mu)$ and $\alpha(0)$ differs by 3.53%.

order	G_μ-scheme	$\alpha(0)$-scheme	$\delta_{G_\mu-\alpha(0)}$ (%)
LO	48882	47215	3.53
NLO QCD (LO + Δ_{10})	55732	53831	3.53
NNLO QCD (LO + Δ_{10} + Δ_{20})	55651	53753	3.53
NLO EW (LO + Δ_{01})	48732	48477	0.53
LO + Δ_{10} + Δ_{01}	55582	55093	0.89

NNLO contributions to neutral current Drell-Yan

Pure Virtual

Real-Virtual

Double Real

NNLO contributions to neutral current Drell-Yan

Pure Virtual

$- \; S^{(1,1)}$

Real-Virtual

Double Real

$\Big\} + d\sigma_{CT}^{(1,1)}$

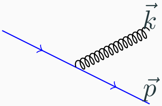
Subtraction : $\quad S^{(1,1)} \sim \int d\sigma_{CT}^{(1,1)}$

NNLO contributions to neutral current Drell-Yan

Pure Virtual

The two-loop virtual amplitudes contain divergences of two types

(a) **Ultraviolet divergences** : UV renormalization of fields and couplings
(b) **Infrared divergences** : Soft (soft gluons & photons) & collinear (collinear partons)



$$\frac{1}{(k+p)^2} = \frac{1}{2k.p} = \frac{1}{2k^0 p^0 (1-\cos\theta)}$$

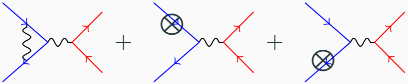
$k^0 \to 0$ Soft divergence
$\theta \to 0$ Collinear divergence

The infrared structure of scattering amplitudes is universal!

Ultraviolet renormalization

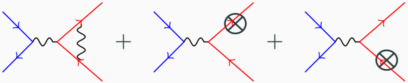
⊛ The Born contribution is zeroth order in α_s, hence no α_s renormalization is needed.

⊛ Renormalization of quark wave function receives one-loop EW and two-loop mixed QCD⊗EW contributions in the on-shell scheme.



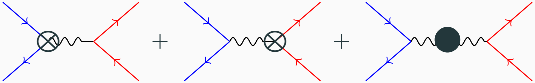
$\Rightarrow$ UV finite

⊛ Renormalization of lepton wave function receives one-loop EW contributions.



$\Rightarrow$ UV finite

⊛ The neutral current vertex is renormalized using background field gauge, with the advantage that the vertex and propagator contributions are separately UV finite.



$\Rightarrow$ UV finite

The infrared divergences and lepton mass

The infrared structure of scattering amplitudes is universal!

Luca's talk

$$\mathcal{M}_{\rm fin}^{(1,1)} = \mathcal{M}^{(1,1)} - \mathcal{I}^{(1,1)}\mathcal{M}^{(0)} - \mathcal{I}^{(0,1)}\mathcal{M}_{\rm fin}^{(1,0)} - \mathcal{I}^{(1,0)}\mathcal{M}_{\rm fin}^{(0,1)}$$

The q_T subtraction requires the leptons to be massive!

The full computation with lepton mass is extremely difficult!

Divergence regulator massless lepton : $\frac{1}{\epsilon}$ massive lepton : $\log m_l$

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(a) When the lepton is attached to a massive boson, it does not generate any collinear divergence. Hence, in all such cases, we can safely assume a massless lepton.

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(b) In a single box diagram, where lepton is attached to one photon and one Z boson, it generates a collinear singularity. However, thanks to [Frenkel, Taylor], once all diagrams are summed up, the collinear divergences cancel.

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(c) Hence, the collinear singularities from leptons ($\log m_l$) come from only the QED-type corrections to the lepton vertex corrections, which we compute with full lepton mass dependence.

