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QCD and advanced Monte Carlo tools

CERN-FNAL HCP summer school – Lecture 1

CERN, 8/6/2011

Plan

- Motivations; basics of perturbative QCD
  $\sim 1^{st}$ lecture

- Fixed-order calculations
  $\sim 2^{nd}$ lecture

- Fixed-order calculations and Event Generators
  $\sim 3^{rd}$ lecture

- Event Generators
  $\sim 4^{th}$ lecture

Minimal bibliography

Textbook stuff

Ellis, Stirling, Webber, *QCD and Collider Physics*, Cambridge Press (1996)

Recent and current work

HCP Summer Schools: `hcpss.web.cern.ch/hcpss`

CTEQ Summer Schools: `www.cteq.org`

Les Houches writeups ("Physics at TeV collider" series)

Check CERN Academic Training lectures (several speakers)

Strong interactions: why bother?

For aesthetic reasons*

For practical reasons

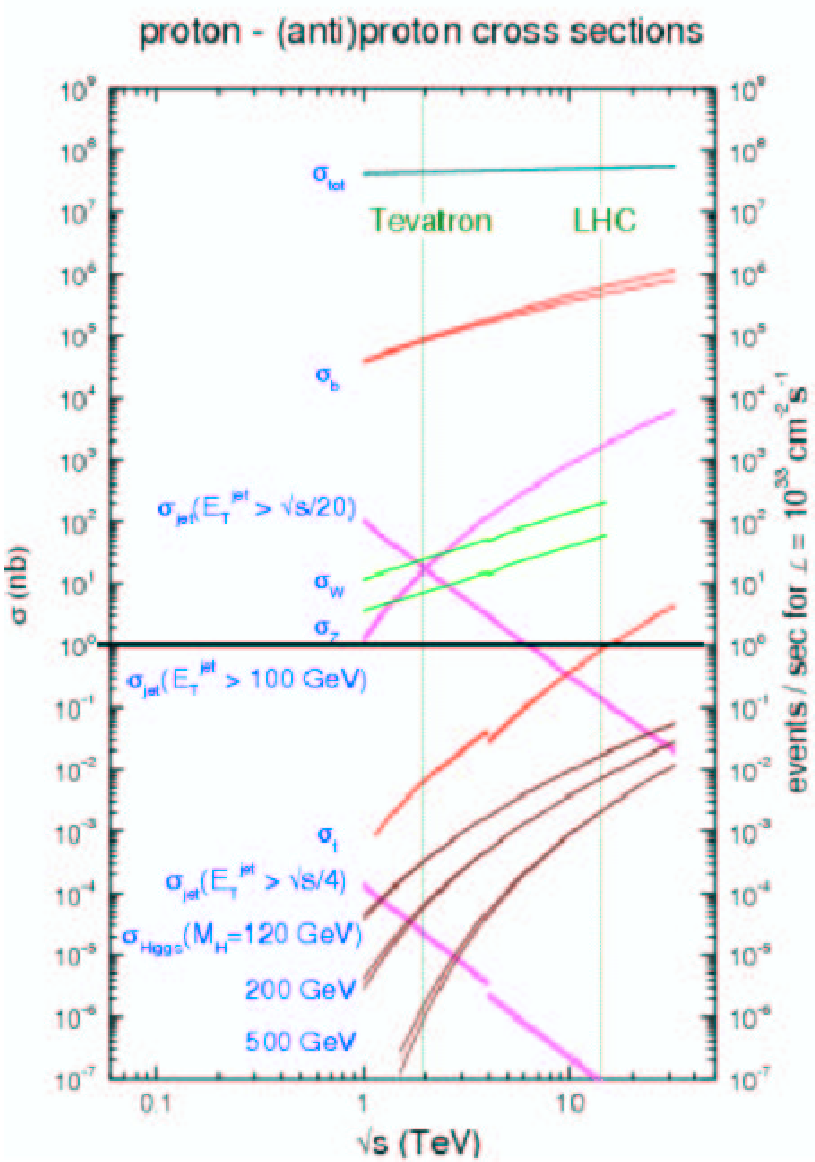
* This result is too beautiful to be false; it is more important to have beauty in one's equations than to have them fit experiment (P. Dirac)

Aesthetic reasons

QCD is a mathematically beautiful, and deceptively simple, theory

- It is a one-parameter theory (α_S)
- It is the only piece of the SM which is not a low-energy theory
- It has a lot of open and very difficult problems (e.g. confinement)

Practical (Dirac may say mundane) reasons



At LHC, one is swamped by QCD processes

For discoveries, this fact can be anything from utterly irrelevant to crucial

But it can hardly be neglected, especially when trying and answer the question "What did we discover?"

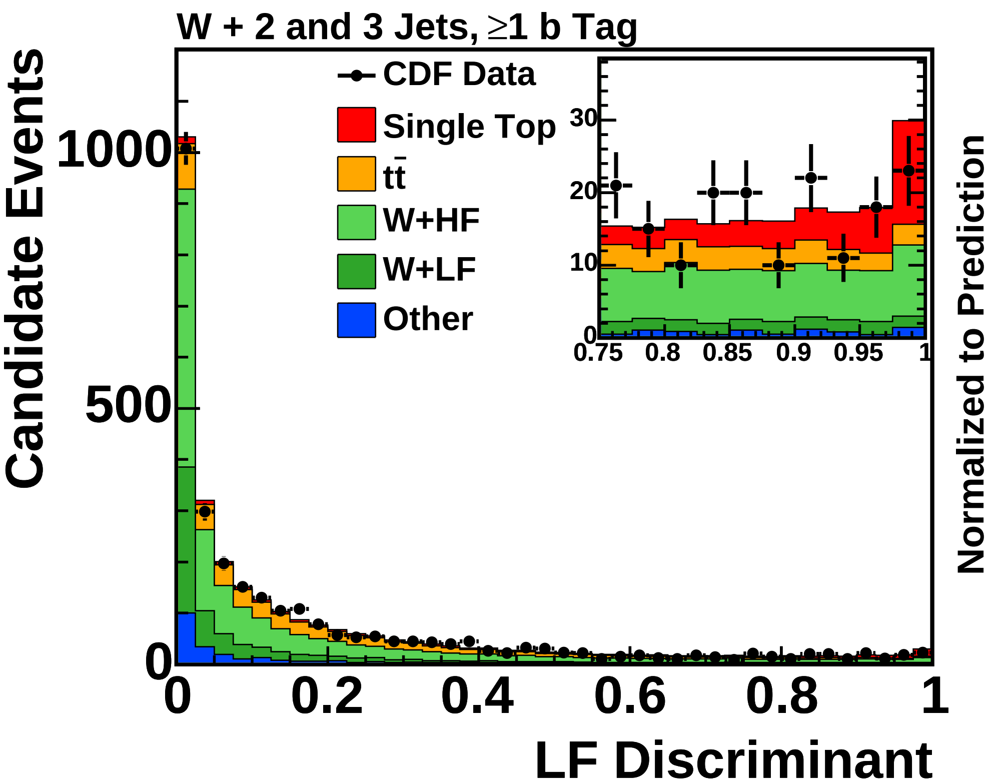
Before understanding the meaning of a discovery, you need to make one

