New Constraints on the Singularities of Feynman Integrals

Andrew McLeod
(CERN)

QCD Journal Club
October 29

arXiv:2109.09744 [hep-th] and arXiv:21xx.xxxxx [hep-th]
with Hofie Hannesdottir, Matthew Schwartz, and Cristian Vergu

Outline

- Motivation
- All-mass Feynman Integrals
- Standard Landau Analyis
- Predicting the Type and Position of Singularities
- Predicting the Allowed Sequences of Singularities

Motivation

Since the sixties, the singularity structure of scattering amplitudes has been known to be strongly constrained by basic physical principles such as causality and locality

$\Rightarrow$ What are the full implications of these physical principles?

The singularities that appear in Feynman integrals also deterimine what types of special functions appear in their evaluation

$\Rightarrow$ How much can be deduced about the functions that appear in a given Feynman integral by looking at its behavior near singularities?

Motivation

Since the sixties, the singularity structure of scattering amplitudes has been known to be strongly constrained by basic physical principles such as causality and locality

$\Rightarrow$ What are the full implications of these physical principles?

The singularities that appear in Feynman integrals also deterimine what types of special functions appear in their evaluation

$\Rightarrow$ How much can be deduced about the functions that appear in a given Feynman integral by looking at its behavior near singularities?

$\Rightarrow$ What restrictions do basic physical principles imply in the simplest case, when amplitudes are expressible as multiple polylogarithms?

Multiple Polylogarithms

- Up to integration constants, these functions are specified by their derivatives, which satisfy

$$dF = \sum_i F^{s_i} d \log s_i$$

for some set of algebraic quantities $\{s_i\}$, where each F^{s_i} is itself either a multiple polylogarithm or algebraic

- Examples of such functions include $\log(z)$, the classical polylogarithms

$$\text{Li}_m(z) = \int_0^z \frac{\text{Li}_{m-1}(t)}{t} dt, \quad \text{Li}_1(z) = -\log(1-z)\,,$$

and transcendental constants such as ζ_m

Multiple Polylogarithms

- Up to integration constants, these functions are specified by their derivatives, which satisfy

$$dF = \sum_i F^{s_i} d\log s_i$$

for some set of algebraic quantities $\{s_i\}$, where each F^{s_i} is itself either a multiple polylogarithm or algebraic

- Most of the analytic structure of these functions are encoded in their 'symbol', which is recursively defined by

$$\mathcal{S}(F) = \sum_i \mathcal{S}(F^{s_i}) \otimes s_i$$

whereupon the s_i are referred to as symbol letters; for example,

$$\mathcal{S}(\mathrm{Li}_m(z)) = -(1-z) \otimes z \otimes \ldots \otimes z$$

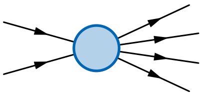
Analytic Constraints ⇒ Bootstrap Calculations

- The symbol encodes all nonzero sequences of logarithmic discontinuities
- It is useful for simplifying expressions and finding identities
- It can also be used as an edifice for 'bootstrapping' amplitudes and Feynman integrals directly, if we know enough about their analytic structure:
  - Construct an ansatz for the symbol consistent with this analytic structure (what symbol letters can appear, appropriate 'transcendental weight', ...)
  - Impose appropriate symmetries and factorization properties
  - Constrain the ansatz in kinematic limits where the amplitude or Feynman integral can be computed directly

Bootstrap Approaches

In planar $\mathcal{N} = 4$, the symbol letters that appear in certain quantities are (conjecturally) known, allowing them to be bootstrapped to high loop order

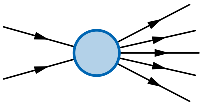
six-particle amplitude



7 loops

[Caron-Huot, Dixon, Dulat, von Hippel, AJM, Papathanasiou]

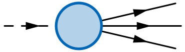
seven-particle amplitude



4 loops

[Dixon, Drummond, Harrington, AJM, Papathanasiou, Spradlin] [Drummond, Foster, Gürdoğan, Papathanasiou]

three-particle form factor



8 loops

[Dixon, Gürdoğan, AJM, Wilhelm] (to appear)

In QCD, this approach has also been applied to rapidity anomalous dimensions and the soft anomalous dimension [Li, Zhu] [Almelid, Duhr, Gardi, AJM, White]

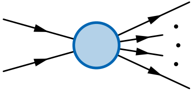
Symbol Letters and Landau Analysis

Beyond these special cases, much less is known about the symbol letters that can appear in polylogarithmic amplitudes

> How much can we learn about the symbol letters that appear in Feynman integrals and amplitudes directly from Landau analysis?