The generic procedure

$d = 4 - 2\epsilon$

- Diagrammatic approach -> `QGRAF` to generate Feynman diagrams
- In-house `FORM` routines for algebraic manipulation :

  Lorentz, Dirac and Color algebra
- Decomposition of the dot products to obtain scalar integrals

$$\frac{2l.p}{l^2(l-p)^2} = \frac{l^2 - (l-p)^2 + p^2}{l^2(l-p)^2} = \frac{1}{(l-p)^2} - \frac{1}{l^2} + \frac{p^2}{l^2(l-p)^2}$$

- Identity relations among scalar integrals : IBPs, LIs & SRs
- Algebraic linear system of equations relating the integrals Simone's talk

$\Downarrow$

Master integrals (MIs)

- Computation of MIs : Method of differential equation & semi-analytic approach
- Ultraviolet renormalization
- Subtraction of the universal infrared poles ($S^{(1,1)}$).
- Numerical evaluation of the hard function to prepare the grid.

The method of differential equations

A Feynman integral is a function of spacetime dimension d and kinematic invariant $z = m^2/q^2$.

$$J_i \sim \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{1}{l_1^2 l_2^2((l_1 - l_2)^2 - m^2)(l_1 - q)^2 (l_2 - q)^2} \equiv f(d,\, z)$$

The idea is to obtain a differential eqn. for the integral $w.r.t.\ z$ and solve it.

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$$\frac{d}{dz} J_i = \text{some combinations of integrals}$$

$$\Downarrow \ \text{IBP identities/reduction}$$

$$= \sum_j c_{ij} J_j$$

c_{ij}'s are rational function of d and z.

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$$d_z \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

$$d_z \mathbb{J} = \mathbb{A}(d, z)\mathbb{J}$$

The black dots ($\bullet$) denote rational functions in d and z.

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To solve such a system, we need to perform series expansion in ϵ and to organize the matrix in each order of ϵ in such a way that it diagonalizes, or at least it takes a block-triangular form. Now, it can be solved using bottom-up approach.

The homogeneous solutions are in general $\log$ or Li_2. Because of the ϵ expansion, the non-homogeneous solutions are recursive integral over the homogeneous solutions.

The results are obtained in terms of iterated integrals (GPLs).

Iterated integrals

From Feynman integrals to iterated integrals : What do we gain?

Parametric Feynman integrals are multi-dimensional. The numerical evaluation is tedious, unstable and not so precise.

Iterated integrals

From Feynman integrals to iterated integrals : What do we gain?

Iterated integrals are one-dimensional. They can be computed with great precision in a short amount of time. Besides, they have the following properties:

(a) **Shuffle algebra** : Allows to obtain a basis for a set of iterated integrals. Reduction to such a basis is extremely effective to reduce the computation time by few times.

(b) **Scaling invariance** : Allows to convert the limit of these integrals from kinematical variables (z) to constants (1). This makes the integration really precise.

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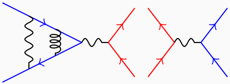
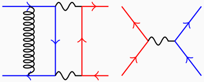
In the two-loop virtual, Chen iterated integrals appear!

Solutions of some MIs contain iterated integrals with square-root kernels $\Rightarrow$ numerical evaluation in physical region is challenging!

The master integrals



[Aglietti, Bonciani, Degrassi, Vicini]
Resulted in simple GPLs: straightforward to evaluate numerically in physical region.



[Bonciani, Di Vita, Mastrolia, Schubert]
Scales : s, t, m_Z
Massive lines introduce square-root letters!

Solved using DE in terms of Chen's iterated integrals! Numerical evaluation possible only in the non-physical region.

(1) Compare to the onshell Z production, many new MIs appear in the full DY scenario.
(2) Even though the new MIs were available, they are not suitable for numerical evaluation in the physical region.

Can we find a different approach to resolve this issue?

Our semi-analytic approach

What do we need for the two-loop virtual amplitudes?

Our semi-analytic approach

What do we need for the two-loop virtual amplitudes?

(a) An analytic formula for the singular part, to perform the infrared subtraction.

(b) A formula for the finite part which should be numerically stable and precise.

(i) The universal subtraction operator indicates that the singular part of the amplitude contains only simple GPLs.