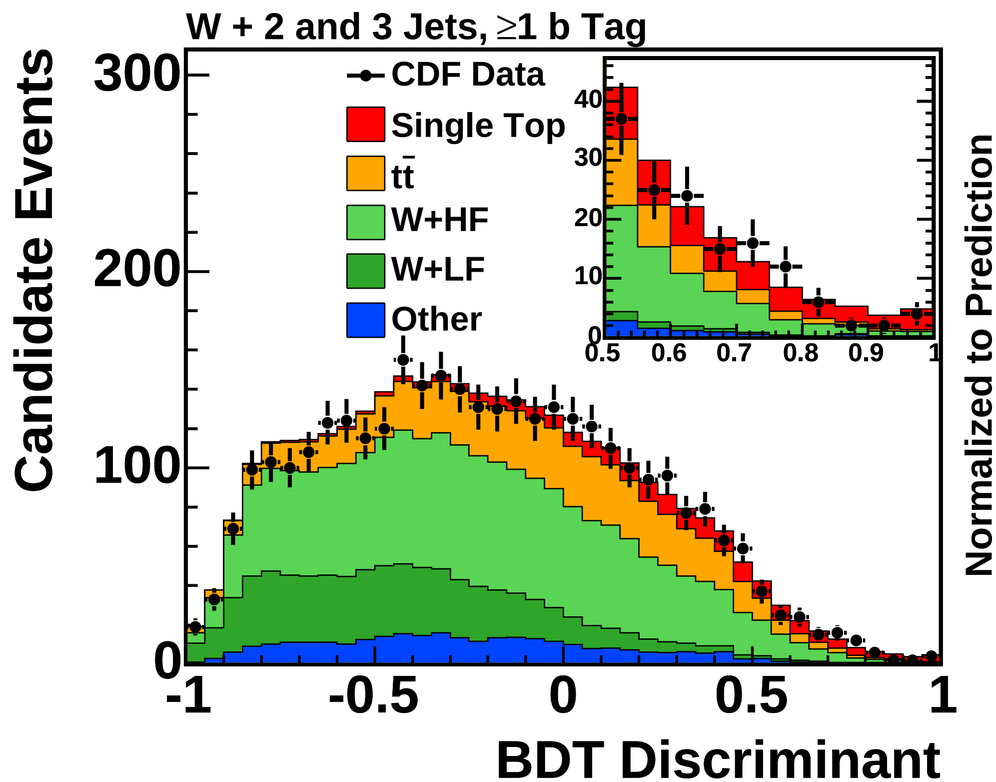
A peak at $M_{\mu\mu} = 2$ TeV: QCD is irrelevant

A SUSY-like case. Lots of backgrounds, but one need not rely on blind theoretical QCD predictions for those: tuning to data (in control regions) will help. Hence, QCD is important, but not crucial

In other cases, you may not be so lucky

(Extreme) example: discovery of single top





CDF (1004.1181)

The signal region is at Discriminant $\longrightarrow 1$, as one sees from the templates of single-top production (not shown here)

Discovery depends on theoretical results for signal

QCD $=$ Strong interactions

QuantumChromoDynamics is

- A non-abelian gauge theory, with gauge group SU(3)
- There are 8 spin-1 massless *gluons* that carry the interaction (adjont representation of SU(3)):

$$A^a, \qquad a = 1, \dots 8$$

- There are $3 \times N_F$ spin-1/2 *quarks*, the matter fields (fundamental representation of SU(3)):

$$\psi_i^{(f)}, \qquad i = 1, 2, 3\,; \;\; f = 1, \dots N_F$$

"Chromo" since $(1, 2, 3) = (r, g, b)$ are called colours

f are the *flavours*. Their number N_F depends on which physics one considers. We call $1, \dots 6 \longrightarrow$ up, down, strange, charm, beauty, top

QCD interactions are flavour blind; differences among quarks are due to EW interactions

The QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} + \sum_{f=1}^{N_F} \bar{\psi}_i^{(f)} \left(i\gamma_\mu D^\mu_{ij} - m_f \delta_{ij} \right) \psi_j^{(f)} + \mathcal{L}_{GF} + \mathcal{L}_{ghost}$$

Covariant derivative:

$$D^\mu_{ij} = \delta_{ij} \partial_\mu + i g t^a_{ij} A^a_\mu$$

Gluon strengh tensor:

$$G_a^{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

Plug the term fAA into the Lagrangian, and you'll get gluon 3- and 4-gluon self interactions – it makes all the difference wrt to QED. This term has a fundamental importance for the very existence of hadrons

SU(3) colour Lie algebra

t^a and T^a are the SU$(N=3)$ generators (in the fundamental and adjoint representations), with

$$[t^a, t^b] = if^{abc}t^c, \quad [T^a, T^b] = if^{abc}T^c, \quad (T^a)_{bc} = -if^{abc}$$

Choosing the normalization

$$\mathrm{Tr}\left(t^a t^b\right) = T_R \delta^{ab} \equiv \frac{1}{2}\delta^{ab}$$

one has

$$\sum_a t^a_{ij} t^a_{jk} = C_F \delta_{ik}\,, \;\; \mathrm{Tr}\left(T^a T^b\right) = C_A \delta^{ab}$$

$$C_F = \frac{N^2-1}{2N} \equiv \frac{4}{3}\,, \;\; C_A = N \equiv 3$$

Fundamental identity (Fierz)

$$\sum_a t^a_{ij} t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$$

Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

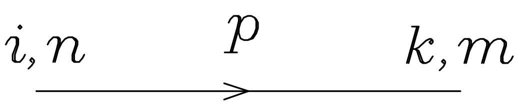
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

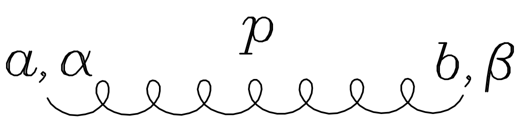
Hermitean, traceless, that provide a representation for SU(3) generators:

$$t^a = \frac{1}{2}\lambda^a$$

Feynman rules (QED like)

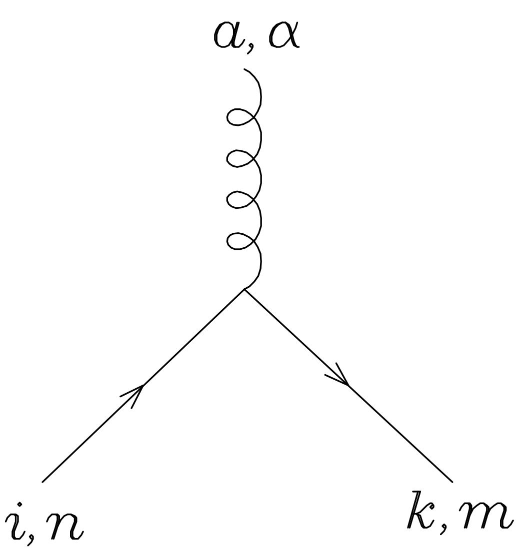


$$i\delta_{ik}\frac{(\gamma_\mu p^\mu + m_f)_{nm}}{p^2 - m_f^2 + i\epsilon}$$



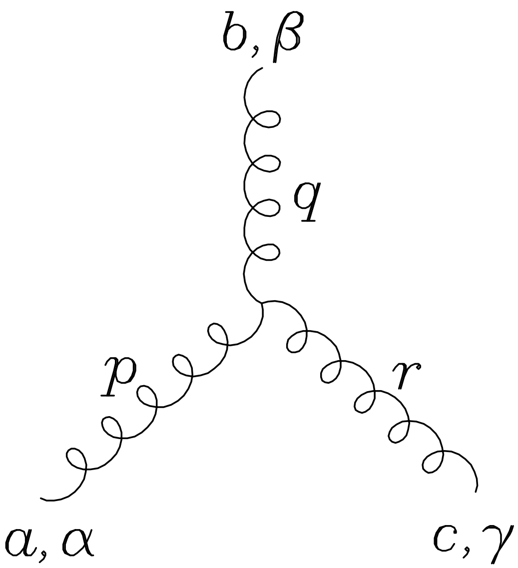
$$\frac{i\delta_{ab}}{p^2 + i\epsilon}\left[-g^{\alpha\beta} + (1-\lambda)\frac{p^\alpha p^\beta}{p^2 + i\epsilon}\right]$$

$$\frac{i\delta_{ab}}{p^2 + i\epsilon}\left[-g^{\alpha\beta} + \frac{p^\alpha n^\beta - p^\beta n^\alpha}{n \cdot p} - n^2\frac{p^\alpha p^\beta}{(n \cdot p)^2}\right]$$