All-Mass Feynman Integrals

We focus on Feynman integrals with generic internal and external masses in D spacetime dimensions:



$$= \int \prod_j \frac{d^D \ell_j}{i\pi^{D/2}} \prod_k \frac{1}{q_k^2 - m_k^2}$$

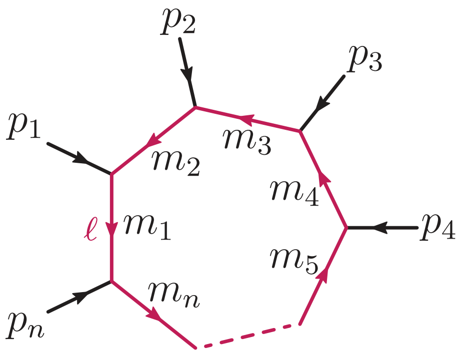
where ℓ_j is the momentum flowing through the j^{th} loop,

m_k is the mass assigned to the k^{th} internal edge,

q_k represents the momentum flowing through the k^{th} internal edge, which depends linearly on the loop momenta $\{\ell_j\}$ and the external momenta $\{p_j\}$

All-mass Feynman Integrals

In particular, our primary example with be the one-loop scalar n-point integrals in n spacetime dimensions:



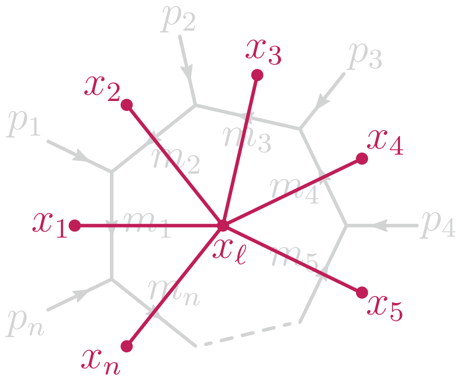
$$x_{ij}^2 = \left(\sum_{k=i}^{j-1} p_k\right)^2$$

$$y_{ij} = \frac{x_{ij}^2 - m_i^2 - m_j^2}{2m_i m_j}$$

$$y = \begin{pmatrix} -1 & y_{12} & y_{13} & \cdots \\ y_{12} & -1 & y_{23} & \cdots \\ y_{13} & y_{23} & -1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

All-mass Feynman Integrals

In particular, our primary example with be the one-loop scalar n-point integrals in n spacetime dimensions:



$$x_{ij}^2 = \left(\sum_{k=i}^{j-1} p_k \right)^2$$

$$y_{ij} = \frac{x_{ij}^2 - m_i^2 - m_j^2}{2 m_i m_j}$$

$$y = \begin{pmatrix} -1 & y_{12} & y_{13} & \cdots \\ y_{12} & -1 & y_{23} & \cdots \\ y_{13} & y_{23} & -1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

All-mass Feynman Integrals

The symbol of these integrals is known for all n:

[Goncharov, Spradlin, Vergu, Volovich] [Schläfli] [Aomoto]

$$\mathcal{S}\big(\mathcal{I}_{n(\text{even})}\big) \propto \frac{1}{\sqrt{-\det y}} \sum \omega^{\emptyset}_{\{i_1,i_2\}} \otimes \omega^{\{i_1,i_2\}}_{\{i_1,i_2,i_3,i_4\}} \otimes \cdots \otimes \omega^{\{i_1,\ldots,i_{n-2}\}}_{\{1,\ldots,n\}}$$

$$\mathcal{S}\big(\mathcal{I}_{n(\text{odd})}\big) \propto \frac{1}{\sqrt{-\det y}} \sum \omega^{\{i_1\}}_{\{i_1,i_2,i_3\}} \otimes \omega^{\{i_1,i_2,i_3\}}_{\{i_1,i_2,i_3,i_4,i_5\}} \otimes \cdots \otimes \omega^{\{i_1,\ldots,i_{n-2}\}}_{\{1,\ldots,n\}}$$

where the sum is over all ways to assign the indices $\{1,\ldots,n\}$ to $\{i_1,\ldots,i_n\}$,

$$\omega^I_J = \frac{-D^{I\cup\{i\}}_{I\cup\{j\}} + i\sqrt{D_I D_J}}{-D^{I\cup\{i\}}_{I\cup\{j\}} - i\sqrt{D_I D_J}}\,, \qquad J = I \cup \{i,j\} \subset \{1,\ldots,n\}$$

and D^I_J is the determinant of the matrix formed by rows I and columns J of y, and $D_I = D^I_I$

All-mass Feynman Integrals

The symbol of these integrals is known for all n:

[Goncharov, Spradlin, Vergu, Volovich] [Schläfli] [Aomoto]

$$\mathcal{S}\big(\mathcal{I}_{n(\text{even})}\big) \propto \frac{1}{\sqrt{-\det y}} \sum \omega^{\emptyset}_{\{i_1,i_2\}} \otimes \omega^{\{i_1,i_2\}}_{\{i_1,i_2,i_3,i_4\}} \otimes \cdots \otimes \omega^{\{i_1,\ldots,i_{n-2}\}}_{\{1,\ldots,n\}}$$

$$\mathcal{S}\big(\mathcal{I}_{n(\text{odd})}\big) \propto \frac{1}{\sqrt{-\det y}} \sum \omega^{\{i_1\}}_{\{i_1,i_2,i_3\}} \otimes \omega^{\{i_1,i_2,i_3\}}_{\{i_1,i_2,i_3,i_4,i_5\}} \otimes \cdots \otimes \omega^{\{i_1,\ldots,i_{n-2}\}}_{\{1,\ldots,n\}}$$

