Perturbative QCD at high densities

Tyler Gorda
TU Darmstadt

Course on Neutron-Star Physics, IGFAE
17 November 2021

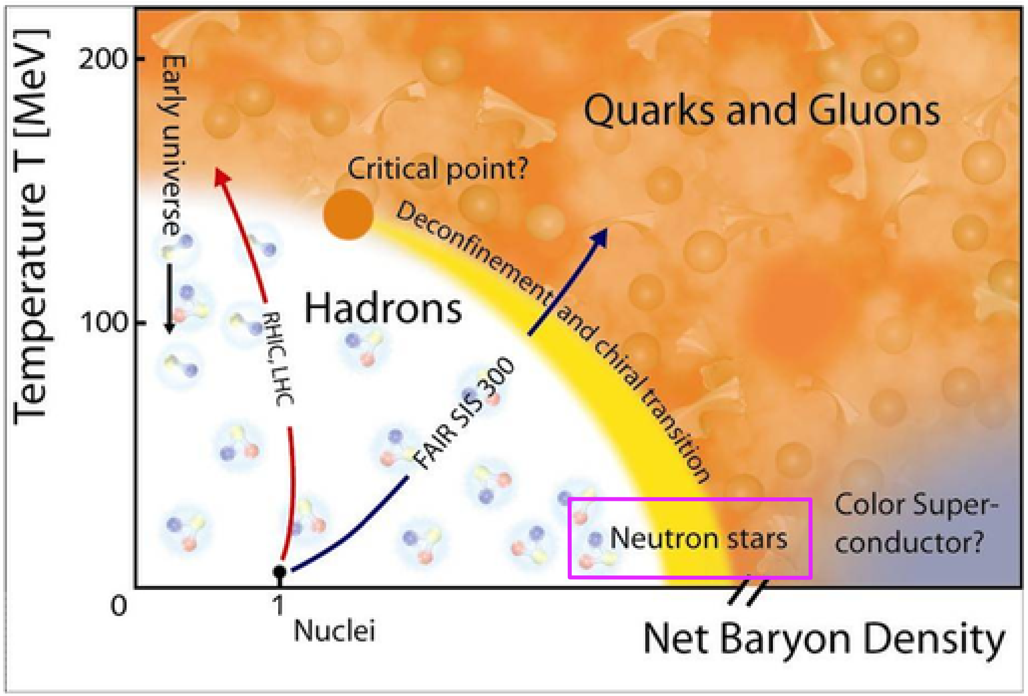
TECHNISCHE UNIVERSITÄT DARMSTADT

Summary / Overview:

The big picture

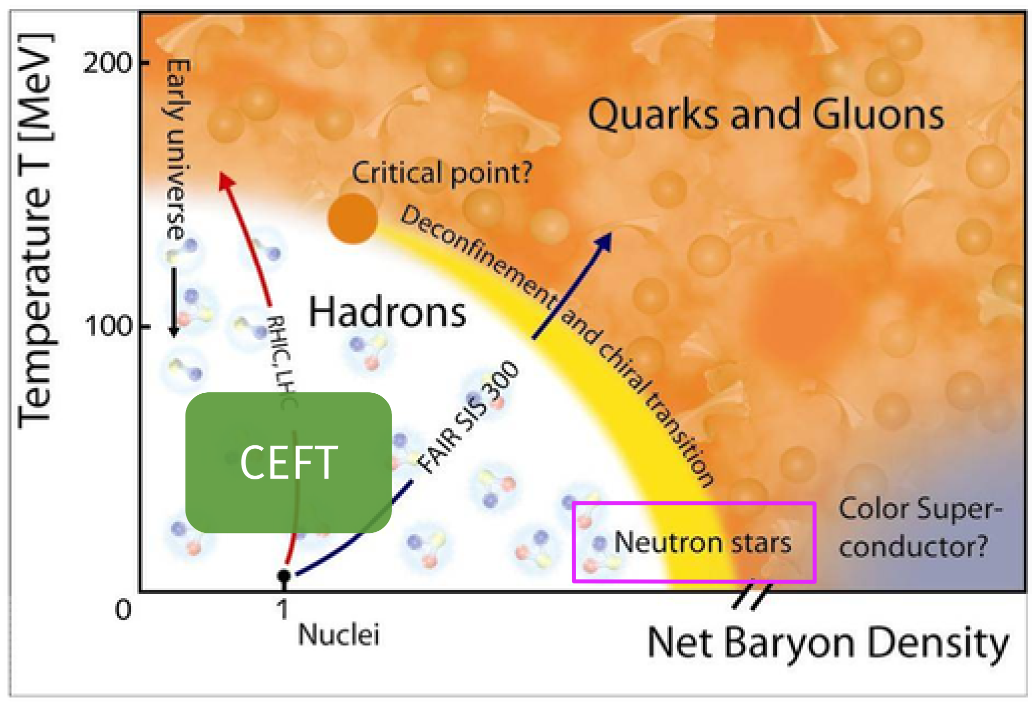
Motivation: Why pQCD at high density?

NSs probe densities beyond nuclear density, but below pQCD densities



Compressed Baryonic Matter (CBM) experiment

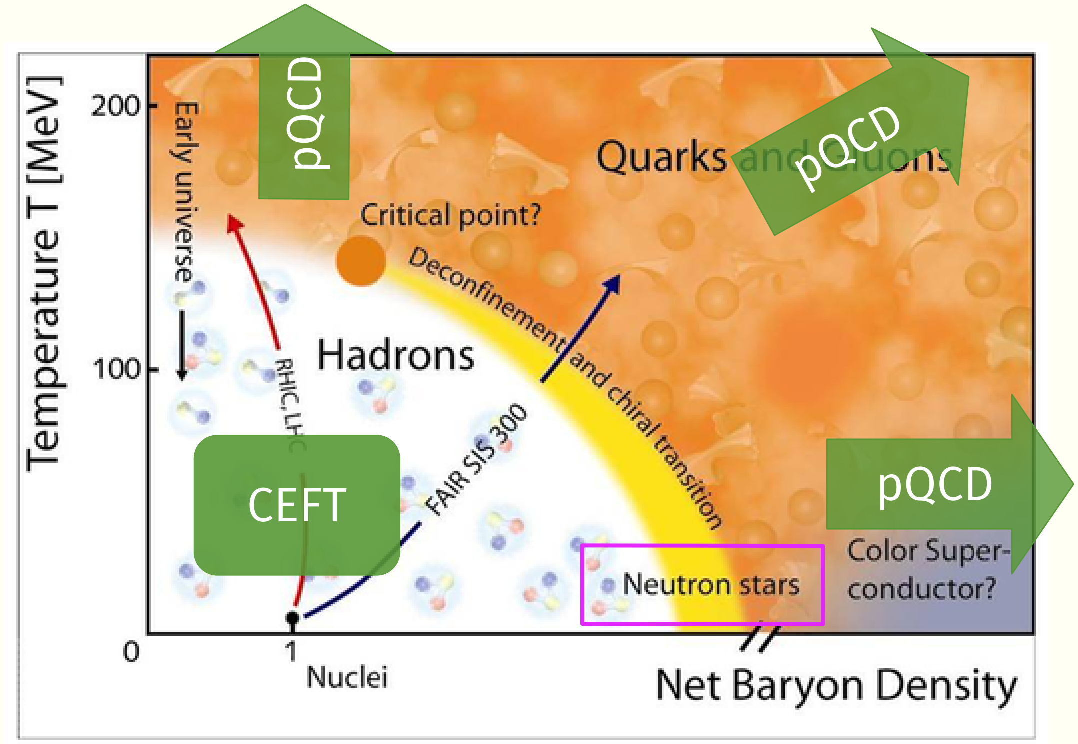
NSs probe densities beyond nuclear density, but below pQCD densities



Compressed Baryonic Matter (CBM) experiment

Motivation: Why pQCD at high density?

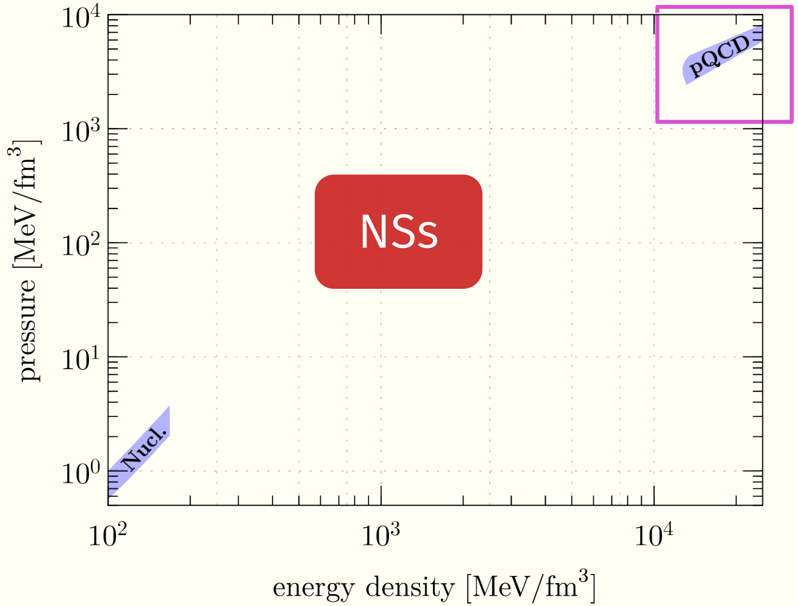
NSs probe densities beyond nuclear density, but below pQCD densities



Compressed Baryonic Matter (CBM) experiment

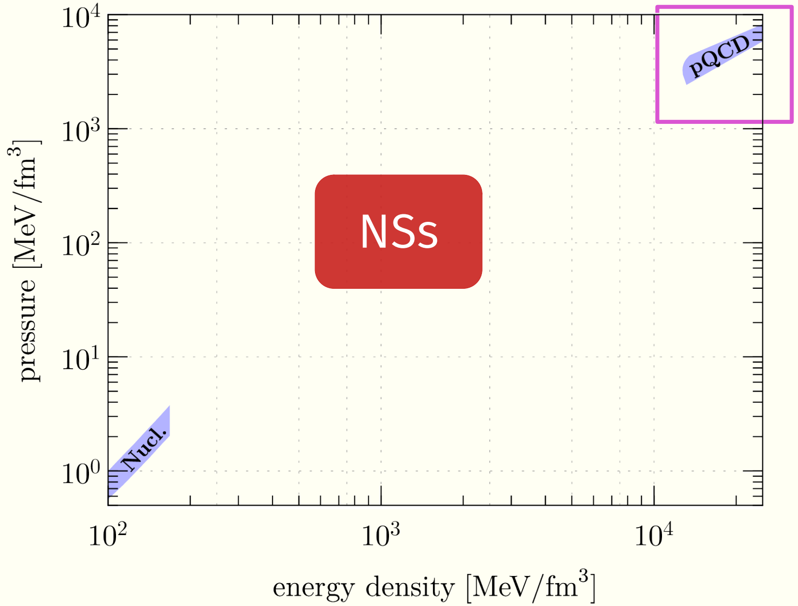
Motivation: Why pQCD at high density?