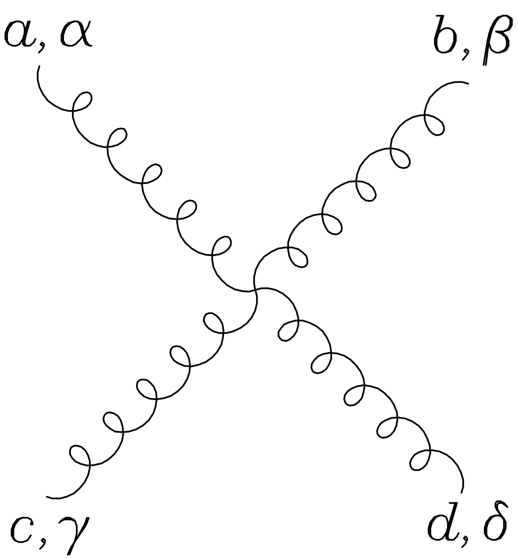


$$-igt^a_{ki}\gamma^\alpha_{mn}$$

Feynman rules



$$-gf^{abc}\left[g^{\alpha\beta}(p-q)^{\gamma}+g^{\beta\gamma}(q-r)^{\alpha}+g^{\gamma\alpha}(r-p)^{\beta}\right]$$



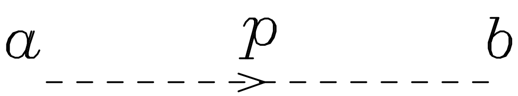
$$-ig^2 f^{eac} f^{ebd}\left(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}\right)$$
$$-ig^2 f^{ead} f^{ebc}\left(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}\right)$$
$$-ig^2 f^{eab} f^{ecd}\left(g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}\right)$$

- Nothing like that in QED

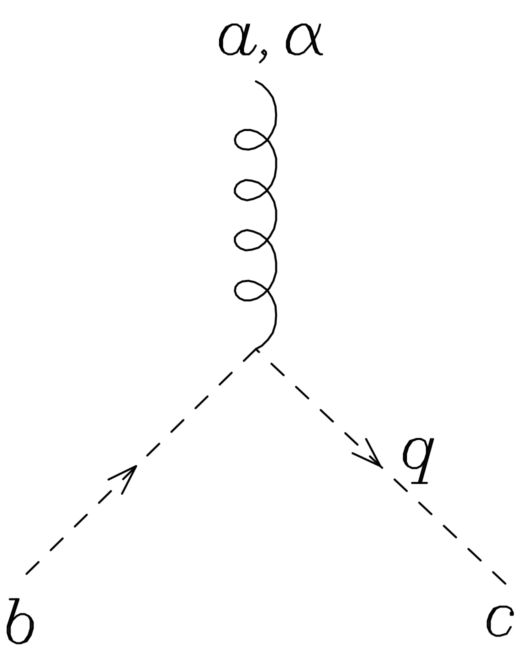
Feynman rules

In axial gauges the gluon has only physical (ie transverse) polarization states $\Longrightarrow$ simpler, intuitive physical picture. Drawback: involved computations become, well, more involved

In covariant gauges of non-abelian theories, ghosts must be included to cancel unphysical polarization states of gluons



$$i\delta_{ab}\frac{1}{p^2 + i\epsilon}$$



$$gf^{abc}q^{\alpha}$$

Light-quark symmetries

$$\mathcal{L}_{matter} = i\sum_{f=1}^{N_F}\left(\bar{\psi}_L^{(f)}\gamma_\mu D^\mu \psi_L^{(f)} + \bar{\psi}_R^{(f)}\gamma_\mu D^\mu \psi_R^{(f)}\right) - \sum_{f=1}^{N_F} m_f\left(\bar{\psi}_L^{(f)}\psi_R^{(f)} + \bar{\psi}_R^{(f)}\psi_L^{(f)}\right)$$

$$\psi_L^{(f)} = \frac{1}{2}(1-\gamma_5)\psi^{(f)}\,, \quad \psi_R^{(f)} = \frac{1}{2}(1+\gamma_5)\psi^{(f)}$$

There is a huge symmetry when $m_f = 0$ *(chiral)*

$$\psi_L^{(f)} \longrightarrow e^{i\phi_L} U_L^{ff'}\psi_L^{(f')}\,, \quad \psi_R^{(f)} \longrightarrow e^{i\phi_R} U_R^{ff'}\psi_R^{(f')}$$

$$\mathrm{SU}_L(N_F)\otimes\mathrm{SU}_R(N_F)\otimes\mathrm{U}_L(1)\otimes\mathrm{U}_R(1)$$

Chiral symmetry is not apparent in the observed spectrum; and, quantization effects may also distroy classical symmetry

- $\mathrm{SU}_L(N_F)\otimes\mathrm{SU}_R(N_F)\longrightarrow\mathrm{SU}_V(N_F)$, "isospin"; $\mathrm{SU}_A(N_F)$ is spontaneously broken, with Goldstone bosons = light mesons (π's for $N_F = 2$, π's+K's+η for $N_F = 3$)
- $\mathrm{U}_L(1)\otimes\mathrm{U}_R(1)\longrightarrow\mathrm{U}_V(1)$, baryon number conservation; $\mathrm{U}_A(1)$ spoilt by quantum effects ($\mathcal{L}_\theta$)

Why SU(3)

An exciting and intricate story, that involved some of the best minds of the 20^{th} century

- From SU(2) to SU(3) – strangeness (Gell-Mann, Nishijima) and the eightfold way (Gell-Mann, Ne'eman)
- Postulation of entities associated with the SU(3) fundamental representation [quarks (Gell-Mann), aces (Zweig)]
- Quarks violate spin-statistics ($\Delta = uuu$) $\implies$ attach hidden d.o.f. (colour) to them (Han, Nambu, Greenberg)
- SLAC experiments hint that Feynman's partons are the same as quarks
- Promote $\mathrm{SU}(3)_c$ to local symmetry (Fritzsch, Gell-Mann, Weinberg)

Note: started from global $\mathrm{SU}(3)_{flav}$ and arrived at local $\mathrm{SU}(3)_c$!

Hadron spectrum

One makes following assumptions

- Hadrons are bound states formed by quarks

- Meson states are

$$\sum_i \psi_i^{(f)^\star} \psi_i^{(f')}$$

- Baryon states are

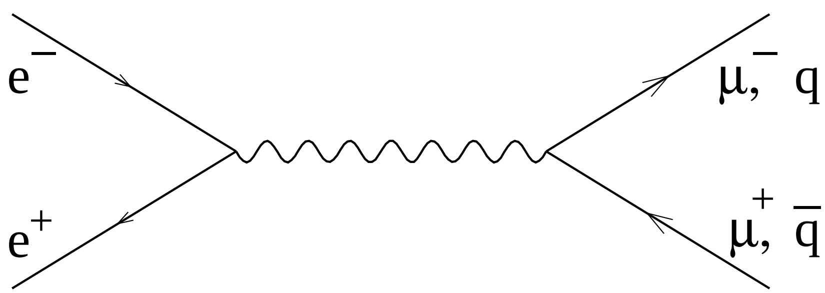
$$\sum_{ijk} \epsilon^{ijk} \psi_i^{(f)} \psi_j^{(f')} \psi_k^{(f'')}$$

- Colour non-singlet states (ie not invariant for $\psi_i \to U_{ij}\psi_j$) are forbidden