For example:

$$\omega^{\{1\}}_{\{1,2,3\}} = \frac{-D^{\{1,2\}}_{\{1,3\}} + i\sqrt{D_{\{1\}} D_{\{1,2,3\}}}}{-D^{\{1,2\}}_{\{1,3\}} - i\sqrt{D_{\{1\}} D_{\{1,2,3\}}}}$$

$$\begin{aligned} D_{\{1\}} &= -1, \\ D_{\{1,2,3\}} &= -1 + y_{12}^2 + y_{13}^2 - 2 y_{12}\, y_{13}\, y_{23} + y_{23}^2 \\ D^{\{1,2\}}_{\{1,3\}} &= y_{12}\, y_{13} - y_{23} \end{aligned}$$

All-mass Feynman Integrals

The codimension-one branch points of these symbol letters can be discerned with the help of the identity

$$D_I D_J = D_{I\cup\{i\}} D_{I\cup\{j\}} - \left(D^{I\cup\{j\}}_{I\cup\{i\}}\right)^2, \qquad J = I \cup \{i,j\}$$

$\Rightarrow$ when either $D_{I\cup\{i\}}$ or $D_{I\cup\{j\}}$ vanishes, the letter

$$\omega^I_J = \frac{-D^{I\cup\{i\}}_{I\cup\{j\}} + i\sqrt{D_I D_J}}{-D^{I\cup\{i\}}_{I\cup\{j\}} - i\sqrt{D_I D_J}}$$

either vanishes or approaches infinity

$\Rightarrow$ when either D_I or D_J vanishes, we have an algebraic branch point

All-mass Feynman Integrals

Thus, all branch points in these integrals occur at $D_J = 0$ for some $J \subset \{1, \ldots, n\}$

$$\text{For even } n\text{, the branch point at } D_J = 0 \text{ is} \begin{cases} \text{logarithmic for } |J| \text{ odd} \\ \text{algebraic for } |J| \text{ even} \end{cases}$$

$$\text{For odd } n\text{, the branch point at } D_J = 0 \text{ is} \begin{cases} \text{logarithmic for } |J| \text{ even} \\ \text{algebraic for } |J| \text{ odd} \end{cases}$$

- Moreover, the sequences of logarithmic branch points in these integrals are always specified by two new dual coordinates (and all previous dual coordinates) at each step

Objective: can we understand the sequences of branch points that appear in these integrals using Landau analysis?

Landau Analysis Review

Landau diagrams can be used to study the varieties on which Feynman integrals become singular [Landau] [Bjorken] [Nakanishi]

- Given a graph G, the associated Landau variety is defined by:

(i) the on-shell condition $q_k^2 = m_k^2$ for each edge in G

(ii) momentum conservation at each vertex in G

(iii) a Landau loop equation

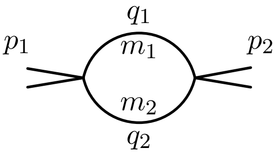
$$\sum_{q_k \in L} \alpha_k q_k = 0$$

for each loop L in G

This variety identifies a kinematic locus on which any Feynman integral that can be contracted to G via some sequence of (internal) edge contractions can become singular

Landau Analysis Review

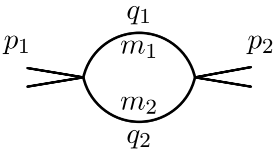
A simple (but relevant) example:



$$\alpha_1\, q_1 + \alpha_2\, q_2 = 0$$

Landau Analysis Review

A simple (but relevant) example:

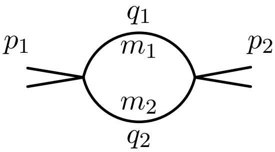


$$\alpha_1\, q_1 + \alpha_2\, q_2 = 0 \qquad \Rightarrow \begin{pmatrix} m_1^2 & q_1 \cdot q_2 \\ q_1 \cdot q_2 & m_2^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$$

- For a nontrivial solution to this equation to exist, the determinant of this matrix must vanish

Landau Analysis Review

A simple (but relevant) example:



$$\alpha_1\, q_1 + \alpha_2\, q_2 = 0 \qquad \Rightarrow \begin{pmatrix} m_1^2 & q_1\cdot q_2 \\ q_1\cdot q_2 & m_2^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$$

- For a nontrivial solution to this equation to exist, the determinant of this matrix must vanish
- But using momentum conservation $p_1 = q_1 - q_2$, we have

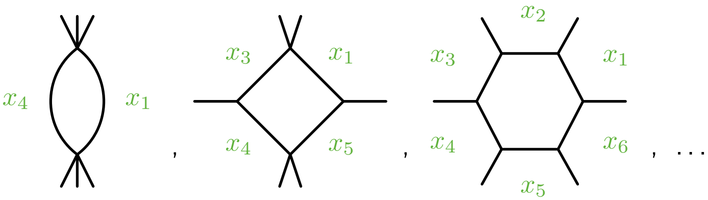
$$p_1^2 = m_1^2 + m_2^2 - 2q_1\cdot q_2$$

so this happens if and only if $\det y = 0$

$$y = \begin{pmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix} \begin{pmatrix} -m_1^2 & \frac{p_1^2-m_1^2-m_2^2}{2} \\ \frac{p_1^2-m_1^2-m_2^2}{2} & -m_2^2 \end{pmatrix} \begin{pmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}$$