Want to use the fact that NS matter (EoS) goes to pQCD EoS at high densities as asymptotic limit



Want to use the fact that NS matter (EoS) goes to pQCD EoS at high densities as asymptotic limit

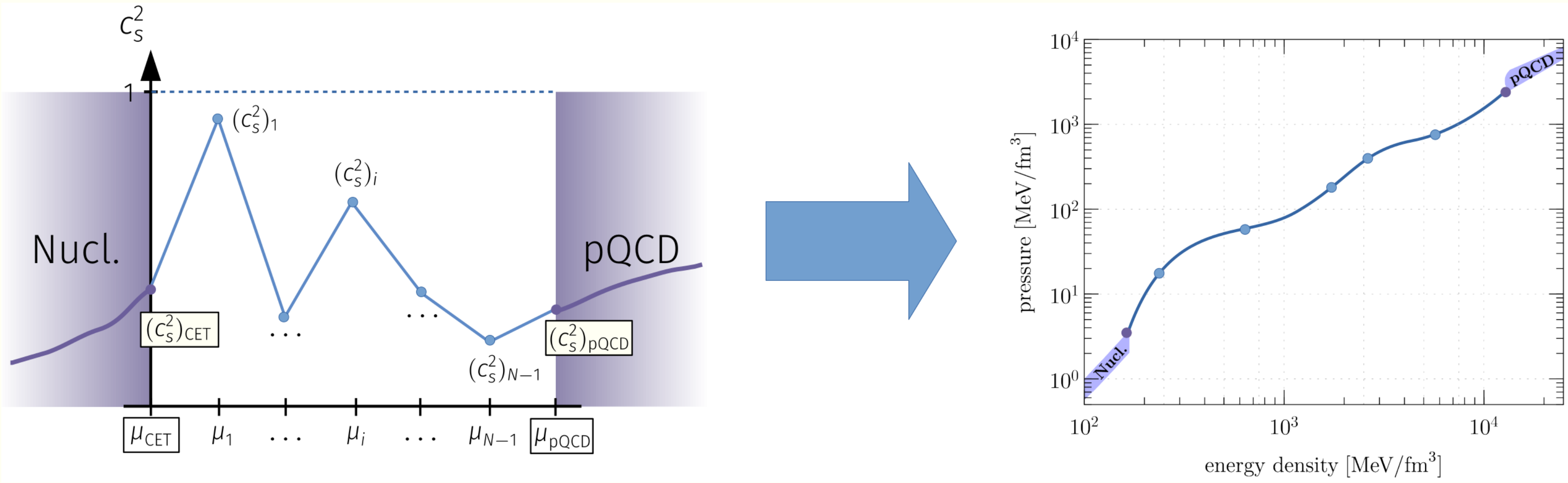
- Can fill in to lower densities by interpolation



Motivation: Why pQCD at high density?

Want to use the fact that NS matter (EoS) goes to pQCD EoS at high densities as asymptotic limit

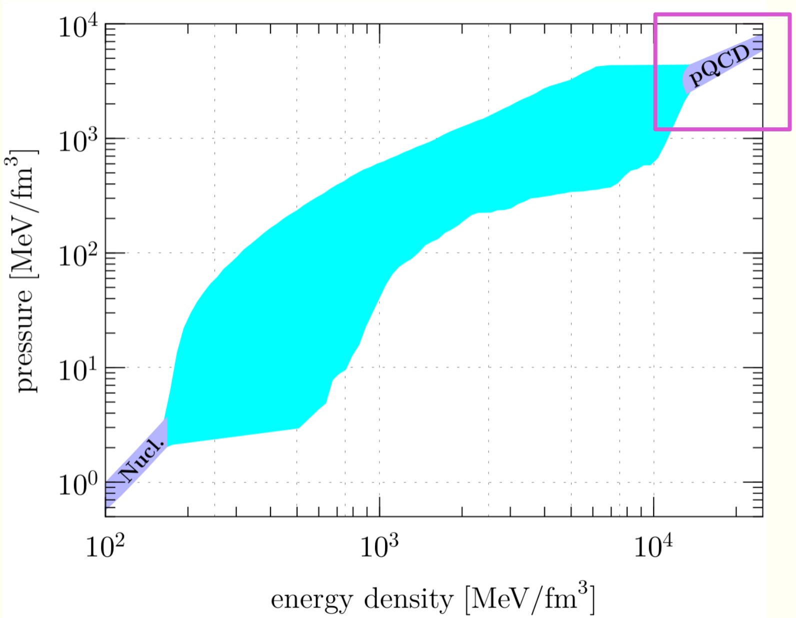
- Can fill in to lower densities by interpolation



Motivation: Why pQCD at high density?

Want to use the fact that NS matter (EoS) goes to pQCD EoS at high densities as asymptotic limit

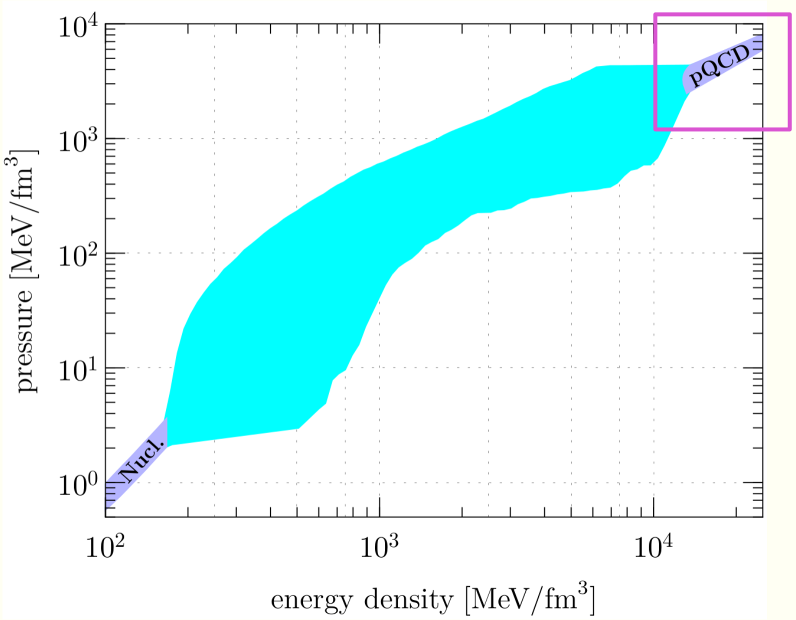
- Can fill in to lower densities by interpolation



Motivation: Why pQCD at high density?

Want to use the fact that NS matter (EoS) goes to pQCD EoS at high densities as asymptotic limit

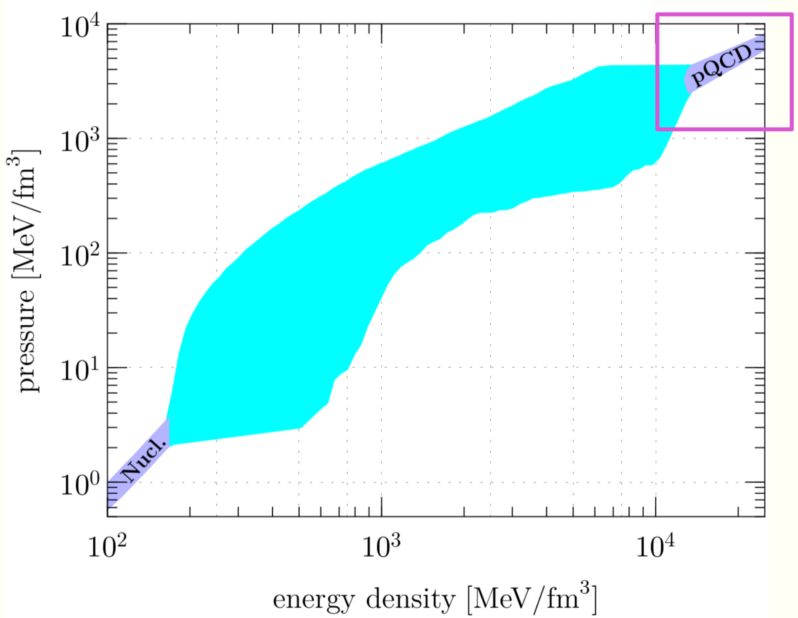
- Can fill in to lower densities by interpolation
- That NS EoS must go to pQCD in (p, ε, n) sets nontrivial constraints when matching
  Komoltsev, Kurkela, arXiv 2111.05350



Motivation: Why pQCD at high density?

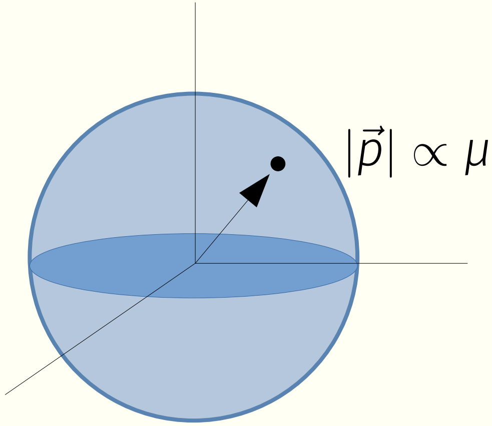
Want to use the fact that NS matter (EoS) goes to pQCD EoS at high densities as asymptotic limit

- Can fill in to lower densities by interpolation
- That NS EoS must go to pQCD in (p, ε, n) sets nontrivial constraints when matching
  Komoltsev, Kurkela, arXiv 2111.05350
- *This is why we are interested in pQCD, and improving pQCD at high density*



Motivation: What is pQCD (cold QM)?

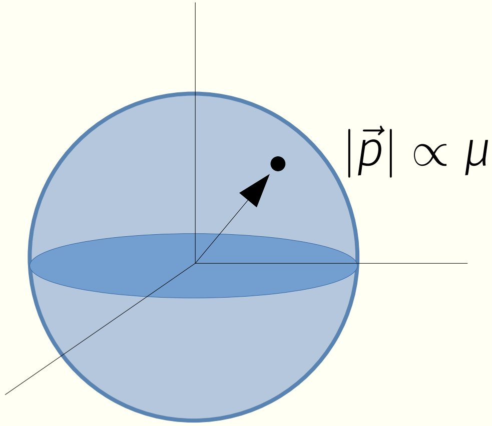
Basic property of cold QM EoS is that it's approximately described by a free electron gas



Motivation: What is pQCD (cold QM)?

Basic property of cold QM EoS is that it's approximately described by a free electron gas

- *QM has colored quarks/gluons as DOF*
- At high density, $\alpha_s \ll 1$, so quarks/gluons quasiparticles, with quark Fermi sea*



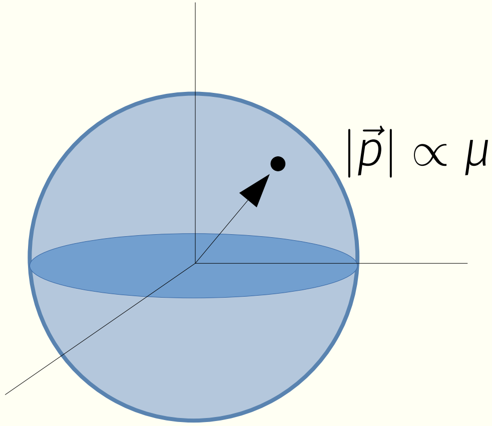
Motivation: What is pQCD (cold QM)?

Basic property of cold QM EoS is that it's approximately described by a free electron gas

- *QM has colored quarks/gluons as DOF*
- At high density, $\alpha_s \ll 1$, so quarks/gluons quasiparticles, with quark Fermi sea*

  $^*(p_{\mathrm{FD}} \propto \mu^4, \quad p_{\mathrm{pairing}} \propto \mu^2\Delta^2)$

  Alford+, Rev. Mod. Phys. 80, 1455 (2008)



Motivation: What is pQCD (cold QM)?

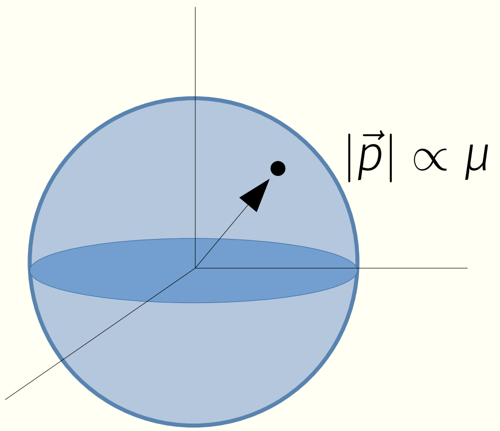
Basic property of cold QM EoS is that it's approximately described by a free electron gas

- *QM has colored quarks/gluons as DOF*
- At high density, $\alpha_s \ll 1$, so quarks/gluons quasiparticles, with quark Fermi sea*

  $^*(p_{\mathrm{FD}} \propto \mu^4, \quad p_{\mathrm{pairing}} \propto \mu^2 \Delta^2)$

  Alford+, Rev. Mod. Phys. 80, 1455 (2008)
- Approximately *conformal* (no mass scales)

  $\implies p = 3\varepsilon$



Motivation: What is pQCD (cold QM)?