Experimental evidence for $N = 3$

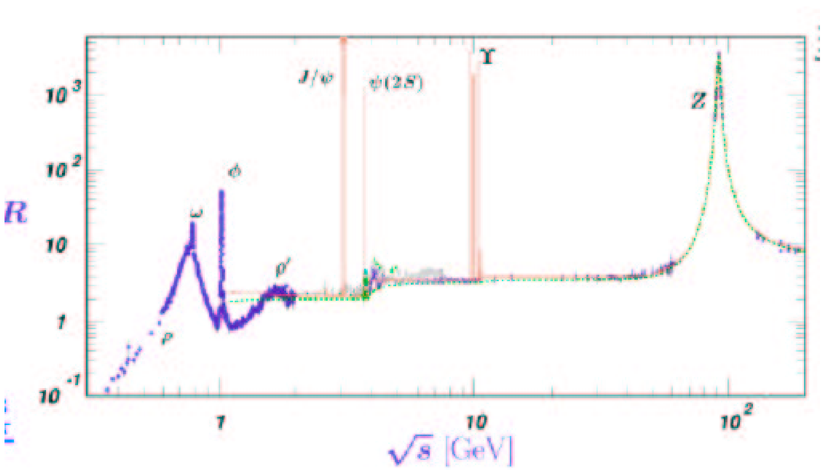
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{i,f} \frac{\sigma(e^+e^- \to q_i^{(f)}\bar{q}_i^{(f)})}{\sigma(e^+e^- \to \mu^+\mu^-)} \longrightarrow \sum_{i=1}^{N}\sum_{f=1}^{N_F} e^2(f)$$

Numerator and denominator are the same diagram (at $\mathcal{O}(\alpha_S^0)$)



$$\Longrightarrow \quad R = N\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right]$$

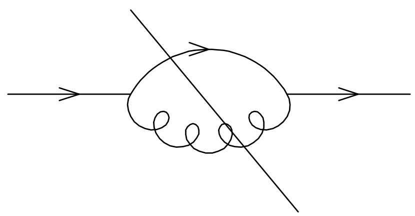
...and so forth for more quark flavours



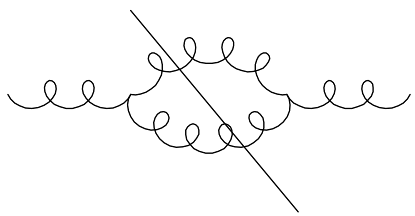
Measurements are amply consistent with $N = 3$

(and test charge assignments)

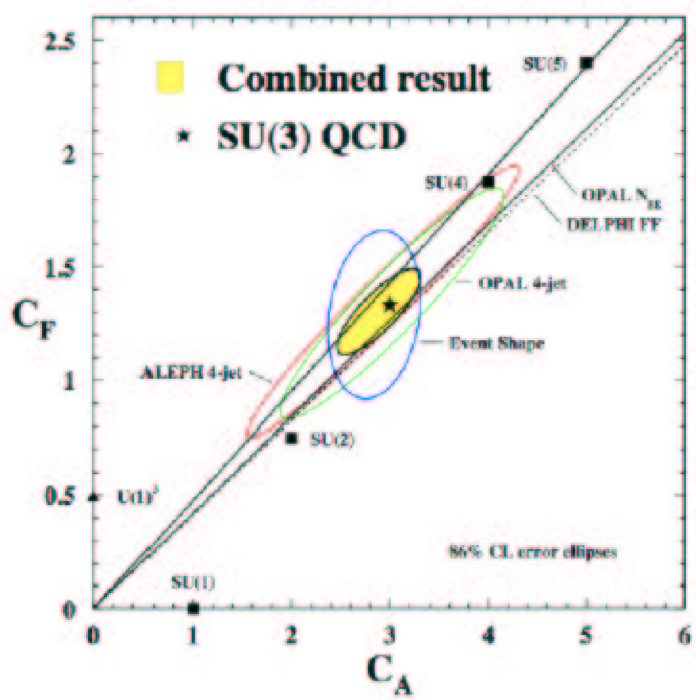
Further experimental evidence



$$\propto \lambda^a \lambda^a \longrightarrow C_F$$



$$\propto f^{abc} f^{dbc} \longrightarrow C_A$$



Four-jet production in e^+e^- is sensitive to both

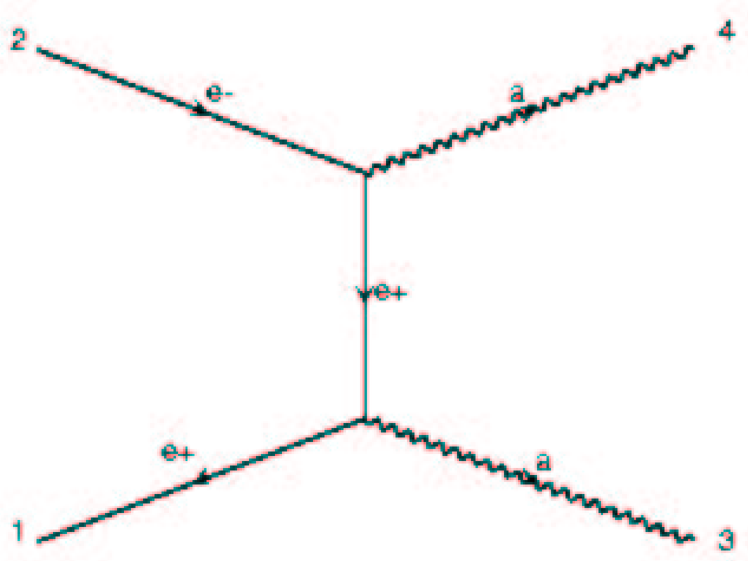
This also shows that clean QCD studies can be done at lepton colliders

Forget about gluon self-interactions. Is what remains just QED, with eight charges rather than one?