(ii) We study the master integrals to find certain internal combinations (at the lowest order in ϵ) which can be solved in terms of simple GPLs.

So, only simple GPLs in the singular part! SOLVED!

Our semi-analytic approach

What do we need for the two-loop virtual amplitudes?

(a) An analytic formula for the singular part, to perform the infrared subtraction.

(b) A formula for the finite part which should be numerically stable and precise.

Most of the MIs, which contribute to the finite part, are known in terms of GPLs. Few MIs, which contain complicated GPLs, we solve them using series expansion.

(i) We consider the system of differential equations for all the MIs (36). Given a boundary point, the system can be solved using series expansion for a nearby point.
(ii) The solution in this new point can now be considered as boundary and thus we can go forward along a path to obtain solution in any phase space point.
(iii) As most of the MIs (31) are known in closed form, they provide crucial checks for the series solution.

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DiffExp : When the path hits a singularity, it performs logarithmic expansion.
Drawback : Works only for real parameters! In our computation, we use complex-mass scheme, hence use of DiffExp can be done with the following approximation:

The complex mass effects mostly the Feynman diagrams with a Z propagator in the s-channel i.e. only vertex-type diagrams. Hence, for the box diagrams, where we have the MIs computed by DiffExp, we use real mass for all the Feynman integrals.

Code by Tommaso : Path is in the complex plane. A Taylor series expansion is enough!

Now, we can use complex mass everywhere!

Finally

We obtain the two-loop virtual amplitude:
(a) The singular part is analytic and contains GPLs. This allows us to successfully check with the universal infrared behaviour of the scattering amplitudes.

(b) The finite part after performing the infrared subtraction contains GPLs and few MIs 'symbolically' which have been computed using our semi-analytic approach.

Finally

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Next?

We need to evaluate the subtracted finite part numerically for few thousand phase-space points. Although evaluation of a single GPL is fast, there are ~ 11000 GPLs in the full expression. Also the expression is extremely large.

Numerical evaluation and the grid

To obtain a fast compilation and successful numerical evaluation, we divide the contributions from various Feynman diagrams in a gauge invariant way by the presence of different EW vector bosons (γ, Z, W). For example, all the diagrams with two γ can be treated separately.

Each such subset, again, can have contributions from Feynman diagrams of different topologies, like two-loop corrections to initial quark vertex, the Box contributions etc.

These subdivisions allow us to parallelize the computation. With a 3000 core cluster, it takes around 2-3 hrs to obtain the full grid of 3250 phase-space points.

Summarizing

- The NNLO QCD-EW contributions to Drell-Yan production are much sought for.
- One of the bottleneck is the computation of two-loop virtual amplitudes.
- Our semi-analytic approach allows us to achieve analytic cancellation of the universal subtraction term, as well as fast and stable numerical evaluation of the finite hard function.
- Compare to DiffExp, the code by Tommaso can incorporate complex mass.

Status for CC DY

- Presence of both W and Z mass in a single diagram makes it more complicated. We expand the Z propagator in terms of W as

$$\frac{1}{q^2 - m_Z^2} = \frac{1}{q^2 - m_W^2} + \frac{\delta_m}{(q^2 - m_W^2)^2} + \cdots$$

- Asymmetric combination of Feynman diagrams does not allow the cancellation of final state collinear divergences in the box diagrams. Massification :

$$|\mathcal{M}_m\rangle = \mathcal{J}_m \mathcal{J}_0^{-1} |\mathcal{M}_0\rangle$$

Update: Computation in δ_m expansion has been done and after massification, the universal infrared subtraction has been successfully performed for $\mathcal{O}(\delta_m^0)$.

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Ongoing:

(i) Checks up to $\mathcal{O}(\delta_m^2)$.
(ii) The computation in exact W and Z within the same code. For this case, the new MIs are not available analytically. We plan to use the output of our semi-analytic approach for a numerical check of subtraction and to obtain directly the numerical hard function.

Thank you for your attention!