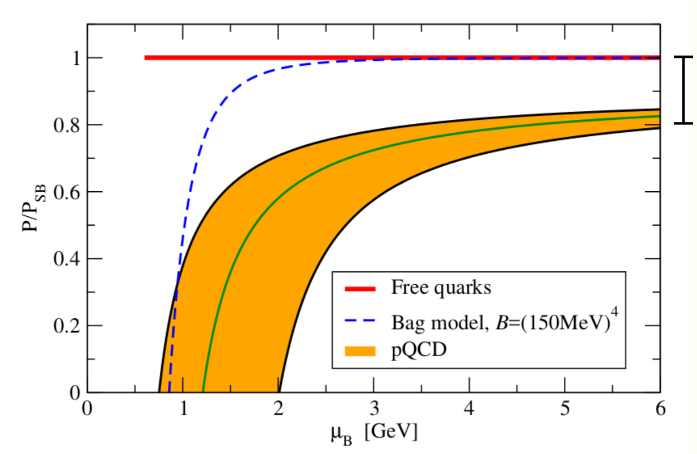
Basic property of cold QM EoS is that it's approximately described by a free electron gas

- *QM has colored quarks/gluons as DOF*
- At high density, $\alpha_s \ll 1$, so quarks/gluons quasiparticles, with quark Fermi sea*

  $^*(p_{\mathrm{FD}} \propto \mu^4, \quad p_{\mathrm{pairing}} \propto \mu^2\Delta^2)$

  Alford+, Rev. Mod. Phys. 80, 1455 (2008)
- Approximately *conformal* (no mass scales)

  $\implies p = 3\varepsilon$
- *However*, the interaction corrections do matter (20% effect!), and $\alpha_s(\Lambda)$ depends on a *renormalization mass scale* (runs with energy of interaction)



Fraga+ Astrophys. J. 781 (2014); see also Kurkela+ Phys. Rev. D 81 (2010).

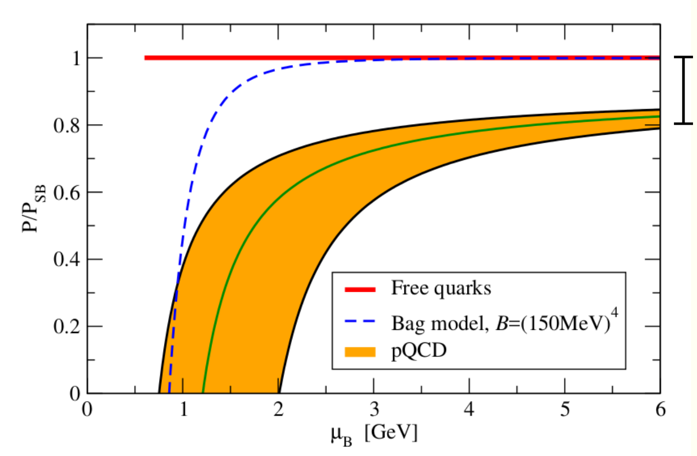
Basic property of cold QM EoS is that it's approximately described by a free electron gas

- *QM has colored quarks/gluons as DOF*
- At high density, $\alpha_s \ll 1$, so quarks/gluons quasiparticles, with quark Fermi sea*

  $^*(p_{\mathrm{FD}} \propto \mu^4, \quad p_{\mathrm{pairing}} \propto \mu^2\Delta^2)$

  Alford+, Rev. Mod. Phys. 80, 1455 (2008)
- Approximately *conformal* (no mass scales)

  $\implies p = 3\varepsilon$
- *However*, the interaction corrections do matter (20% effect!), and $\alpha_s(\Lambda)$ depends on a *renormalization mass scale* (runs with energy of interaction)



Fraga+ Astrophys. J. 781 (2014); see also Kurkela+ Phys. Rev. D 81 (2010).

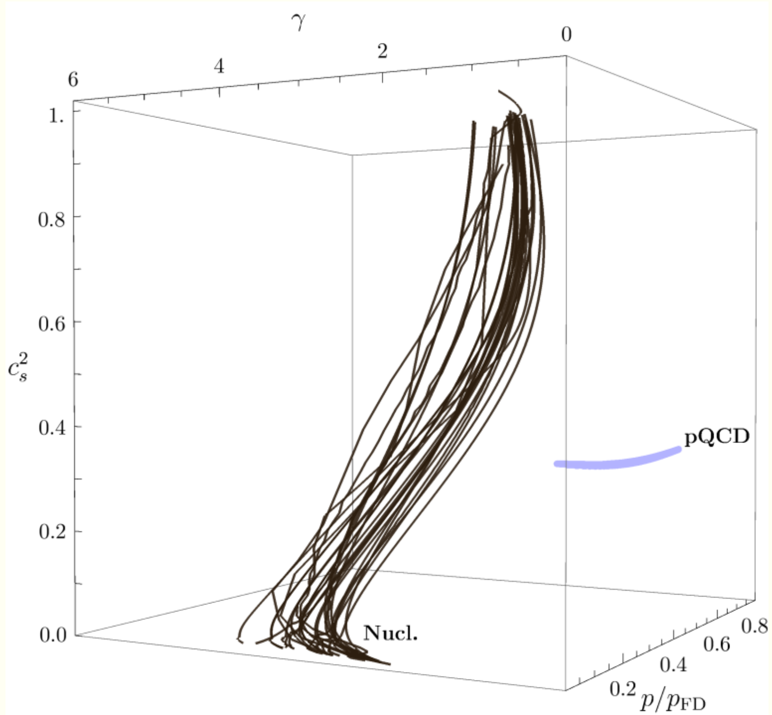
So we want to calculate these corrections accurately!

Motivation: what is Quark Matter? *Physical properties*

- *QM has different physical properties than hadronic matter:*

  Fortin+ Phys. Rev. C 94, (2016), Lattimer & Prakash, Astrophys. J. 550 (2001), Gandolfi+ Phys. Rev. C 85 (2012)

	Hadronic	Quark
c_s^2	increases	$\lesssim 1/3$
$\gamma \equiv \frac{\mathrm{d}\ln p}{\mathrm{d}\ln \varepsilon}$	≈ 2.5	≈ 1
p/p_{FD}	$\approx 0.1 - 0.3$	$\approx 0.5 - 0.8$



Annala, TG, Kurkela, Nättilä, Vuorinen Nat. Phys. 16 (2020)

Motivation: what is Quark Matter? *Physical properties*

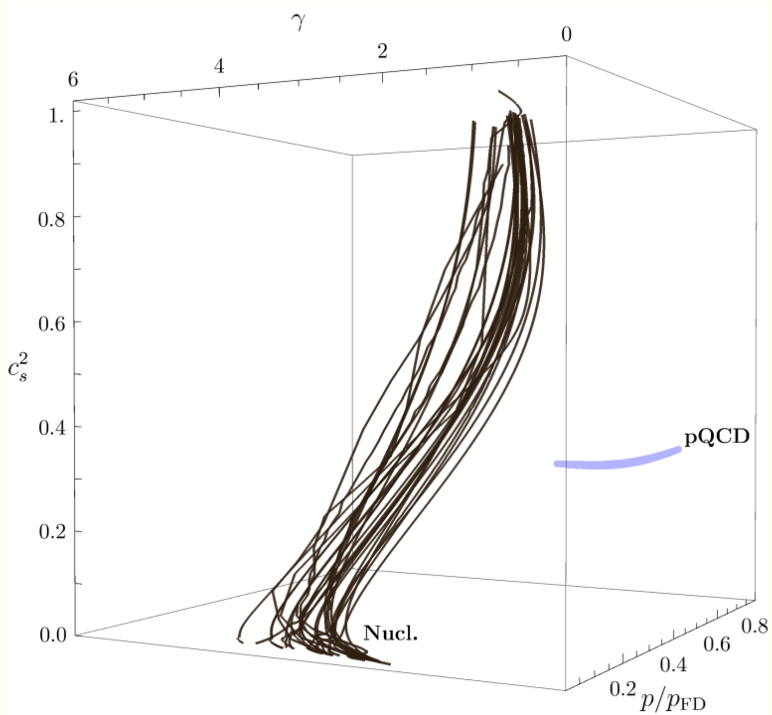
- *QM has different physical properties than hadronic matter:*

  Fortin+ Phys. Rev. C 94, (2016), Lattimer & Prakash, Astrophys. J. 550 (2001), Gandolfi+ Phys. Rev. C 85 (2012)

	Hadronic	Quark
c_s^2	increases	$\lesssim 1/3$
$\gamma \equiv \frac{\mathrm{d}\ln p}{\mathrm{d}\ln \varepsilon}$	≈ 2.5	≈ 1
p/p_{FD}	$\approx 0.1 - 0.3$	$\approx 0.5 - 0.8$

- *Strategy:*

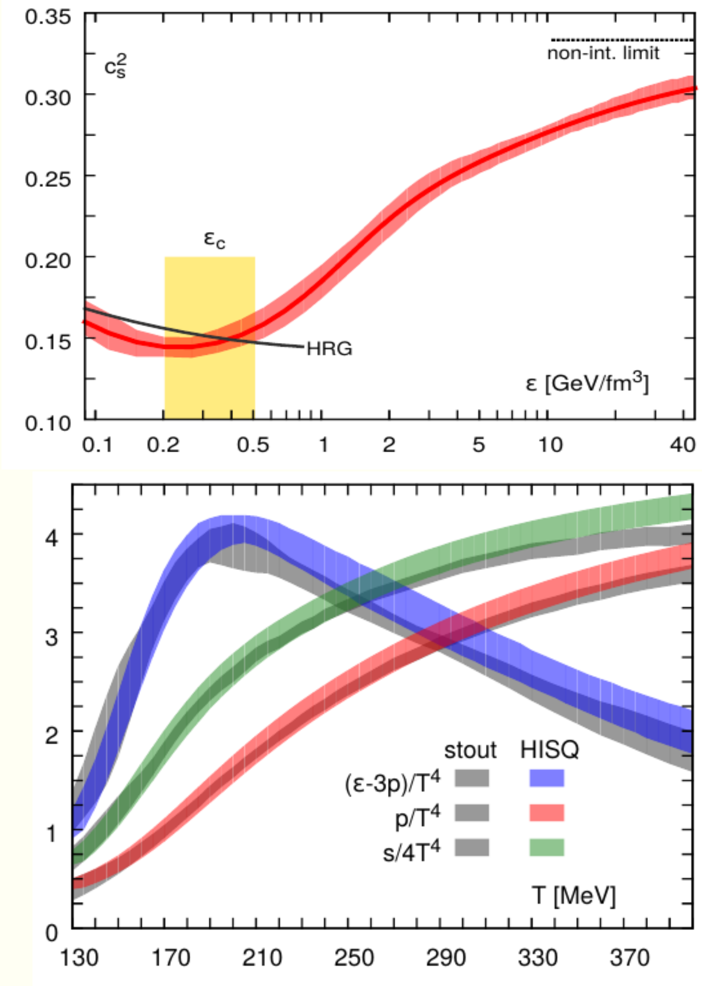
  Identify where EoS changes physical properties from hadronic → quark



Annala, TG, Kurkela, Nättilä, Vuorinen Nat. Phys. 16 (2020)

- Similar to looking for change in behavior of lattice results at high T.