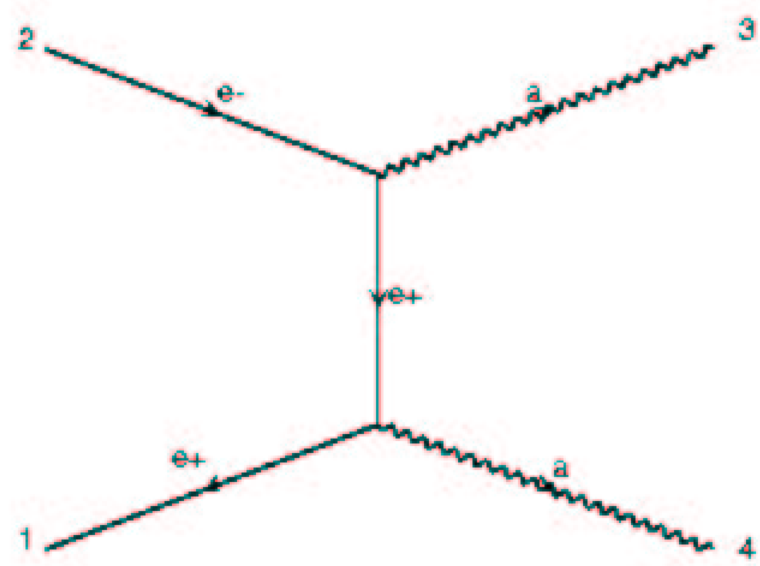
Unfortunately, not

Consider $e^+e^- \to \gamma\gamma$ and $u\bar{u} \to gg$

$$e^+e^- \to \gamma\gamma$$



D_1

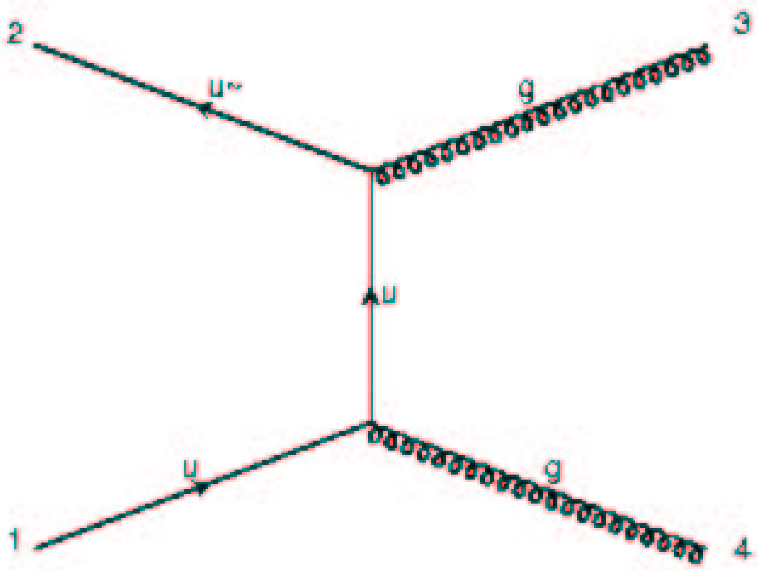


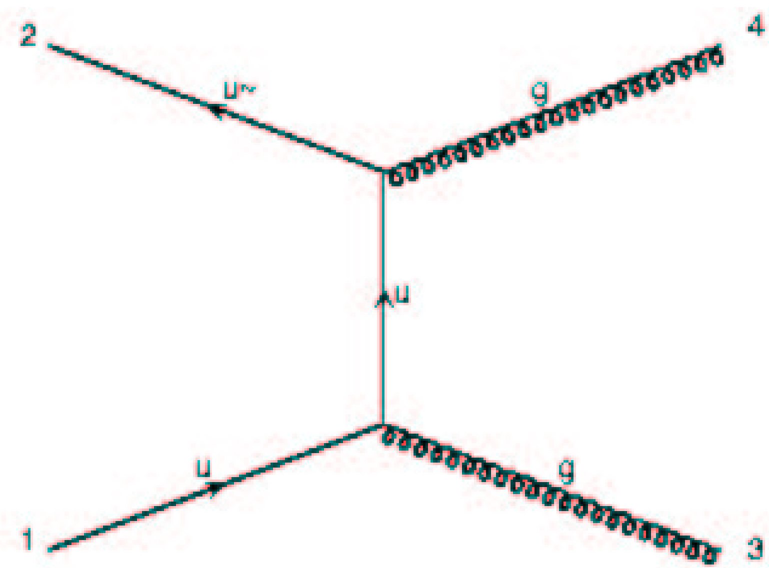
D_2

Check gauge invariance. Take photon #4 to be definite:

$$D_1 \cdot k_4 = -D_2 \cdot k_4$$

$u\bar{u} \to gg$





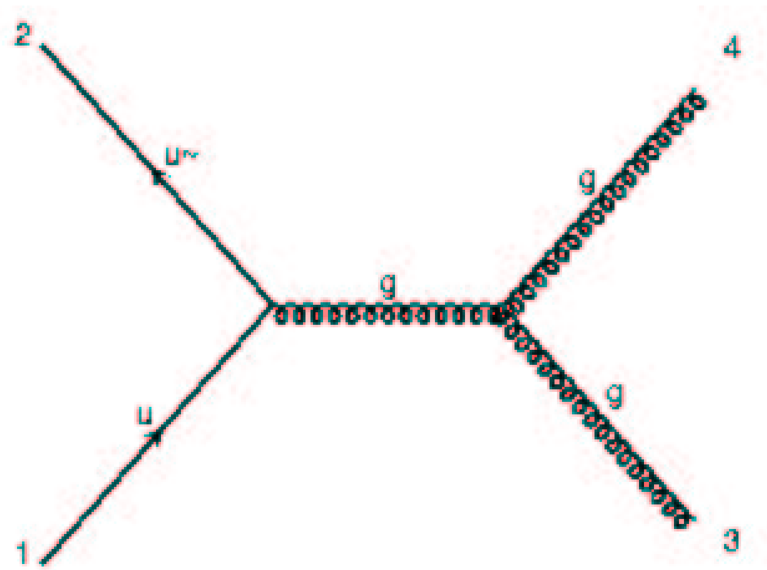
$$G_1 = \frac{g^2}{e^2}(t^{a_3}t^{a_4})_{c_2c_1}D_1 \qquad G_2 = \frac{g^2}{e^2}(t^{a_4}t^{a_3})_{c_2c_1}D_2$$

Check gauge invariance. Take gluon #4 in analogy with the QED case:

$$D_1 \cdot k_4 = -D_2 \cdot k_4 \qquad \Longrightarrow \qquad G_1 \cdot k_4 \neq -G_2 \cdot k_4$$

QCD does not violate gauge invariance, obviously. The calculation above is wrong, since one Feynman diagram has been left out

$u\bar{u} \to gg$



$$G_3 = \frac{g^2}{e^2} f^{a_3 a_4 b} t^b_{c_2 c_1} \, D_3$$

$1/e^2$ is just to have the same normalization in D_3 as in D_1 and D_2

Now use SU(3) algebra

$$f^{a_3 a_4 b} t^b_{c_2 c_1} = -i \left(t^{a_3} t^{a_4}\right)_{c_2 c_1} + i \left(t^{a_4} t^{a_3}\right)_{c_2 c_1}$$

Thus the complete amplitude is:

$$(e^2/g^2)\mathcal{A} = \left(t^{a_3} t^{a_4}\right)_{c_2 c_1} [D_1 - iD_3] + \left(t^{a_4} t^{a_3}\right)_{c_2 c_1} [D_2 + iD_3]$$

By direct computation:

$$D_1 \cdot k_4 = -D_2 \cdot k_4 = iD_3 \cdot k_4 \qquad \Longrightarrow \qquad \mathcal{A} \cdot k_4 = 0$$

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- True, QCD is not QED. But amplitudes can be decomposed into sums of QED-like terms, which factor out colour structures naturally associated with diagrams with no gluon self-interactions
- Does this matter? What we observe is not an amplitude, but an amplitude squared.

It actually does matter, since it suggests a beautiful re-organization of amplitudes, and an approximation of amplitudes squared which is parametric and systematically improvable

Take again the $u\bar{u} \to gg$ amplitude, square it and sum over colours

(I neglect pre-factors now)

$$\mathcal{A} = (t^{a_3} t^{a_4})_{c_2 c_1} \widehat{\mathcal{A}}_1 + (t^{a_4} t^{a_3})_{c_2 c_1} \widehat{\mathcal{A}}_2$$

$$\widehat{\mathcal{A}}_1 = D_1 - iD_3 \qquad \widehat{\mathcal{A}}_2 = D_2 + iD_3$$

Then (implicit sums over identical indices):

$$\sum_{colours} |\mathcal{A}|^2 = \mathrm{Tr}\left(t^a t^b t^b t^a\right) \left|\widehat{\mathcal{A}}_1\right|^2 + \mathrm{Tr}\left(t^a t^b t^b t^a\right) \left|\widehat{\mathcal{A}}_2\right|^2 + 2\,\mathrm{Tr}\left(t^a t^b t^a t^b\right) \Re\left(\widehat{\mathcal{A}}_1 \widehat{\mathcal{A}}_2^\star\right)$$

By direct computation (try it!):

$$4\mathrm{Tr}\left(t^a t^b t^b t^a\right) = N^3 - 2N + 1/N$$

$$4\mathrm{Tr}\left(t^a t^b t^a t^b\right) = -N + 1/N$$

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- Gauge invariance of $\widehat{\mathcal{A}}_i$ implies enormous simplifications in their computations. By using recursion relations* one reduces the complexity from factorial to polynomial in the number of legs (at least for gluon amplitudes)

* There are beautiful results which I'll not be able to present. If interested, check papers by Berend & Giele, Cachazo, Svrcek, & Witten, Britto, Cachazo & Feng

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- This is a general property of QCD amplitudes
- Gauge invariance of $\widehat{\mathcal{A}}_i$ implies enormous simplifications in their computations. By using recursion relations* one reduces the complexity from factorial to polynomial in the number of legs (at least for gluon amplitudes)
- It may be useful to think that QCD=QED$\pm\mathcal{O}(10\%)$, but sometimes it is misleading. Subleading terms can be $\mathcal{O}(1/N)$, and interference contributions may actually be dominant for certain observables. Handle with care!