Landau Analysis Review

- In fact, the one-loop Landau diagram with r internal propagators more generally encodes the locus $D_J = 0$ for $|J| = r$ [Eden, Landshoff, Olive, Polkinghorne]
- The sequences of logarithmic branch points we hope to be able to 'predict' therefore correspond to sequences of Landau diagrams involving two more dual points at each step, for instance



and not sequences in which changing sets of indices appear, or different successive polygons

(this iterative structure can also be seen using the diagrammatic coaction [Abreu, Britto, Duhr, Gardi])

Locations of Singularities in the Symbol

First, we ask whether we can determine *where* in the symbol different singularities can appear

- Landau also showed how to estimate the leading non-analytic behavior of a Feynman integral near each of these singular loci

$$F(\varphi \to 0) \sim \begin{cases} \varphi^{\gamma} \log \varphi & \text{if } \gamma \geq 0 \text{ is an integer} \\ \varphi^{\gamma} & \text{otherwise} \end{cases}$$

$$\gamma = \frac{1}{2}(D\ell - n - 1)$$

Locations of Singularities in the Symbol

First, we ask whether we can determine *where* in the symbol different singularities can appear

- Landau also showed how to estimate the leading non-analytic behavior of a Feynman integral near each of these singular loci

$$F(\varphi \to 0) \sim \begin{cases} \varphi^\gamma \log \varphi & \text{if } \gamma \geq 0 \text{ is an integer} \\ \varphi^\gamma & \text{otherwise} \end{cases}$$

$$\gamma = \frac{1}{2}(D\ell - n - 1)$$

- Note that only single powers of $\log \varphi$ appear, and the only algebraic branch points that appear are square roots

Locations of Singularities in the Symbol

- Since the leading non-analytic behavior can involve at most a single singular logarithm, we focus on symbol terms of the form

$$b_1 \otimes \cdots \otimes b_p \otimes \varphi \otimes c_1 \otimes \cdots \otimes c_q$$

Locations of Singularities in the Symbol

- Since the leading non-analytic behavior can involve at most a single singular logarithm, we focus on symbol terms of the form

$$b_1 \otimes \cdots \otimes b_p \otimes \varphi \otimes c_1 \otimes \cdots \otimes c_q$$

- This encodes a contribution to the integral that can be put into the form

$$\int_0^1 U(t)\, d\log\varphi(t)\, V(t)\,,$$

$$U(t) = \int_0^t d\log b_1(t_1) \int_{t_1}^t d\log b_2(t_2) \cdots \int_{t_{p-1}}^t d\log b(t_p)$$

$$V(t) = \int_t^1 d\log c_1(t_1) \int_{t_1}^1 d\log c_2(t_2) \cdots \int_{t_{q-1}}^1 d\log c(t_q)$$

Locations of Singularities in the Symbol

- Since the leading non-analytic behavior can involve at most a single singular logarithm, we focus on symbol terms of the form

$$b_1 \otimes \cdots \otimes b_p \otimes \varphi \otimes c_1 \otimes \cdots \otimes c_q$$

- This encodes a contribution to the integral that can be put into the form

$$\int_0^1 U(t)\, d\log\varphi(t)\, V(t)\,, \qquad \begin{aligned} U(t) &= \int_0^t d\log b_1(t_1) \int_{t_1}^t d\log b_2(t_2) \cdots \int_{t_{p-1}}^t d\log b(t_p) \\ V(t) &= \int_t^1 d\log c_1(t_1) \int_{t_1}^1 d\log c_2(t_2) \cdots \int_{t_{q-1}}^1 d\log c(t_q) \end{aligned}$$

$$\varphi(t) = (1-t) + \varphi_0 t \qquad \Rightarrow \qquad d\log\varphi(t) = \frac{1+\varphi_0}{(1-t)+\varphi_0 t} dt$$

Locations of Singularities in the Symbol

To determine the leading non-analytic contribution near $\varphi_0 \to 0$, we expand $U(t)$ and $V(t)$ to leading order as $t \to 1$ and carry out the integral over t to get a contribution

$$U(1)\frac{1}{q!}\frac{d^q V}{dt^q}(1)\,\varphi_0^q \log \varphi_0$$

Locations of Singularities in the Symbol

To determine the leading non-analytic contribution near $\varphi_0 \to 0$, we expand $U(t)$ and $V(t)$ to leading order as $t \to 1$ and carry out the integral over t to get a contribution

$$U(1)\frac{1}{q!}\frac{d^q V}{dt^q}(1)\,\varphi_0^q \log \varphi_0$$

- This reproduces the form predicted by Landau, when $\gamma = \frac{1}{2}(D\ell - n - 1) = q$

In other words, we can predict the position in the symbol at which a logarithmic singularity associated with a Landau diagram can appear [Hannesdottir, AJM, Schwartz, Vergu]