- Identify change in phase from *change in physical properties* of matter

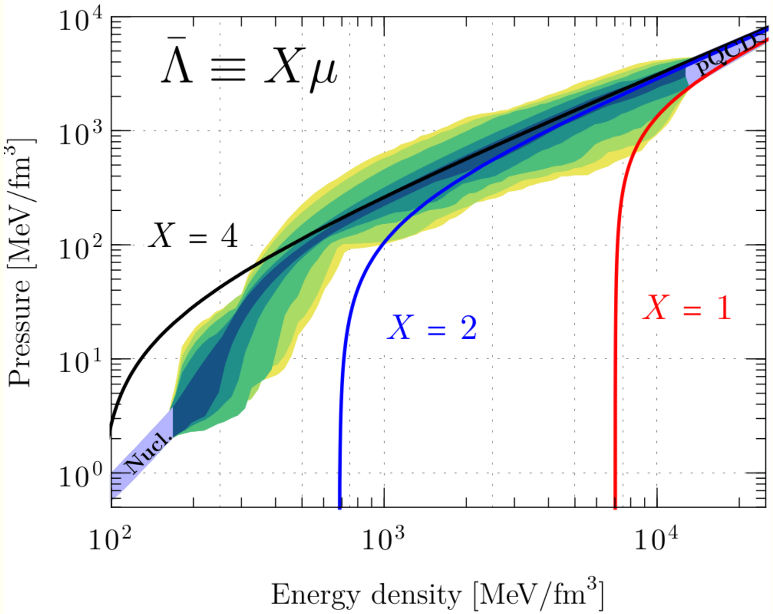


HotQCD Phys.Rev.D 90 (2014), Borsanyi+ Phys. Lett. B 370 (2014)

Motivation: What is pQCD (cold QM)?

Currently, have *large* renorm.-scale dependence

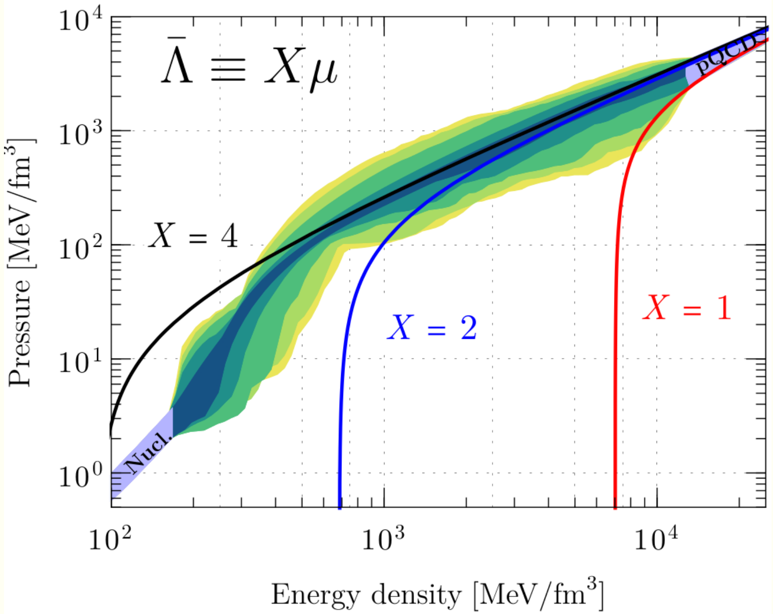
- Can't use cold QM EoS below $45n_s$; factor of 5-10 higher than is relevant in NSs
- Hope that higher-order pQCD calculations will allow us to fix renorm. scale by, e.g., Principle of minimum sensitivity
  P.M. Stevenson. Nucl. Phys. B 231 (1984)



Motivation: What is pQCD (cold QM)?

Currently, have *large* renorm.-scale dependence

- Can't use cold QM EoS below $45n_s$; factor of 5-10 higher than is relevant in NSs
- Hope that higher-order pQCD calculations will allow us to fix renorm. scale by, e.g., Principle of minimum sensitivity
  P.M. Stevenson. Nucl. Phys. B 231 (1984)
- *Want to improve cold QM EoS to use down to more relevant densities*



Framework of high-density calculations

Framework for cold QM computations is relativistic thermal QFT.

- Systemmatic framework for calculating corrections in a series expansion in α_s* (*important caveats to come!*)

$$p = \underbrace{p_0}_{\textit{free quark gas}} + p_1\alpha_s + p_2\alpha_s^2 + \cdots$$

Framework of high-density calculations

Framework for cold QM computations is relativistic thermal QFT.

- Systemmatic framework for calculating corrections in a series expansion in α_s* (*important caveats to come!*)

$$p = \underbrace{p_0}_{\textit{free quark gas}} + p_1\alpha_s + p_2\alpha_s^2 + \cdots$$

- Language for this expansion is ***Feynman diagrams***

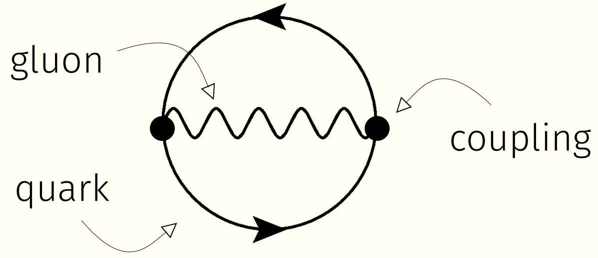
Framework of high-density calculations

Framework for cold QM computations is relativistic thermal QFT.

- Systemmatic framework for calculating corrections in a series expansion in α_s* (*important caveats to come!*)

$$p = \underbrace{p_0}_{\textit{free quark gas}} + p_1\alpha_s + p_2\alpha_s^2 + \cdots$$

- Language for this expansion is ***Feynman diagrams***



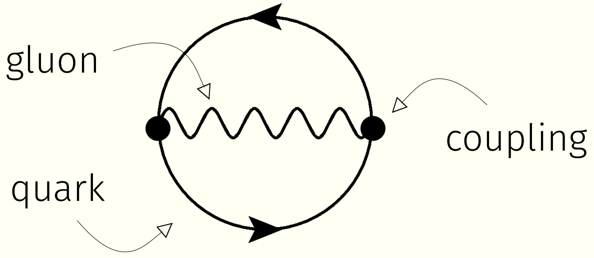
Framework of high-density calculations

Framework for cold QM computations is relativistic thermal QFT.

- Systemmatic framework for calculating corrections in a series expansion in α_s* (*important caveats to come!*)

$$p = \underbrace{p_0}_{\textit{free quark gas}} + p_1\alpha_s + p_2\alpha_s^2 + \cdots$$

- Language for this expansion is ***Feynman diagrams***

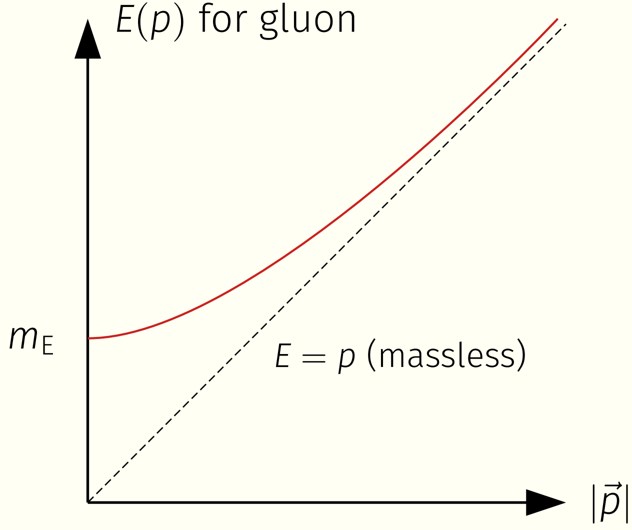


*(no external lines because this is the vacuum with μ>0)

IR problems...

Important caveat is that TQFT has IR (long-wavelength) differences from what you would expect

e.g.:



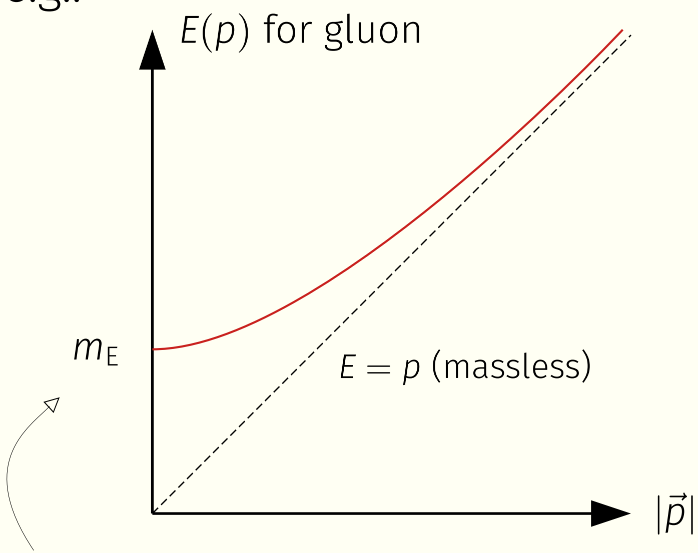
$$-E(\vec{p})^2 + \vec{p}^2 + \overbrace{\Pi\big(E(\vec{p}), \vec{p}\big)}^{\text{“self-energy”}} = 0$$

*describes quantum + statistical corrections to particle propagation

IR problems...

Important caveat is that TQFT has IR (long-wavelength) differences from what you would expect

e.g.:



gluon has a ***thermal mass***!

$$m_E = O(\alpha_s^{1/2}\mu, \alpha_s^{1/2}T)$$

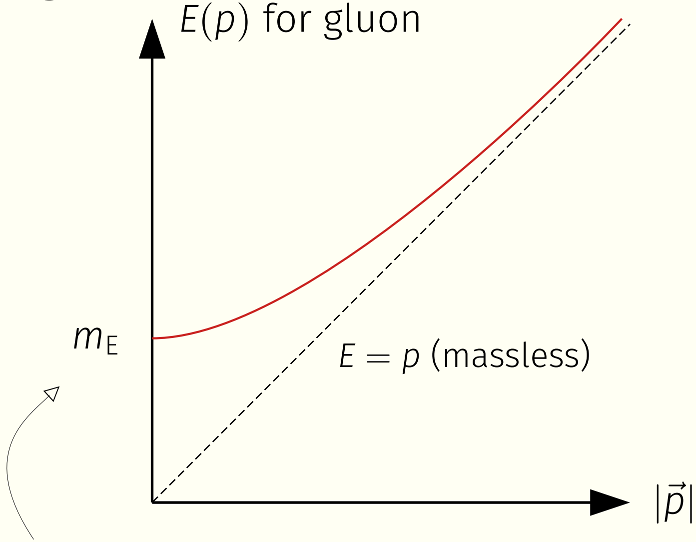
$$-E(\vec{p})^2 + \vec{p}^2 + \overbrace{\Pi\big(E(\vec{p}), \vec{p}\big)}^{\text{“self-energy”}} = 0$$

*describes quantum + statistical corrections to particle propagation

IR problems...

Important caveat is that TQFT has IR (long-wavelength) differences from what you would expect

e.g.:



gluon has a ***thermal mass***!

$m_E = O(\alpha_s^{1/2}\mu, \alpha_s^{1/2}T)$

$$-E(\vec{p})^2 + \vec{p}^2 + \overbrace{\Pi\big(E(\vec{p}), \vec{p}\big)}^{\text{“self-energy”}} = 0$$

*describes quantum + statistical corrections to particle propagation

- Mass screws up naive Feynman-diagram expansions

Loop expansion ≠ coupling expansion

...and their effects

This will modify the naive expectations:

$$p = \boxed{\underbrace{p_0}_{\textit{free quark gas}} + p_1^h\alpha_s + p_2^h\alpha_s^2 + p_3^h\alpha_s^3} \leftarrow \text{scale } |P| \gtrsim \mu$$

$$\boxed{+ \underbrace{p_2^s\alpha_s^2}_{\textit{free soft pressure (screened)}} + p_3^s\alpha_s^3} \leftarrow \text{scale } |P| \lesssim m_E$$

$$\boxed{+ p_3^m\alpha_s^3} \leftarrow \textit{mixed}\text{; both scales}$$

TG+ Phys. Rev. D 104 (2021), Phys. Rev. Lett. 127 (2021);
see also TG+ Phys. Rev. Lett. 121 (2018); $O(\alpha_s^2)$: Freedman & McLerran Phys. Rev. D 16 (1977)

Let's dive in!