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Decomposition of amplitudes

- Only gluons

$$\mathcal{A}(a_1,\ldots a_n)=\sum_{\sigma\in P'_n}\mathrm{Tr}\left(t^{a_{\sigma(1)}}\ldots t^{a_{\sigma(n)}}\right)\widehat{\mathcal{A}}\left(k_{\sigma(1)},\ldots k_{\sigma(n)}\right)$$

  Sum over non-cyclic permutations. By using the SU(3) Lie algebra, a QED-like structure emerges, even if there are no fermions around. Physical meaning: gluon $\sigma(i)$ is colour-connected to gluons $\sigma(i-1)$ and $\sigma(i+1)$. I'll show later that this determines the singularity structure of $\widehat{\mathcal{A}}$ in Lorentz space

- One quark-antiquark line, and n gluons

$$\mathcal{A}(i;a_1,\ldots a_n;j)=\sum_{\sigma\in P_n}\left(t^{a_{\sigma(1)}}\ldots t^{a_{\sigma(n)}}\right)_{ij}\widehat{\mathcal{A}}\left(k_q;k_{\sigma(1)},\ldots k_{\sigma(n)};k_{\bar{q}}\right)$$

  Sum over all permutations (including non-cyclic ones)

- More than one quark-antiquark line, and n gluons: same as before, with combinatorics

Decomposition of amplitudes

- Only gluons

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Sum over all permutations (including non-cyclic ones)

- Review: Mangano & Parke, hep-th/0509223

On colour transformations

For $U \in$ SU(3), the definition of Lie algebra implies:

$$U = I + i\delta\theta_a X^a + \mathcal{O}\left((\delta\theta)^2\right) \,, \qquad X^a = t^a, T^a$$

By construction, quarks, antiquarks, and gluons transform according to 3,$\bar{3}$, and 8. If one denotes

$$\{c_i\}_{i=1}^{3} \qquad \{\bar{c}_i\}_{i=1}^{3} \qquad \{g_a\}_{a=1}^{8}$$

the possible colours of a quark, antiquark, and gluon, an infinitesimal colour transformation then amounts to:

$$\begin{aligned} c_i' &= c_i + i\delta\theta_a t^a_{ij} c_j \\ \bar{c}_i' &= \bar{c}_i - i\delta\theta_a t^a_{ji} \bar{c}_j \\ g_a' &= g_a + i\delta\theta_c T^c_{ab} g_b \end{aligned}$$

One can use a matrix notation

$$c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad \bar{c} = (\bar{c}_1, \bar{c}_2, \bar{c}_3) \qquad g = g_a t^a$$

The transformations given before are

$$c' = U c \qquad \bar{c}' = \bar{c}\, U^\star \qquad g' = U\, g\, U^\star$$

Now consider the colour of a $q\bar{q}$ “state”

$$c\bar{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} (\bar{c}_1, \bar{c}_2, \bar{c}_3) = \begin{pmatrix} c_1\bar{c}_1 & c_1\bar{c}_2 & c_1\bar{c}_3 \\ c_2\bar{c}_1 & c_2\bar{c}_2 & c_2\bar{c}_3 \\ c_3\bar{c}_1 & c_3\bar{c}_2 & c_3\bar{c}_3 \end{pmatrix}$$

Hence

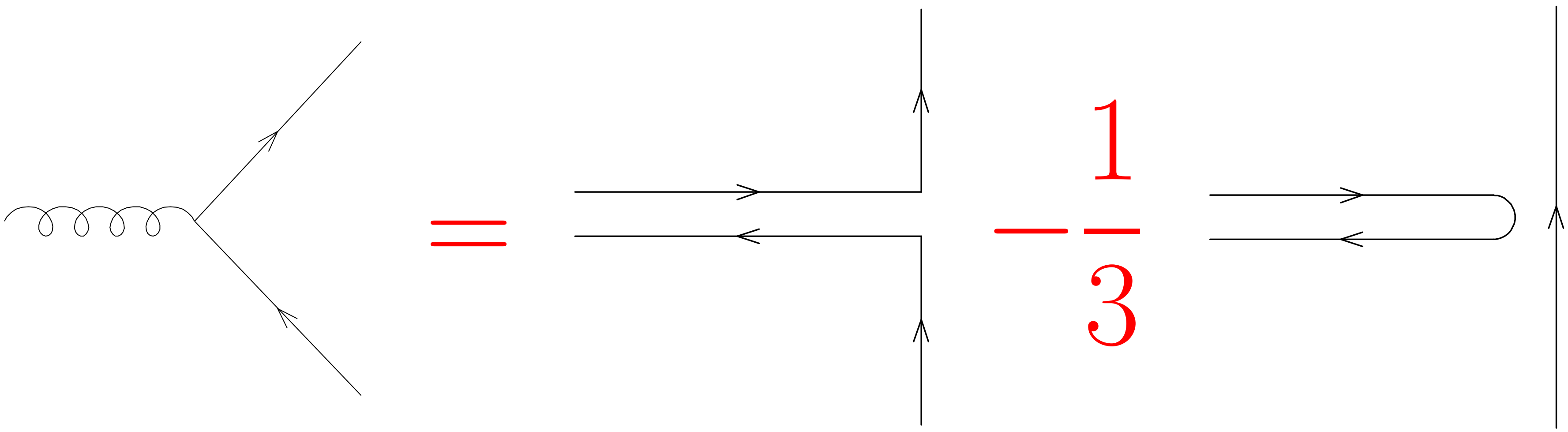
$$(c\bar{c})' = U\,(c\bar{c})\,U^\star$$

Same as for gluons

From the colour viewpoint, a gluon then behaves *almost* as if it were a $q\bar{q}$ pair. One talks about the colour and anticolour of a gluon, as one talks about the colour of a quark and the anticolour of an antiquark. Note:

$$(c\bar{c}) = f_a t^a + f_9 I \qquad \Longrightarrow \qquad \mathrm{Tr}\,(c\bar{c}) = 3 f_9$$

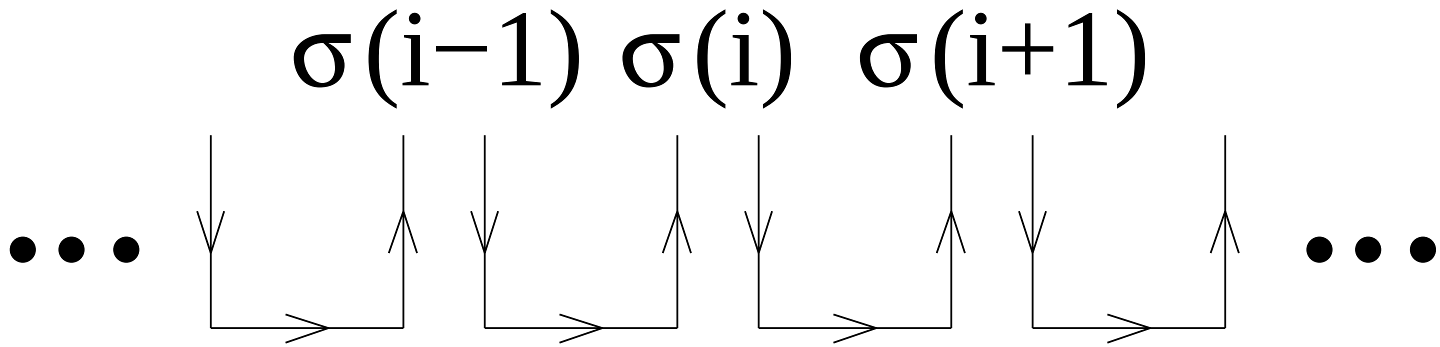
The f_9 component (singlet) is obviously not there in the case of gluons. A graphical representation of this is:



and the last term drops out in the $N \to \infty$ limit

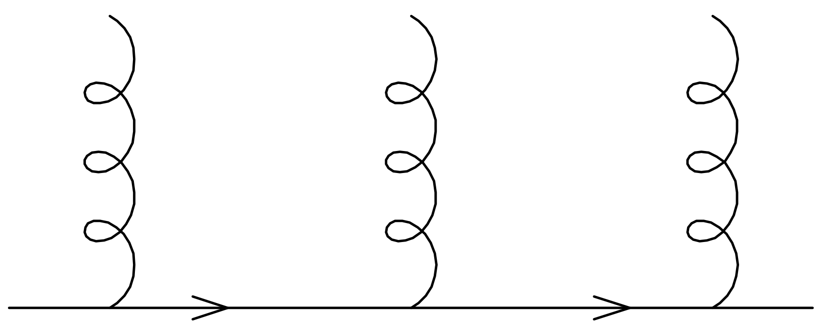
These rules are used to draw colour flows pictorially