Locations of Singularities in the Symbol

For example, consider the hexagon integral in six dimensions, where we have

$$\gamma = \frac{1}{2}(D\ell - n - 1) = \frac{1}{2}(5 - n)\,,$$

where n is the number of internal lines in the Landau diagram

- For $n = 5$, we have $\gamma = 0$, which predicts the appearance of pentagon Landau diagrams zero entries from the end of the symbol
- For $n = 3$, we have $\gamma = 1$, which predicts the appearance of triangle Landau diagrams one entry from the end of the symbol

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

This correctly predicts the graph corresponding to each Landau diagram. However, it doesn't tell us about which branch points appear in sequence (i.e. which sequences of discontinuities will be nonzero)

Landau Varieties as Graph Contractions

To study which *sequences* of branch points can appear in the symbol, we generalize our notion of Landau varieties by associating them with graph contractions [Pham]

- Given a graph contraction $\kappa : G_1 \twoheadrightarrow G_2$, we construct a variety using the equations:

  (i) the on-shell condition $q_k^2 = m_k^2$ for each edge in G_1

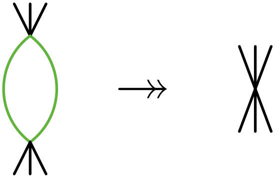
  (ii) momentum conservation at each vertex in G_1

  (iii) a Landau loop equation for each loop L in the kernel of κ

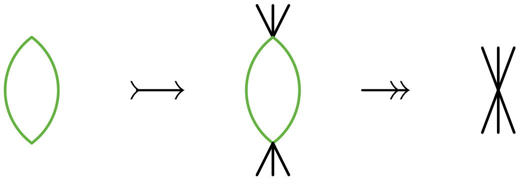
  where the kernel of κ is the graph formed by all edges that are contracted

This variety identifies a kinematic locus where the Landau diagram G_2 can become singular

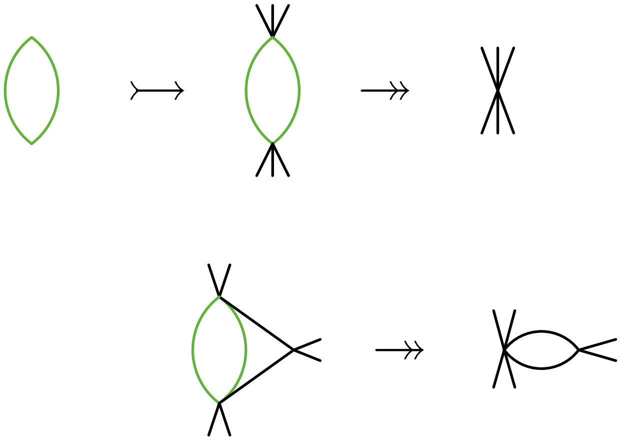
Landau Varieties as Graph Contractions



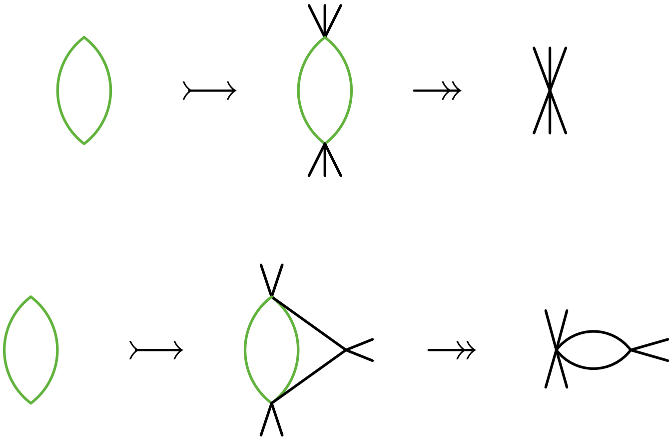
Landau Varieties as Graph Contractions



Landau Varieties as Graph Contractions



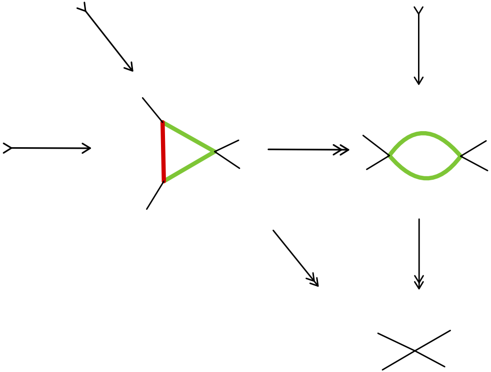
Landau Varieties as Graph Contractions



Iterated Discontinuities and Contractions

To study the sequences of discontinuities that can appear in a given Feynman integral, we study sequences of these types of graph contractions

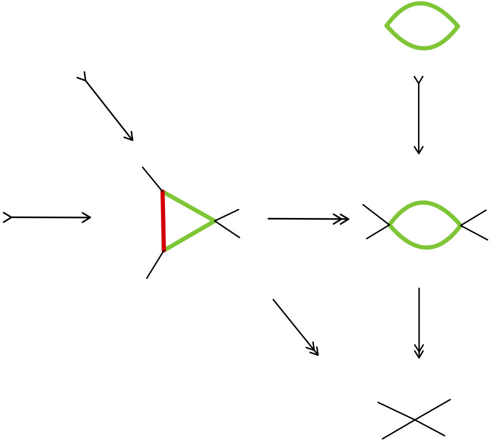
- For instance:



Iterated Discontinuities and Contractions

To study the sequences of discontinuities that can appear in a given Feynman integral, we study sequences of these types of graph contractions

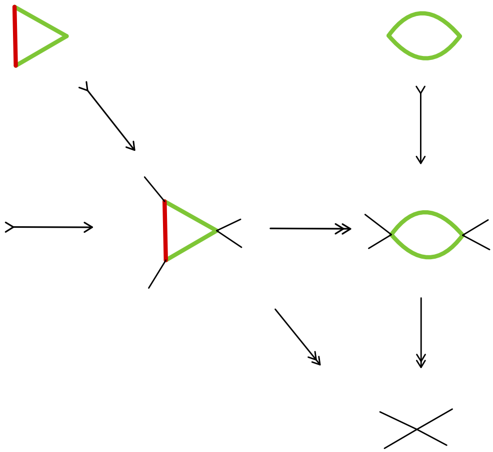
- For instance:



Iterated Discontinuities and Contractions

To study the sequences of discontinuities that can appear in a given Feynman integral, we study sequences of these types of graph contractions

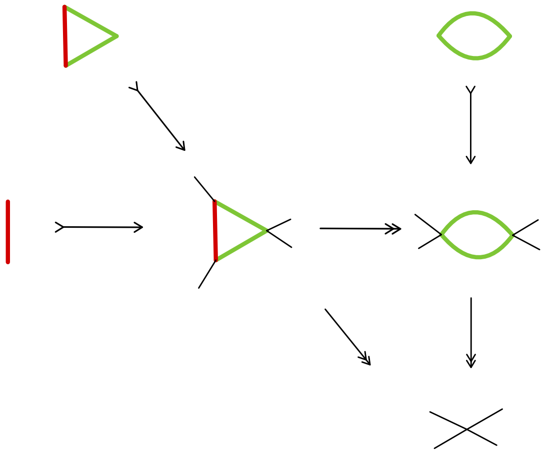
- For instance:



Iterated Discontinuities and Contractions

To study the sequences of discontinuities that can appear in a given Feynman integral, we study sequences of these types of graph contractions

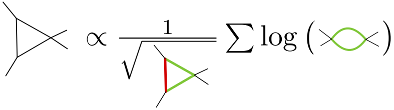
- For instance:

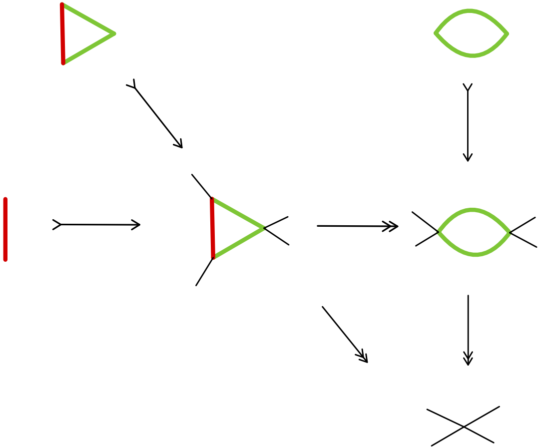


Iterated Discontinuities and Contractions

To study the sequences of discontinuities that can appear in a given Feynman integral, we study sequences of these types of graph contractions

- For instance:

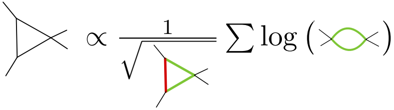




Iterated Discontinuities and Contractions

To study the sequences of discontinuities that can appear in a given Feynman integral, we study sequences of these types of graph contractions

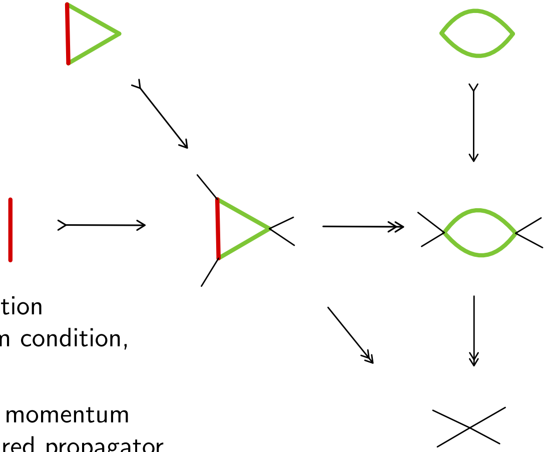
- For instance:



$\Downarrow$

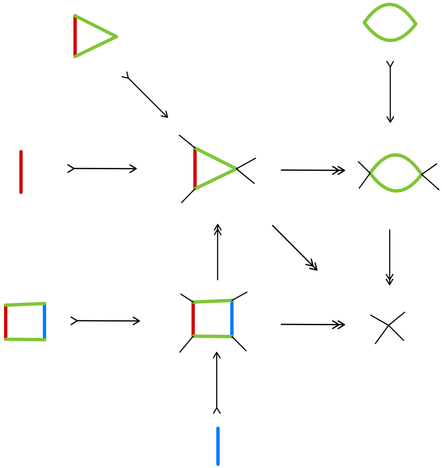
If we plug the bubble Landau diagram solution $y_{12} \to \pm 1$ into the triangle Landau diagram condition, this condition becomes $(y_{13} \mp y_{23})^2 = 0$