Defining equations of QCD as a theory

Defining equations (Minkowski space):

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$
$$= \sum_f \overline{\psi}^i_f\left(\delta_{ij}\left(\mathrm{i}\gamma^\mu\partial_\mu - m_f\right) - \boxed{g\gamma^\mu A^a_\mu T^a_{ij}\Big)\psi^j_f} - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu},$$

interactions

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \boxed{g f^{abc} A^b_\mu A^c_\nu}$$

$T^a = \lambda^a/2$ generators of SU(3) in the fundamental representation.

$$f^{111} = 1, f^{458} = f^{678} = \frac{\sqrt{3}}{2}.$$
$$f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix},$$
$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Defining equations of QCD as a theory

Defining equations (Minkowski space):

$$\begin{aligned}\mathcal{L}_{\mathrm{QCD}} &= \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \\ &= \sum_f \overline{\psi}^i_f\Big(\delta_{ij}\left(\mathrm{i}\gamma^\mu\partial_\mu - m_f\right) - \boxed{g\gamma^\mu A^a_\mu T^a_{ij}\Big)\psi^j_f} - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu},\end{aligned}$$

with

interactions

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \boxed{gf^{abc}A^b_\mu A^c_\nu}$$

$T^a = \lambda^a/2$ generators of SU(3) in the fundamental representation.

Usually, the algebraic identities below are more important than the actual matrices: $(T_F = \frac{1}{2}, C_A = 8, C_F = \frac{4}{3})$

$$[T^a, T^b] = if^{abc}T^c \qquad \mathrm{tr}(T^aT^b) = T_F\delta^{ab}, \qquad (T^aT^a)_{ij} = C_f\delta_{ij}, \qquad f^{acd}f^{bcd} = C_A\delta^{ab},$$

Defining equations of QCD as a theory

Defining equations (Minkowski space):

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$
$$= \sum_f \overline{\psi}^i_f\Big(\delta_{ij}\big(\mathrm{i}\gamma^\mu\partial_\mu - m_f\big) - \boxed{g\gamma^\mu A^a_\mu T^a_{ij}\Big)\psi^j_f} - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu},$$

interactions

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \boxed{g f^{abc} A^b_\mu A^c_\nu}$$

Note that only u,d,s active in dense matter!

$T^a = \lambda^a/2$ generators of SU(3) in the fundamental representation.

Usually, the algebraic identities below are more important than the actual matrices: $(T_F = \frac{1}{2}, C_A = 8, C_F = \frac{4}{3})$

$$[T^a, T^b] = i f^{abc} T^c \qquad \mathrm{tr}(T^a T^b) = T_F \delta^{ab}, \qquad (T^a T^a)_{ij} = C_f \delta_{ij}, \qquad f^{acd} f^{bcd} = C_A \delta^{ab},$$

Thermodynamics of relativistic QFTs: partition function

Want to evaluate partition function:

conserved current

thermodynamic grand potential

$$Z = \underbrace{\mathrm{tr}\left[e^{-\beta\hat{H}}\right]}_{(T>0)} \to \underbrace{\mathrm{tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]}_{(T>0,\mu>0)} = e^{-\Omega}$$

Thermodynamics of relativistic QFTs: partition function

Want to evaluate partition function:

conserved current

thermodynamic grand potential

$$Z = \underbrace{\mathrm{tr}\left[e^{-\beta\hat{H}}\right]}_{(T>0)} \to \underbrace{\mathrm{tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]}_{(T>0,\mu>0)} = e^{-\Omega}$$

- Note that $e^{-\beta\hat{H}}$ quite similar to $e^{-\mathrm{i}\hat{H}t}$ used in QM evolution.

Thermodynamics of relativistic QFTs: partition function

Want to evaluate partition function:

conserved current

thermodynamic grand potential

$$Z = \underbrace{\mathrm{tr}\left[e^{-\beta\hat{H}}\right]}_{(T>0)} \rightarrow \underbrace{\mathrm{tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]}_{(T>0,\mu>0)} = e^{-\Omega}$$

- Note that $e^{-\beta\hat{H}}$ quite similar to $e^{-\mathrm{i}\hat{H}t}$ used in QM evolution.
- Suggests $it \leftrightarrow \beta = \frac{1}{T}$; $-(\Delta t)^2 + (\Delta\vec{x})^2$ (Minkowski) $\rightarrow (\Delta\tau)^2 + (\Delta\vec{x})^2$ (Euclidean)

Thermodynamics of relativistic QFTs: partition function

Want to evaluate partition function:

conserved current

thermodynamic grand potential

$$Z = \underbrace{\mathrm{tr}\left[e^{-\beta\hat{H}}\right]}_{(T>0)} \to \underbrace{\mathrm{tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]}_{(T>0,\mu>0)} = e^{-\Omega}$$

- Note that $e^{-\beta\hat{H}}$ quite similar to $e^{-\mathrm{i}\hat{H}t}$ used in QM evolution.
- Suggests $it \leftrightarrow \beta = \frac{1}{T}$; $-(\Delta t)^2 + (\Delta\vec{x})^2$ (Minkowski) $\to (\Delta\tau)^2 + (\Delta\vec{x})^2$ (Euclidean)
- Like in normal QFT, simplest to construct a ***path-integral*** representation of the partition function by dividing up the "time" interval into equal pieces:

$$e^{-\beta(\hat{H}-\mu\hat{N})} = \underbrace{e^{-\Delta\tau(\hat{H}-\mu\hat{N})}e^{-\Delta\tau(\hat{H}-\mu\hat{N})}\cdots e^{-\Delta\tau(\hat{H}-\mu\hat{N})}}_{N \text{ equal pieces}}, \quad \Delta\tau \equiv \frac{\beta}{N}$$

Path integral for the partition function: Grassman variables

First need a quick summary of how to deal with fermionic (Grassman) variables.

ψ is an anticommuting variable. We define it to behave the following way:

Path integral for the partition function: Grassman variables

First need a quick summary of how to deal with fermionic (Grassman) variables.

ψ is an anticommuting variable. We define it to behave the following way:

$$\text{Anticommuting:} \quad \psi_1\psi_2 = -\psi_2\psi_1 \qquad (\implies \psi^2 = 0)$$

Path integral for the partition function: Grassman variables

First need a quick summary of how to deal with fermionic (Grassman) variables.

ψ is an anticommuting variable. We define it to behave the following way:

Anticommuting: $\psi_1\psi_2 = -\psi_2\psi_1 \qquad (\implies \psi^2 = 0)$

Functions: *defined* by Taylor series

$f(\psi) \equiv f(0) + \psi f'(0)$ truncates!

Path integral for the partition function: Grassman variables

First need a quick summary of how to deal with fermionic (Grassman) variables.

ψ is an anticommuting variable. We define it to behave the following way:

Anticommuting: $\psi_1\psi_2 = -\psi_2\psi_1 \qquad (\implies \psi^2 = 0)$

Functions: *defined* by Taylor series

$f(\psi) \equiv f(0) + \psi f'(0)$ truncates!

Differentiation: $\dfrac{\mathrm{d}}{\mathrm{d}\psi}(c) = 0, \quad \dfrac{\mathrm{d}}{\mathrm{d}\psi}\,\psi = 1;$ linear

(Definite) Integration: $\displaystyle\int \mathrm{d}\psi = 0, \quad \int \mathrm{d}\psi\,\psi = 1;$ linear

Path integral for the partition function: Grassman variables

First need a quick summary of how to deal with fermionic (Grassman) variables.

ψ is an anticommuting variable. We define it to behave the following way:

Anticommuting: $\psi_1\psi_2 = -\psi_2\psi_1 \qquad (\implies \psi^2 = 0)$

Functions: *defined* by Taylor series

$f(\psi) \equiv f(0) + \psi f'(0)$ truncates!

Differentiation: $\dfrac{\mathrm{d}}{\mathrm{d}\psi}(c) = 0, \quad \dfrac{\mathrm{d}}{\mathrm{d}\psi}\psi = 1;$ linear

(Definite) Integration: $\displaystyle\int \mathrm{d}\psi = 0, \quad \int \mathrm{d}\psi\, \psi = 1;$ linear

(Essentially, all operations defined algebraically)

Path integral for the partition function: Coherent states

Follow Altland and Simons and use ***coherent states*** to evaluate [then bosons (ζ=+1) and fermions (ζ=-1) are very similar]:

$$|\varphi\rangle \equiv e^{\pm\varphi\hat{a}^\dagger}|0\rangle \qquad \text{so that}$$

Path integral for the partition function: Coherent states

Follow Altland and Simons and use *coherent states* to evaluate [then bosons (ζ=+1) and fermions (ζ=-1) are very similar]:

$$|\varphi\rangle \equiv e^{\pm\varphi\hat{a}^\dagger}|0\rangle \qquad \text{so that} \qquad \left.\begin{aligned} \hat{a}|\phi\rangle &= [a, e^{\phi\hat{a}^\dagger}]|0\rangle \\ &= \sum_{n=1}^{\infty}\frac{\phi^n}{n!}n(\hat{a}^\dagger)^{n-1}|0\rangle \\ &= \phi e^{\phi\hat{a}^\dagger}|0\rangle \\ &= \phi|\phi\rangle \end{aligned}\right\} \quad \textit{bosons}$$

Path integral for the partition function: Coherent states

Follow Altland and Simons and use *coherent states* to evaluate [then bosons (ζ=+1) and fermions (ζ=-1) are very similar]:

$$|\varphi\rangle \equiv e^{\pm\varphi\hat{a}^\dagger}|0\rangle \qquad \text{so that}$$

$$\left.\begin{aligned}
\hat{a}|\phi\rangle &= [a, e^{\phi\hat{a}^\dagger}]|0\rangle \\
&= \sum_{n=1}^{\infty}\frac{\phi^n}{n!}n(\hat{a}^\dagger)^{n-1}|0\rangle \\
&= \phi e^{\phi\hat{a}^\dagger}|0\rangle \\
&= \phi|\phi\rangle
\end{aligned}\right\} \quad \textit{bosons}$$

$$\left.\begin{aligned}
\hat{a}|\eta\rangle &= \hat{a}(1-\eta\hat{a}^\dagger)|0\rangle \\
&= \eta|0\rangle = \eta(1-\eta\hat{a}^\dagger)|0\rangle \\
&= \eta|\eta\rangle
\end{aligned}\right\} \quad \textit{fermions}$$

Path integral for the partition function: Coherent states

Follow Altland and Simons and use ***coherent states*** to evaluate [then bosons (ζ=+1) and fermions (ζ=-1) are very similar]:

$$|\varphi\rangle \equiv e^{\pm\varphi\hat{a}^\dagger}|0\rangle \qquad \text{so that} \qquad \hat{a}|\varphi\rangle = \varphi|\varphi\rangle \text{ (both cases)}$$

Path integral for the partition function: Coherent states

Follow Altland and Simons and use ***coherent states*** to evaluate [then bosons (ζ=+1) and fermions (ζ=-1) are very similar]:

$$|\varphi\rangle \equiv e^{\pm\varphi\hat{a}^\dagger}|0\rangle \qquad \text{so that} \qquad \hat{a}|\varphi\rangle = \varphi|\varphi\rangle \text{ (both cases)}$$

Need the following relations:

Completeness: $\quad \mathrm{Id} = \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger\varphi}|\varphi\rangle\langle\varphi|$

Overlap: $\quad \langle\varphi'|\varphi\rangle = \exp\left(\varphi'^\dagger\varphi\right)$

Measure: $\quad d(\varphi^\dagger, \varphi) = \dfrac{d\varphi^\dagger d\varphi}{\pi^{(1+\zeta)/2}}$

Path integral for the partition function: Derivation

First we want to write the trace in the partition function in terms of an integral over these coherent states at the beginning and final "times":

$$
\begin{aligned}
Z = \mathrm{tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right] &= \sum_{n>0}\langle n|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle \\
&= \int d(\varphi^\dagger,\varphi)e^{-\varphi^\dagger\varphi}\sum_{n>0}\underbrace{\langle n|\varphi\rangle}\langle\varphi|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle
\end{aligned}
$$

Path integral for the partition function: Derivation

First we want to write the trace in the partition function in terms of an integral over these coherent states at the beginning and final "times":

$$
\begin{aligned}
Z = \mathrm{tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right] &= \sum_{n>0} \langle n|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle \\
&= \int d(\varphi^\dagger,\varphi)e^{-\varphi^\dagger\varphi} \sum_{n>0} \underbrace{\overset{\varphi\,\leftrightarrow\,\varphi^\dagger}{\langle n|\varphi\rangle}} \langle\varphi|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle
\end{aligned}
$$

bosonic operator

Path integral for the partition function: Derivation

First we want to write the trace in the partition function in terms of an integral over these coherent states at the beginning and final "times":

$$Z = \mathrm{tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right] = \sum_{n>0} \langle n| e^{-\beta(\hat{H}-\mu\hat{N})} |n\rangle$$

$$= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \sum_{n>0} \underbrace{\langle n|\varphi\rangle}_{} \overset{\varphi \leftrightarrow \varphi^\dagger}{} \langle \varphi | e^{-\beta(\hat{H}-\mu\hat{N})} |n\rangle$$

bosonic operator

move to end; exchanges Grassman variables!