Example:



$$\mathrm{Tr}\left(t^{a_{\sigma(1)}} \ldots t^{a_{\sigma(n)}}\right) \widehat{\mathcal{A}}\left(k_{\sigma(1)}, \ldots k_{\sigma(n)}\right)$$

You may read this as:



...but don't take it too literally: it is misleading

Representation of colour algebra

The transformation rules given before

$$\begin{aligned} c'_i &= c_i + i\delta\theta_a t^a_{ij} c_j \\ \bar{c}'_i &= \bar{c}_i - i\delta\theta_a t^a_{ji} \bar{c}_j \\ g'_a &= g_a + i\delta\theta_c T^c_{ab} g_b \end{aligned}$$

can be compactly written by introducing the following representation of the Lie colour algebra:

$$\vec{Q}_p = \{t^a\}_{a=1}^{8} \;,\; \{-t^{a\mathrm{T}}\}_{a=1}^{8} \;,\; \{T^a\}_{a=1}^{8} \;, \qquad p = q, \bar{q}, g$$

$$\vec{Q}_{p1} \cdot \vec{Q}_{p2} = \vec{Q}_{p2} \cdot \vec{Q}_{p1} \;, \qquad \vec{Q}_p \cdot \vec{Q}_p \equiv Q_p^2 = C(p) I$$

$$C(q) = C(\bar{q}) = C_F \qquad C(g) = C_A$$

so that for any colour configuration $x_p = \{c_i\}, \{\bar{c}_i\}, \{g_a\}$

$$x'_p = \left(I + i\delta\vec{\theta} \cdot \vec{Q}_p \right) x_p$$

Summary

- QCD is a non-abelian gauge theory with gauge group SU(3)
- Matter fields (quarks) and gauge bosons (gluons) carry colour charges
- Colour can only be observed indirectly, through its static (spectroscopy) and dynamic effects
- A QCD amplitude can be decomposed into a sum of linearly-independent colour structures times gauge-invariant dual amplitudes
- Dual amplitudes are orthogonal at the leading order in N

Memo on RGE and beta functions

Suppose A is a dimensionless quantity which depends on a single large energy scale $Q \gg m$, with m any mass. If the limit $m \to 0$ exists, then by dimensional analysis A is independent of Q

$$A = A(Q/m, \alpha_S) \stackrel{m \to 0}{\longrightarrow} A(\alpha_S)$$

This elegant derivation does not survive quantization. Because of the presence of ultraviolet divergences, the theory must be renormalized, and this always introduce an arbitrary mass scale μ (in A and α_S renormalized)

$$A \stackrel{\text{quantization}}{\longrightarrow} A(Q^2/\mu^2, \alpha_S)$$

The scale μ is arbitrary, and physical results cannot depend on it

$$\frac{d}{d\mu^2} A(Q^2/\mu^2, \alpha_S) = \left(\frac{\partial}{\partial \mu^2} + \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S} \right) A = 0$$

which is a Renormalization Group Equation

In order to solve RGE's, one defines

$$t = \log \frac{Q^2}{\mu^2}\,, \qquad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2}$$

$$\left(-\frac{\partial}{\partial t} + \beta(\alpha_S) \frac{\partial}{\partial \alpha_S} \right) A = 0$$

The *running coupling* $\alpha_S(Q)$ is then introduced

$$t = \int_{\alpha_S}^{\alpha_S(Q^2)} da \frac{1}{\beta(a)}\,, \qquad \alpha_S(\mu^2) = \alpha_S$$

from which it follows that

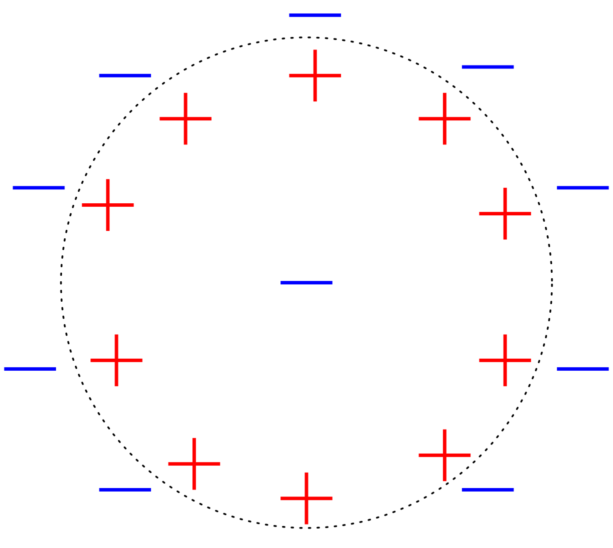
$$A(Q^2/\mu^2, \alpha_S) = A(1, \alpha_S(Q^2))$$

Thus, the scale dependence of A is known if that of $\alpha_S(Q^2)$ is known

The computation of β functions in QFTs has profound implications

The case of QED...

...is relatively simple, and allows a graphical explanation of the running coupling



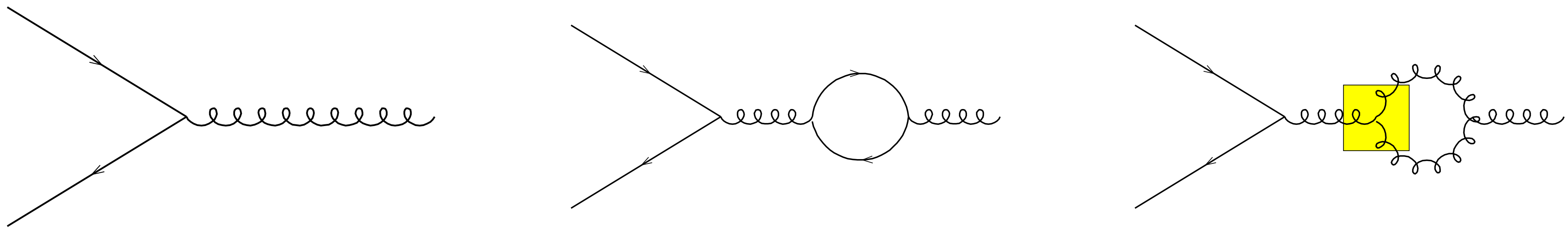
In a relativistic framework, an electron is surrounded by a cloud of virtual electrons and positrons. From the distance, one may not see their charges. By looking closer (probe with larger momenta), one starts to resolve them, and electron charge appears larger

$$Q^2 \frac{d\alpha}{dQ^2} = \beta_{QED}(\alpha) \,, \quad \beta_{QED}(\alpha) = \frac{\alpha^2}{3\pi} + \mathcal{O}(\alpha^3) \quad \Longrightarrow \quad \alpha(Q^2) = \frac{1}{137 - \frac{1}{3\pi}\log(Q^2/m_e^2)}$$

Since $\alpha \to \infty$ for $Q^2 \to e^{411\pi} m_e^2$, Landau (1954) thought QED was ill-defined

The case of QCD

In QCD there are additional contributions from gluon self-interaction...



that have a dramatic effect on the β function

$$\beta_{QCD}(\alpha_S) = -\beta_0 \alpha_S^2 + \mathcal{O}(\alpha_S^3)\,, \quad \beta_0 = \frac{11 C_A - 2 N_F}{12\pi}\,, \quad C_A = N_C \equiv 3$$

Basically, the gluonic contribution to the vacuum polarization reverses the sign of the β function, in such a way that $\alpha_S(Q^2)$ *decreases* when Q^2 *increases* (for $N_F \leq 16$...)