- This is equivalent to the on-shell and momentum conservation conditions involving the red propagator



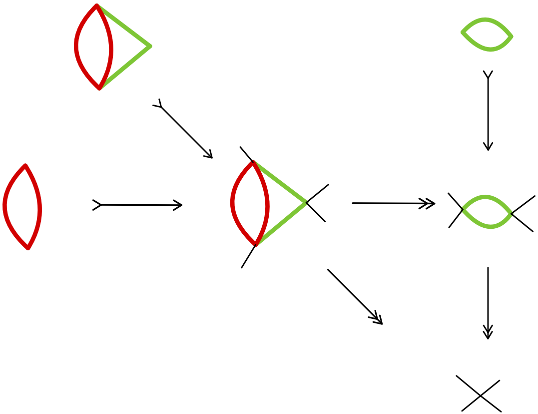
Iterated Discontinuities and Contractions

- We can also iterate this type of reasoning to contractions of graphs involving more propagators:



Iterated Discontinuities and Contractions

- At higher loops we can also get more interesting sequences of contractions:



Why Sequenceable Contractions?

The sequences of discontinuities in the all-mass n-gons are clearly allowed, since they correspond to sequences of graph contractions. But are other sequences actually disallowed?

Why Sequenceable Contractions?

The sequences of discontinuities in the all-mass n-gons are clearly allowed, since they correspond to sequences of graph contractions. But are other sequences actually disallowed?

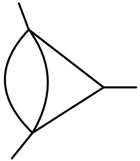
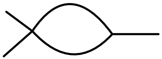
Yes: A theorem by Pham states that the Landau diagram associated with a graph contraction $\kappa_1 : G_1 \twoheadrightarrow G_2$ is analytic above the threshold and outside of the Landau varieties corresponding to contractions $\kappa_0 : G_0 \twoheadrightarrow G_1$ that can be put in sequence with κ_1

Why Sequenceable Contractions?

The sequences of discontinuities in the all-mass n-gons are clearly allowed, since they correspond to sequences of graph contractions. But are other sequences actually disallowed?

Yes: A theorem by Pham states that the Landau diagram associated with a graph contraction $\kappa_1 : G_1 \twoheadrightarrow G_2$ is analytic above the threshold and outside of the Landau varieties corresponding to contractions $\kappa_0 : G_0 \twoheadrightarrow G_1$ that can be put in sequence with κ_1

- Note that this doesn't quite imply the 'naïve hierarchy principle', since Landau diagrams can also have singularities on loci identified by their subdiagrams [Landshoff, Olive, Polkinghorne]

 can become singular on the locus represented by 

Constraints on Sequential Discontinuities

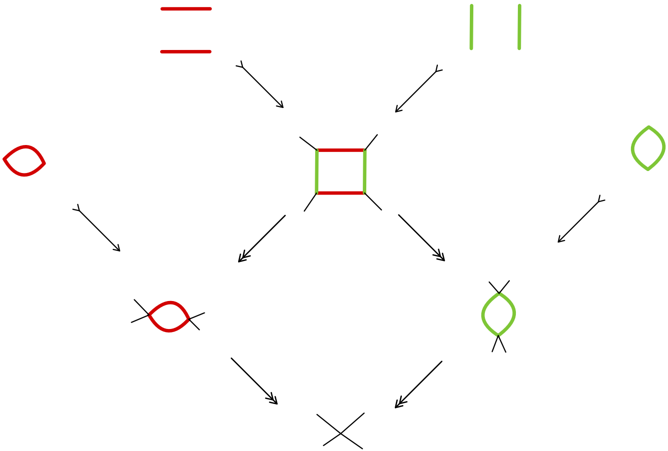
What types of iterated discontinuities does this forbid?

- As observed in the previous examples, the singular loci associated with a given Landau diagram can be ‘shifted’ when we consider it on the support of another Landau diagram
- Compare this to the Steinmann relations, which forbid double discontinuities in partially overlapping channels—and in particular forbids a second discontinuity in a given channel from forming at the kinematic locus where we would expect a first discontinuity in this channel
- For this to be allowed, the singular locus associated with one Landau diagram would have to remain unshifted by the condition imposed by the other Landau diagram

More generally, when do there exist cuts that ‘don’t know about each other’ in a Feynman diagram?