Path integral for the partition function: Derivation

First we want to write the trace in the partition function in terms of an integral over these coherent states at the beginning and final "times":

$$Z = \mathrm{tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right] = \sum_{n>0} \langle n| e^{-\beta(\hat{H}-\mu\hat{N})} |n\rangle$$

bosonic operator

$$= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \sum_{n>0} \underbrace{\langle n|\varphi\rangle}_{} \overset{\varphi \leftrightarrow \varphi^\dagger}{} \langle \varphi | e^{-\beta(\hat{H}-\mu\hat{N})} |n\rangle$$

move to end; exchanges Grassman variables!

$$= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \sum_{n>0} \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} |n\rangle \langle n|\varphi\rangle$$

$$= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} |\varphi\rangle$$

Path integral for the partition function: Derivation

First we want to write the trace in the partition function in terms of an integral over these coherent states at the beginning and final "times":

$$Z = \mathrm{tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right] = \sum_{n>0} \langle n|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle$$

bosonic operator

$$= \int d(\varphi^\dagger,\varphi)e^{-\varphi^\dagger\varphi} \sum_{n>0} \overbrace{\underbrace{\langle n|\varphi\rangle}}^{\varphi \leftrightarrow \varphi^\dagger} \langle\varphi|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle$$

move to end; exchanges Grassman variables!

$$= \int d(\varphi^\dagger,\varphi)e^{-\varphi^\dagger\varphi} \sum_{n>0} \langle\pm\varphi|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle\langle n|\varphi\rangle$$

$$= \int d(\varphi^\dagger,\varphi)e^{-\varphi^\dagger\varphi} \langle\pm\varphi|e^{-\beta(\hat{H}-\mu\hat{N})}|\varphi\rangle$$

Bosons return to same state; fermions to negative the state!

Path integral for the partition function: Derivation

Now break up into little pieces, inserting identities along the way (following Laine and Vuorinen)

$$Z = \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

Path integral for the partition function: Derivation

Now break up into little pieces, inserting identities along the way (following Laine and Vuorinen)

$$
\begin{aligned}
Z &= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} | \varphi \rangle \\
&= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\Delta\tau_N(\hat{H}-\mu\hat{N})} e^{-\Delta\tau_{N-1}(\hat{H}-\mu\hat{N})} \cdots e^{-\Delta\tau_1(\hat{H}-\mu\hat{N})} | \varphi \rangle
\end{aligned}
$$

Path integral for the partition function: Derivation

Now break up into little pieces, inserting identities along the way (following Laine and Vuorinen)

$$Z = \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

$$= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\Delta\tau_N(\hat{H}-\mu\hat{N})} e^{-\Delta\tau_{N-1}(\hat{H}-\mu\hat{N})} \cdots e^{-\Delta\tau_1(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

$$= \int d(\varphi^\dagger, \varphi) d(\varphi_N^\dagger, \varphi_N) e^{-\varphi^\dagger \varphi} e^{-\varphi_N^\dagger \varphi_N} \langle \pm\varphi | e^{-\Delta\tau_N(\hat{H}-\mu\hat{N})} | \varphi_N \rangle \langle \varphi_N | e^{-\Delta\tau_{N-1}(\hat{H}-\mu\hat{N})} \cdots e^{-\Delta\tau_1(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

Path integral for the partition function: Derivation

Now break up into little pieces, inserting identities along the way (following Laine and Vuorinen)

$$Z = \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

$$= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\Delta\tau_N(\hat{H}-\mu\hat{N})} e^{-\Delta\tau_{N-1}(\hat{H}-\mu\hat{N})} \cdots e^{-\Delta\tau_1(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

$$= \int d(\varphi^\dagger, \varphi) d(\varphi_N^\dagger, \varphi_N) e^{-\varphi^\dagger \varphi} e^{-\varphi_N^\dagger \varphi_N} \langle \pm\varphi | e^{-\Delta\tau_N(\hat{H}-\mu\hat{N})} | \varphi_N \rangle \langle \varphi_N | e^{-\Delta\tau_{N-1}(\hat{H}-\mu\hat{N})} \cdots e^{-\Delta\tau_1(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

Now deal with each term:

$$\mathrm{e}^{-\varphi_i^\dagger \varphi_i} \langle \varphi_i | e^{-\Delta\tau(\hat{H}[\hat{a}^\dagger, \hat{a}] - \mu\hat{N}[\hat{a}^\dagger, \hat{a}])} | \varphi_{i-1} \rangle \approx e^{-\varphi_i^\dagger \varphi_i} \langle \varphi_i | \varphi_{i-1} \rangle \, e^{-\Delta\tau(H[\varphi_i^\dagger, \varphi_{i-1}] - \mu N[\varphi_i^\dagger, \varphi_{i-1}])}$$

Path integral for the partition function: Derivation

Now break up into little pieces, inserting identities along the way (following Laine and Vuorinen)

$$Z = \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

$$= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\Delta\tau_N(\hat{H}-\mu\hat{N})} e^{-\Delta\tau_{N-1}(\hat{H}-\mu\hat{N})} \cdots e^{-\Delta\tau_1(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

$$= \int d(\varphi^\dagger, \varphi) d(\varphi_N^\dagger, \varphi_N) e^{-\varphi^\dagger \varphi} e^{-\varphi_N^\dagger \varphi_N} \langle \pm\varphi | e^{-\Delta\tau_N(\hat{H}-\mu\hat{N})} | \varphi_N \rangle \langle \varphi_N | e^{-\Delta\tau_{N-1}(\hat{H}-\mu\hat{N})} \cdots e^{-\Delta\tau_1(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

Now deal with each term:

$$\mathrm{e}^{-\varphi_i^\dagger \varphi_i} \langle \varphi_i | e^{-\Delta\tau(\hat{H}[\hat{a}^\dagger,\hat{a}]-\mu\hat{N}[\hat{a}^\dagger,\hat{a}])} | \varphi_{i-1} \rangle \approx e^{-\varphi_i^\dagger \varphi_i} \langle \varphi_i | \varphi_{i-1} \rangle \, e^{-\Delta\tau(H[\varphi_i^\dagger,\varphi_{i-1}]-\mu N[\varphi_i^\dagger,\varphi_{i-1}])}$$

$$= \exp\left\{ -\varphi_i^\dagger \varphi_i + \varphi_i^\dagger \varphi_{i-1} - \Delta\tau (H[\varphi_i^\dagger, \varphi_{i-1}] - \mu N[\varphi_i^\dagger, \varphi_{i-1}]) \right\}$$

$$= \exp\left\{ -\Delta\tau \left[\varphi^\dagger \left(\frac{\varphi_i - \varphi_{i-1}}{\Delta\tau} \right) + H[\varphi_i^\dagger, \varphi_{i-1}] - \mu N[\varphi_i^\dagger, \varphi_{i-1}] \right] \right\}$$

Path integral for the partition function: Derivation

Taking the limit of a large N, gives:

$$
\begin{aligned}
Z &= \int d(\varphi^\dagger, \varphi) e^{-\varphi^\dagger \varphi} \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} | \varphi \rangle \\
&= \int\limits_{\substack{\varphi^\dagger(\beta)=\pm\varphi^\dagger(0) \\ \varphi(\beta)=\pm\varphi(0)}} \mathcal{D}\varphi^\dagger(\tau)\mathcal{D}\varphi(\tau) \exp\left\{-\int_0^\beta \mathrm{d}\tau \left[\varphi^\dagger(\tau)\frac{\mathrm{d}\varphi(\tau)}{\mathrm{d}\tau} + H[\varphi^\dagger(\tau), \varphi(\tau)] - \mu N[\varphi^\dagger(\tau), \varphi(\tau)]\right]\right\}
\end{aligned}
$$

Path integral for the partition function: Derivation

Taking the limit of a large N, gives:

$$Z = \int d(\varphi^\dagger,\varphi) e^{-\varphi^\dagger\varphi} \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} | \varphi \rangle$$
$$= \int\limits_{\substack{\varphi^\dagger(\beta)=\pm\varphi^\dagger(0)\\ \varphi(\beta)=\pm\varphi(0)}} \mathcal{D}\varphi^\dagger(\tau)\mathcal{D}\varphi(\tau) \exp\left\{-\int_0^\beta \mathrm{d}\tau \left[\varphi^\dagger(\tau)\frac{\mathrm{d}\varphi(\tau)}{\mathrm{d}\tau} + H[\varphi^\dagger(\tau),\varphi(\tau)] - \mu N[\varphi^\dagger(\tau),\varphi(\tau)]\right]\right\}$$

For usual Hamiltonians, Legendre transformation gives ***Euclidean Lagrangian*** (now go back to fields as well):

$$\boxed{Z = \int\limits_{\substack{\varphi^\dagger(\beta,\vec{x})=\pm\varphi^\dagger(0,\vec{x})\\ \varphi(\beta,\vec{x})=\pm\varphi(0,\vec{x})}} \mathcal{D}\varphi^\dagger\mathcal{D}\varphi \exp\left\{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\mathcal{L}_\mathrm{E} - \mu\mathcal{N}\right]\right\}}$$

Path integral: Summary

$$\boxed{Z = \int\limits_{\substack{\varphi^\dagger(\beta,\vec{x})=\pm\varphi^\dagger(0,\vec{x}) \\ \varphi(\beta,\vec{x})=\pm\varphi(0,\vec{x})}} \mathcal{D}\varphi^\dagger \mathcal{D}\varphi \exp\left\{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\mathcal{L}_\mathrm{E} - \mu\mathcal{N}\right]\right\}}$$

- Compact “time” integral
- Bosons periodic in imaginary time (energies $\omega_n = 2\pi n T$)
- Fermions anti-periodic in imaginary time [energies $\omega_n = 2\pi(n + \frac{1}{2})T$]
- Path integral with Euclidean Lagrangian ($t \to -i\tau$)

$$\mathcal{L}^\mathrm{E}_\mathrm{QCD} = \sum_f \overline{\psi}^i_f \Big(\delta_{ij}(\gamma^\mathrm{E}_\mu \partial_\mu + m_f) - ig\gamma^\mathrm{E}_\mu A^a_\mu T^a_{ij}\Big)\psi^j_f + \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu},$$

High density

$$\boxed{Z = \int\limits_{\substack{\varphi^\dagger(\beta,\vec{x})=\pm\varphi^\dagger(0,\vec{x})\\ \varphi(\beta,\vec{x})=\pm\varphi(0,\vec{x})}} \mathcal{D}\varphi^\dagger\mathcal{D}\varphi \exp\left\{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\mathcal{L}_\mathrm{E} - \mu\mathcal{N}\right]\right\}}$$

- As $T \to 0$, "time" interval becomes infinite again (periodic/antiperiodic doesn't matter)