This is called Asymptotic Freedom

Gross, Politzer, Wilczek (1973) Nobel prize 2004

This is the opposite as in QED, which implies that QCD is not an effective low-energy theory of something unknown

$$\alpha_S(Q^2) = \frac{\alpha_S(\mu^2)}{1 + \alpha_S(\mu^2)\beta_0 \log(Q^2/\mu^2)}$$

The (perturbative) computation of β_{QCD}

Time after 1973 has not passed in vain. We have now

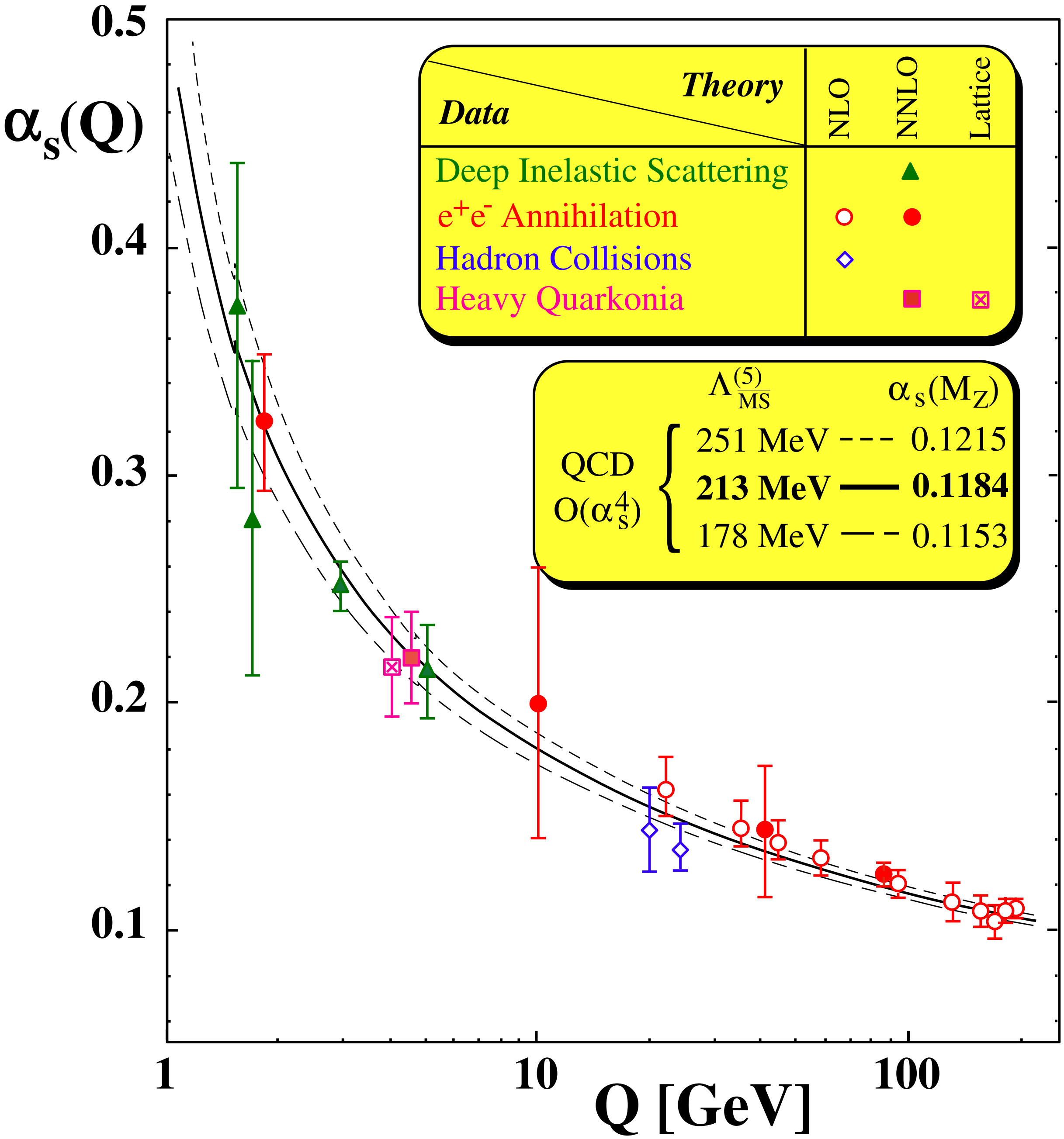
$$\frac{1}{4\pi}\beta(\alpha_S) = -\hat{\beta}_0 \left(\frac{\alpha_S}{4\pi}\right)^2 - \hat{\beta}_1 \left(\frac{\alpha_S}{4\pi}\right)^3 - \hat{\beta}_2 \left(\frac{\alpha_S}{4\pi}\right)^4 - \hat{\beta}_3 \left(\frac{\alpha_S}{4\pi}\right)^5 + \mathcal{O}(\alpha_S^6)$$

thanks to

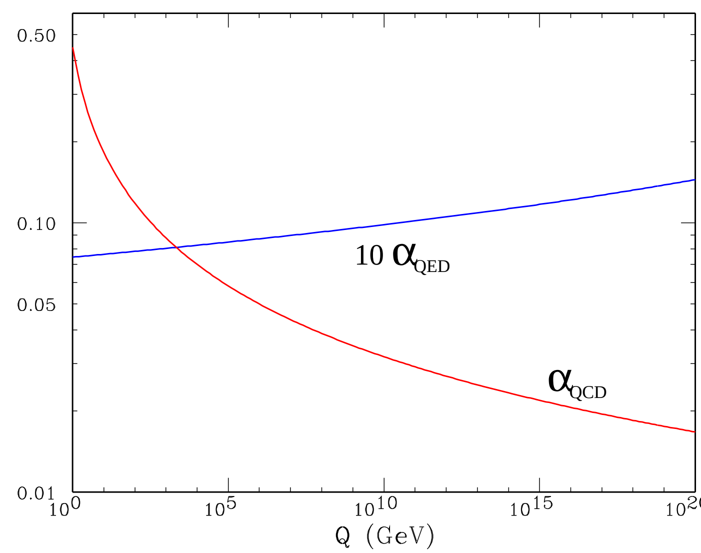
- $\hat{\beta}_0$: Gross, Wilczek, Politzer (1973)
- $\hat{\beta}_1$: Caswell, Jones (1974)
- $\hat{\beta}_2$: Tarasov, Vladimirov, Zharkov (1980)
- $\hat{\beta}_3$: van Ritbergen, Vermaseren, Larin (1997)

Note that $\hat{\beta}_3$ requires a four-loop computation, and the evaluation of about 50000 Feynman diagrams – there are a lot of spinoffs from a computation like this one (computing and mathematics)

Comparisons with data



There is nowadays a very solid evidence that α_S runs as predicted by QCD with $N_C = 3$



The discovery of asymptotic freedom proved that indeed quarks can behave as free particles in DIS (and elsewhere), as suggested by SLAC results

It also allows one to use standard perturbation techniques, as the case of β_{QCD} determination spectacularly shows

We also have *hints* on why quarks/gluons cannot be seen in isolation (i.e. confinement). Naively, large distances $\equiv$ small scales $\Longrightarrow$ inter-parton force grows

Lattice gives further (solid) evidence

Summary

- In certain kinematic regimes, strong interactions are weakly coupled: asymptotic freedom allows us to use the perturbative machinery
- We know (we suspect) that QCD can describe physical hadrons and explain confinement

This is not sufficient for us to give predictions for physical observables. What we can compute (quark and gluon reactions) is non-observable, and what is observable (hadrons) we cannot compute

We need three additional concepts to proceed:

- Hadron-parton duality
- Infrared safety
- Factorization theorems