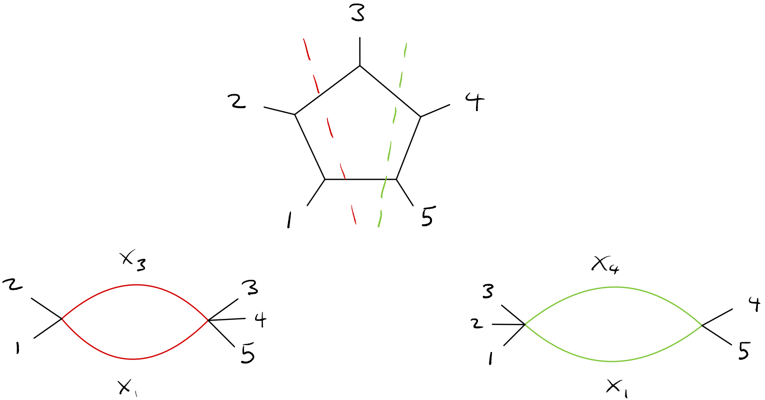
Constraints on Sequential Discontinuities

Stated differently: when can pairs of discontinuities commute?



Constraints on Sequential Discontinuities

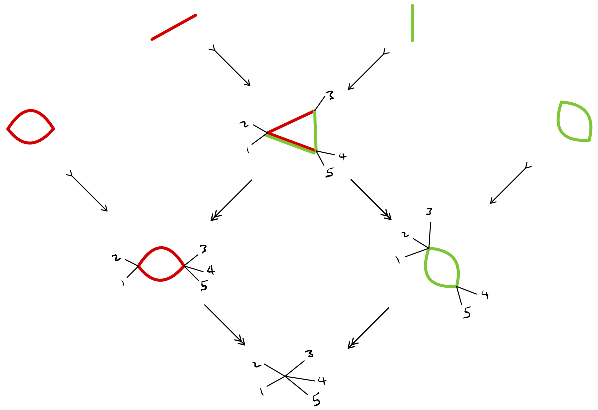
This also implies new constraints that go beyond Steinmann



- For example, at one loop we have $\text{Disc}_{s_{12}}\text{Disc}_{s_{123}}(\mathcal{I}_n) = \text{Disc}_{s_{123}}\text{Disc}_{s_{12}}(\mathcal{I}_n) = 0$ since the contractions corresponding to these cuts are not sequenceable

Constraints on Sequential Discontinuities

This also implies new constraints on the discontinuity structure of all-mass integrals

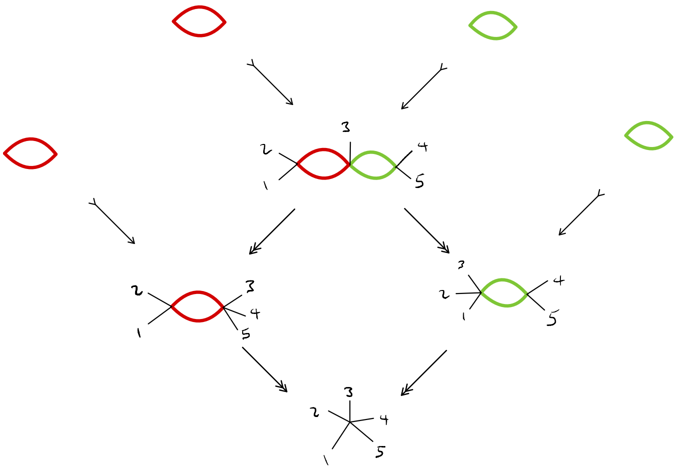


- For example, at one loop we have $\text{Disc}_{s_{12}}\text{Disc}_{s_{123}}(\mathcal{I}_n) = \text{Disc}_{s_{123}}\text{Disc}_{s_{12}}(\mathcal{I}_n) = 0$ since the contractions corresponding to these cuts are not sequenceable

Constraints on Sequential Discontinuities

However, each such constraint only applies to integrals with a certain topology

$\Rightarrow$ $\text{Disc}_{s_{12}}\text{Disc}_{s_{123}}$ can be nonzero in (some) two- and higher-loop diagrams:



Conclusions and Future Directions

A great deal can be learned about the analytic structure of Feynman integrals directly from Landau analysis

- The methods presented here also extend to square root branch points... will this provide enough constraints to reconstruct full symbol letters and bootstrap all-mass integrals?
- This technology doesn't say anything about analytic continuation paths—it may be the case that not all of these branch points are accessible
- How do elliptic polylogarithms (and more complicated special functions) fit into this picture?
- How can this be generalized to Feynman integrals involving massless particles?

Thanks!

Steinmann Constraints

- We can explore the intuition behind the Steinmann constraint by considering a kinematic configuration in which the two contractions do in fact commute

  $\Rightarrow$ To simultaneously solve both bubble Landau loop equations, we take the momentum flowing through the two bubbles to be perpendicular. Defining a pair of unit vectors

$$e_1^2 = e_2^2 = 1 \qquad e_1 \cdot e_2 = 0$$

we set

$$q_1 = m_1\, e_1, \quad q_3 = m_3\, e_2,$$
$$q_2 = m_2\, e_1, \quad q_4 = m_4\, e_2,$$
$$\Downarrow$$
$$y = \begin{pmatrix} -1 & 0 & y_{12} & 0 \\ 0 & -1 & 0 & y_{34} \\ y_{12} & 0 & -1 & 0 \\ 0 & y_{34} & 0 & -1 \end{pmatrix}, \quad \begin{matrix} y_{12} \to 1 \\ y_{34} \to 1 \end{matrix}$$