High density

$$\boxed{Z = \int\limits_{\substack{\varphi^\dagger(\beta,\vec{x})=\pm\varphi^\dagger(0,\vec{x})\\ \varphi(\beta,\vec{x})=\pm\varphi(0,\vec{x})}} \mathcal{D}\varphi^\dagger \mathcal{D}\varphi \exp\left\{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\,[\mathcal{L}_\mathrm{E} - \mu\mathcal{N}]\right\}}$$

- As $T \to 0$, "time" interval becomes infinite again (periodic/antiperiodic doesn't matter)
- Chemical potential is like imaginary A^0 field in Euclidean Lagrangian

$$\mathcal{L}_E \ni (\overline{\psi}\partial\!\!\!/_\mu\psi) - (\overline{\psi}\gamma_\mathrm{E}^0\psi)igA^0 \quad \longleftrightarrow \quad (\overline{\psi}\partial\!\!\!/_\mu\psi) - (\overline{\psi}\gamma_\mathrm{E}^0\psi)\mu$$

$$\boxed{\partial_0 - igA^0 \quad \longleftrightarrow \quad \partial_0 - \mu}$$

High density

$$\boxed{Z = \int\limits_{\substack{\varphi^\dagger(\beta,\vec{x})=\pm\varphi^\dagger(0,\vec{x}) \\ \varphi(\beta,\vec{x})=\pm\varphi(0,\vec{x})}} \mathcal{D}\varphi^\dagger \mathcal{D}\varphi \exp\left\{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \, [\mathcal{L}_\mathrm{E} - \mu \mathcal{N}]\right\}}$$

- As $T \to 0$, "time" interval becomes infinite again (periodic/antiperiodic doesn't matter)
- Chemical potential is like imaginary A^0 field in Euclidean Lagrangian

$$\mathcal{L}_E \ni (\overline{\psi} \partial\!\!\!/_\mu \psi) - (\overline{\psi} \gamma^0_\mathrm{E} \psi) \mathrm{i} g A^0 \quad \longleftrightarrow \quad (\overline{\psi} \partial\!\!\!/_\mu \psi) - (\overline{\psi} \gamma^0_\mathrm{E} \psi) \mu$$

$$\boxed{\partial_0 - \mathrm{i} g A^0 \quad \longleftrightarrow \quad \partial_0 - \mu}$$

$$\boxed{\mathrm{i}\omega_n \mapsto \mathrm{i}\omega_n - \mu = \mathrm{i}(\omega_n + \mathrm{i}\mu) \text{ imaginary shift to the frequency!}}$$

Perturbation theory

Interacting fields / perturbation theory

Write the action as a free part plus an interacting part, $S[\varphi] = S_0[\varphi] + S_I[\varphi]$
assume interaction is small:

Interacting fields / perturbation theory

Write the action as a free part plus an interacting part, $S[\varphi] = S_0[\varphi] + S_I[\varphi]$
assume interaction is small:

$$e^{-\Omega} = \int \mathcal{D}\varphi e^{-S_0[\varphi]-S_I[\varphi]} \simeq \int \mathcal{D}\varphi e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2}S_I^2 + \cdots\right)$$

Interacting fields / perturbation theory

Write the action as a free part plus an interacting part, $S[\varphi] = S_0[\varphi] + S_I[\varphi]$ assume interaction is small:

$$e^{-\Omega} = \int \mathcal{D}\varphi e^{-S_0[\varphi]-S_I[\varphi]} \simeq \int \mathcal{D}\varphi e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2}S_I^2 + \cdots\right)$$

$$\implies \Omega = -\ln Z = -\ln\left[\int \mathcal{D}\varphi e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2}S_I^2 + \cdots\right)\right]$$

Interacting fields / perturbation theory

Write the action as a free part plus an interacting part, $S[\varphi] = S_0[\varphi] + S_I[\varphi]$
assume interaction is small:

$$e^{-\Omega} = \int \mathcal{D}\varphi e^{-S_0[\varphi]-S_I[\varphi]} \simeq \int \mathcal{D}\varphi e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2} S_I^2 + \cdots\right)$$

$$\Longrightarrow \Omega = -\ln Z = -\ln\left[\int \mathcal{D}\varphi e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2} S_I^2 + \cdots\right)\right]$$

$$= \Omega_0 - \ln\left[\left\langle 1 - S_I + \tfrac{1}{2} S_I^2 + \cdots \right\rangle_0\right] \longleftarrow \boxed{\langle O\rangle_0 = \frac{\int \mathcal{D}\varphi\, O\, e^{-S_0[\varphi]}}{\int \mathcal{D}\varphi e^{-S_0[\varphi]}}}$$

Interacting fields / perturbation theory

Write the action as a free part plus an interacting part, $S[\varphi] = S_0[\varphi] + S_I[\varphi]$ assume interaction is small:

$$e^{-\Omega} = \int \mathcal{D}\varphi e^{-S_0[\varphi] - S_I[\varphi]} \simeq \int \mathcal{D}\varphi e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2} S_I^2 + \cdots\right)$$

$$\implies \Omega = -\ln Z = -\ln \left[\int \mathcal{D}\varphi e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2} S_I^2 + \cdots\right)\right]$$

$$= \Omega_0 - \ln \left[\left\langle 1 - S_I + \tfrac{1}{2} S_I^2 + \cdots \right\rangle_0\right]$$

$$= \Omega_0 + \langle S_I \rangle_0 - \tfrac{1}{2} \underbrace{\left[\langle S_I^2 \rangle_0 - \langle S_I \rangle_0^2\right]}_{\textit{connected}} + \cdots$$

$$\boxed{\langle O \rangle_0 = \frac{\int \mathcal{D}\varphi \, O \, e^{-S_0[\varphi]}}{\int \mathcal{D}\varphi e^{-S_0[\varphi]}}}$$

Interacting fields / perturbation theory

Write the action as a free part plus an interacting part, $S[\varphi] = S_0[\varphi] + S_I[\varphi]$ assume interaction is small:

$$e^{-\Omega} = \int \mathcal{D}\varphi e^{-S_0[\varphi] - S_I[\varphi]} \simeq \int \mathcal{D}\varphi e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2} S_I^2 + \cdots\right)$$

$$\implies \Omega = -\ln Z = -\ln \left[\int \mathcal{D}\varphi e^{-S_0[\varphi]} \left(1 - S_I + \tfrac{1}{2} S_I^2 + \cdots\right) \right]$$

$$= \Omega_0 - \ln \left[\left\langle 1 - S_I + \tfrac{1}{2} S_I^2 + \cdots \right\rangle_0 \right]$$

$$\boxed{\langle O \rangle_0 = \frac{\int \mathcal{D}\varphi \, O \, e^{-S_0[\varphi]}}{\int \mathcal{D}\varphi e^{-S_0[\varphi]}}}$$

$$= \Omega_0 + \langle S_I \rangle_0 - \tfrac{1}{2} \underbrace{\left[\langle S_I^2 \rangle_0 - \langle S_I \rangle_0^2 \right]}_{\textit{connected}} + \cdots$$

$$\boxed{\Omega = \Omega_0 + \left\langle S_I - \tfrac{1}{2} S_I^2 + \cdots \right\rangle_{0,c}}$$

connected corrections!

Interacting fields: Wick's theorem

Q: How do we compute these connected corrections?

A: Introduce a source and differentiate:

Write $\quad S_0[\varphi] = \int_{x,y} \frac{1}{2}\varphi_x A_{x,y} \varphi_y,$

Interacting fields: Wick's theorem

Q: How do we compute these connected corrections?

A: Introduce a source and differentiate:

complete square and shift φ by constant

Write $\quad S_0[\varphi] = \int_{x,y} \frac{1}{2}\varphi_x A_{x,y} \varphi_y,$

Define $\quad Z[J] = \int \mathcal{D}\varphi \exp\left[\int_{x,y}(-\frac{1}{2}\varphi_x A_{x,y}\varphi_y + J_x\varphi_x)\right] = \exp\left[\int_{x,y}\frac{1}{2}J_x A^{-1}_{x,y} J_y\right] Z[0]$

Interacting fields: Wick's theorem

Q: How do we compute these connected corrections?

A: Introduce a source and differentiate:

complete square and shift φ by constant

Write $S_0[\varphi] = \int_{x,y} \frac{1}{2}\varphi_x A_{x,y}\varphi_y,$

Define $Z[J] = \int \mathcal{D}\varphi \exp\left[\int_{x,y}(-\frac{1}{2}\varphi_x A_{x,y}\varphi_y + J_x\varphi_x)\right] = \exp\left[\int_{x,y}\frac{1}{2}J_x A^{-1}_{x,y}J_y\right] Z[0]$

Then $\langle\varphi_1\varphi_2\cdots\varphi_N\rangle_{0,c} = \frac{1}{Z[0]}\,\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\cdots\frac{\delta}{\delta J_N}Z[J]\Big|_{J=0}$

Interacting fields: Wick's theorem

Q: How do we compute these connected corrections?

A: Introduce a source and differentiate:

complete square and shift φ by constant

Write $\quad S_0[\varphi] = \int_{x,y} \frac{1}{2}\varphi_x A_{x,y} \varphi_y,$

Define $\quad Z[J] = \int \mathcal{D}\varphi \exp\left[\int_{x,y}(-\frac{1}{2}\varphi_x A_{x,y}\varphi_y + J_x\varphi_x)\right] = \exp\left[\int_{x,y} \frac{1}{2} J_x A^{-1}_{x,y} J_y\right] Z[0]$

Then $\quad$
$$\langle \varphi_1 \varphi_2 \cdots \varphi_N \rangle_{0,c} = \frac{1}{Z[0]}\,\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\cdots\frac{\delta}{\delta J_N} Z[J]\Big|_{J=0}$$
$$= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\cdots\frac{\delta}{\delta J_N}\left(1 + \int_{a,b}\frac{1}{2}J_a A^{-1}_{a,b} J_b + \frac{1}{2}\int_{a,b}\frac{1}{2}J_a A^{-1}_{a,b} J_b \int_{c,d}\frac{1}{2}J_c A^{-1}_{c,d} J_d + \cdots\right)\Big|_{J=0}$$

Q: How do we compute these connected corrections?

A: Introduce a source and differentiate:

complete square and shift φ by constant

Write $\quad S_0[\varphi] = \int_{x,y} \frac{1}{2}\varphi_x A_{x,y}\varphi_y,$

Define $\quad Z[J] = \int \mathcal{D}\varphi \exp\left[\int_{x,y}(-\frac{1}{2}\varphi_x A_{x,y}\varphi_y + J_x\varphi_x)\right] = \exp\left[\int_{x,y}\frac{1}{2}J_x A^{-1}_{x,y}J_y\right] Z[0]$

Then $\quad \langle\varphi_1\varphi_2\cdots\varphi_N\rangle_{0,c} = \frac{1}{Z[0]}\,\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\cdots\frac{\delta}{\delta J_N}Z[J]\Big|_{J=0}$

$$= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\cdots\frac{\delta}{\delta J_N}\left(1+\int_{a,b}\frac{1}{2}J_a A^{-1}_{a,b}J_b + \frac{1}{2}\int_{a,b}\frac{1}{2}J_a A^{-1}_{a,b}J_b\int_{c,d}\frac{1}{2}J_c A^{-1}_{c,d}J_d + \cdots\right)\Big|_{J=0}$$

$$= \sum_{\text{all combinations, c}} A^{-1}_{1,i}A^{-1}_{i,j}\cdots A^{-1}_{k,l}A^{-1}_{l,N}$$

Interacting fields: Wick's theorem

Q: How do we compute these connected corrections?

A: Introduce a source and differentiate:

Write $S_0[\varphi] = \int_{x,y} \frac{1}{2} \varphi_x A_{x,y} \varphi_y$,

complete square and shift φ by constant

Define $Z[J] = \int \mathcal{D}\varphi \exp\left[\int_{x,y}(-\frac{1}{2}\varphi_x A_{x,y}\varphi_y + J_x \varphi_x)\right] = \exp\left[\int_{x,y} \frac{1}{2} J_x A^{-1}_{x,y} J_y\right] Z[0]$

Then
$$\langle \varphi_1 \varphi_2 \cdots \varphi_N \rangle_{0,c} = \frac{1}{Z[0]} \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \cdots \frac{\delta}{\delta J_N} Z[J] \Big|_{J=0}$$
$$= \frac{\delta}{\delta J_1} \frac{\delta}{\delta J_2} \cdots \frac{\delta}{\delta J_N} \left(1 + \int_{a,b} \frac{1}{2} J_a A^{-1}_{a,b} J_b + \frac{1}{2} \int_{a,b} \frac{1}{2} J_a A^{-1}_{a,b} J_b \int_{c,d} \frac{1}{2} J_c A^{-1}_{c,d} J_d + \cdots \right)\Big|_{J=0}$$
$$= \sum_{\text{all combinations, c}} A^{-1}_{1,i} A^{-1}_{i,j} \cdots A^{-1}_{k,l} A^{-1}_{l,N}$$

$$\boxed{\langle \varphi_1 \varphi_2 \cdots \varphi_N \rangle_{0,c} = \sum_{\text{all combinations, c}} \langle \varphi_1 \varphi_i \rangle_0 \langle \varphi_i \varphi_j \rangle_0 \cdots \langle \varphi_k \varphi_l \rangle_0 \langle \varphi_l \varphi_N \rangle_0}$$

Feynman diagrams, propagator, self-energy

An example, from ϕ^4 theory:

$$\langle S_I \rangle_{0,c} = \frac{\lambda}{4!} \langle \phi\phi\phi\phi \rangle_{0,c}$$

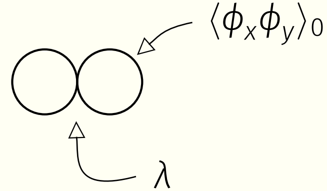
Feynman diagrams, propagator, self-energy

An example, from ϕ^4 theory:

$$\langle S_I \rangle_{0,c} = \frac{\lambda}{4!} \langle \phi\phi\phi\phi \rangle_{0,c} = \frac{3\lambda}{4!} \langle \phi\phi \rangle_0 \langle \phi\phi \rangle_0$$

An example, from ϕ^4 theory:

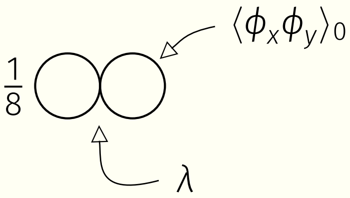
$$\langle S_I \rangle_{0,c} = \frac{\lambda}{4!}\langle \phi\phi\phi\phi \rangle_{0,c} = \frac{3\lambda}{4!}\langle \phi\phi \rangle_0 \langle \phi\phi \rangle_0 \quad = \frac{1}{8}$$



Feynman diagrams, propagator, self-energy

An example, from ϕ^4 theory:

$$\langle S_I \rangle_{0,c} = \frac{\lambda}{4!}\langle \phi\phi\phi\phi \rangle_{0,c} = \frac{3\lambda}{4!}\langle \phi\phi \rangle_0 \langle \phi\phi \rangle_0 \quad = \frac{1}{8}$$



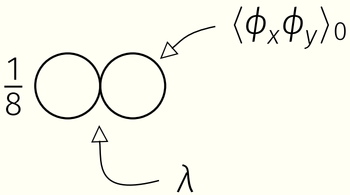
These are ***Feynman Diagrams***. This fundamental correlation is called the ***propagator***

$$\langle \phi_x \phi_y \rangle_0 = \int_P e^{-P\cdot(X-Y)} \frac{1}{P^2 + m^2}$$

Feynman diagrams, propagator, self-energy

An example, from ϕ^4 theory:

$$\langle S_I\rangle_{0,c} = \frac{\lambda}{4!}\langle\phi\phi\phi\phi\rangle_{0,c} = \frac{3\lambda}{4!}\langle\phi\phi\rangle_0\langle\phi\phi\rangle_0 \quad = \frac{1}{8}$$



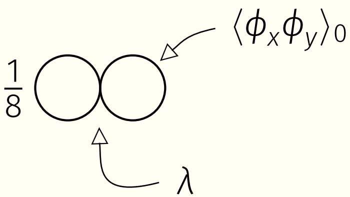
These are ***Feynman Diagrams***. This fundamental correlation is called the ***propagator***

$$\langle\phi_x\phi_y\rangle_0 = \int_P e^{-P\cdot(X-Y)}\frac{1}{P^2+m^2}$$

The ***self-energy*** are corrections to the propagator: [expanding $(P^2+\Pi)^{-1}$]

An example, from ϕ^4 theory:

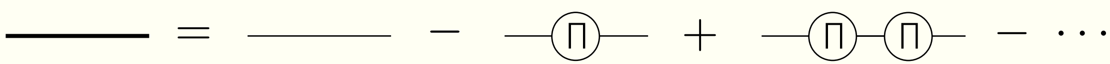
$$\langle S_I \rangle_{0,c} = \frac{\lambda}{4!}\langle \phi\phi\phi\phi \rangle_{0,c} = \frac{3\lambda}{4!}\langle \phi\phi \rangle_0 \langle \phi\phi \rangle_0 \quad = \frac{1}{8}$$



These are ***Feynman Diagrams***. This fundamental correlation is called the ***propagator***

$$\langle \phi_x \phi_y \rangle_0 = \int_P e^{-P\cdot(X-Y)} \frac{1}{P^2 + m^2}$$

The ***self-energy*** are corrections to the propagator: $[\text{expanding } (P^2 + \Pi)^{-1}]$



Full correlation function

$$\langle \phi\phi \rangle_c = \langle \phi\phi \rangle_0 - \langle \phi\phi \rangle_0 \Pi \langle \phi\phi \rangle_c \quad \longleftrightarrow \quad \langle \phi\phi \rangle_c^{-1} = \langle \phi\phi \rangle_0^{-1} + \Pi$$

Self-energy evaluation

The self energy has a nontrivial IR limit; let's calculate it in QCD:

Rearranging prev. expression for the one-loop gluon self energy at high density:

$$\langle \phi\phi \rangle_c - \langle \phi\phi \rangle_0 = \langle \phi\phi(e^{-S_i} - 1) \rangle_{0,c} = -\langle \phi\phi \rangle_0 \Pi \langle \phi\phi \rangle_c$$

Self-energy evaluation

The self energy has a nontrivial IR limit; let's calculate it in QCD:

Rearranging prev. expression for the one-loop gluon self energy at high density:

$$\langle AA \rangle_0 \Pi(P) \langle AA \rangle_0 = \langle A(S_I - \tfrac{1}{2}S_I^2)A \rangle_{0,c}$$

Self-energy evaluation

The self energy has a nontrivial IR limit; let's calculate it in QCD:

Rearranging prev. expression for the one-loop gluon self energy at high density:

$$\langle AA\rangle_0 \Pi(P) \langle AA\rangle_0 = \langle A(S_I - \tfrac{1}{2}S_I^2)A\rangle_{0,c} \quad = \langle A(-\tfrac{1}{2}S_I^2)A\rangle_{0,c}$$

(need fermion loop)

Self-energy evaluation

The self energy has a nontrivial IR limit; let's calculate it in QCD:

Rearranging prev. expression for the one-loop gluon self energy at high density:

$$\langle AA\rangle_0 \Pi(P)\langle AA\rangle_0 = \langle A(S_I - \tfrac{1}{2}S_I^2)A\rangle_{0,c} \quad = \langle A(-\tfrac{1}{2}S_I^2)A\rangle_{0,c}$$

(need fermion loop)

$$= -\frac{(-ig)^2}{2}\mathrm{tr}(T^a T^b)\langle A(\bar{\psi}\not{A}\psi)(\bar{\psi}\not{A}\psi)A\rangle_{0,c}$$

Self-energy evaluation

The self energy has a nontrivial IR limit; let's calculate it in QCD:

Rearranging prev. expression for the one-loop gluon self energy at high density:

$$\langle AA\rangle_0 \Pi(P) \langle AA\rangle_0 = \langle A(S_I - \tfrac{1}{2}S_I^2)A\rangle_{0,c} \quad = \langle A(-\tfrac{1}{2}S_I^2)A\rangle_{0,c}$$

(need fermion loop)

$$= -\frac{(-\mathrm{i}g)^2}{2}\mathrm{tr}(T^a T^b)\langle A(\bar{\psi}\not{A}\psi)(\bar{\psi}\not{A}\psi)A\rangle_{0,c}$$

$$= \langle AA\rangle_0 \left[g^2 T_f \delta^{ab} \left\langle (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\nu\psi)\right\rangle_{0,c}\right] \langle AA\rangle_0$$

Self-energy evaluation

The self energy has a nontrivial IR limit; let's calculate it in QCD:

Rearranging prev. expression for the one-loop gluon self energy at high density:

$$\langle AA\rangle_0 \Pi(P) \langle AA\rangle_0 = \langle A(S_I - \tfrac{1}{2}S_I^2)A\rangle_{0,c} \quad = \langle A(-\tfrac{1}{2}S_I^2)A\rangle_{0,c}$$

(need fermion loop)

$$= -\frac{(-\mathrm{i}g)^2}{2}\mathrm{tr}(T^a T^b)\langle A(\bar\psi \not{A}\psi)(\bar\psi \not{A}\psi)A\rangle_{0,c}$$

$$= \langle AA\rangle_0 \left[g^2 T_f \delta^{ab} \left\langle (\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\nu\psi)\right\rangle_{0,c}\right] \langle AA\rangle_0$$

$$\Longrightarrow \qquad \Pi(P) = g^2 T_f \delta^{ab} \left\langle (\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\nu\psi)\right\rangle_{0,c}$$

Self-energy evaluation

The self energy has a nontrivial IR limit; let's calculate it in QCD:

Rearranging prev. expression for the one-loop gluon self energy at high density:

$$\langle AA\rangle_0 \Pi(P)\langle AA\rangle_0 = \langle A(S_I - \tfrac{1}{2}S_I^2)A\rangle_{0,c} \quad = \langle A(-\tfrac{1}{2}S_I^2)A\rangle_{0,c}$$

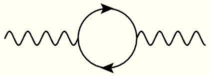
(need fermion loop)

$$= -\frac{(-ig)^2}{2}\mathrm{tr}(T^a T^b)\langle A(\bar\psi \not{A}\psi)(\bar\psi\not{A}\psi)A\rangle_{0,c}$$

$$= \langle AA\rangle_0 \left[g^2 T_f \delta^{ab}\left\langle(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\nu\psi)\right\rangle_{0,c}\right]\langle AA\rangle_0$$

$$\Longrightarrow \qquad \Pi(P) = g^2 T_f \delta^{ab}\left\langle(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\nu\psi)\right\rangle_{0,c} \quad = -g^2 T_f \delta^{ab}\mathrm{tr}[\langle\psi\bar\psi\rangle_0\gamma^\mu\langle\psi\bar\psi\rangle_0\gamma^\nu]$$

(only connected contraction; reordered the fermions)

$$=$$ 

Self-energy evaluation

The self energy has a nontrivial IR limit; let's calculate it in QCD:

Rearranging prev. expression for the one-loop gluon self energy at high density:

$$\langle AA\rangle_0 \Pi(P) \langle AA\rangle_0 = \langle A(S_I - \tfrac{1}{2}S_I^2)A\rangle_{0,c} \quad = \langle A(-\tfrac{1}{2}S_I^2)A\rangle_{0,c}$$

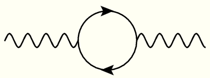
(need fermion loop)

$$= -\frac{(-\mathrm{i}g)^2}{2}\mathrm{tr}(T^a T^b)\langle A(\bar\psi \not{A}\psi)(\bar\psi \not{A}\psi)A\rangle_{0,c}$$

$$= \langle AA\rangle_0 \left[g^2 T_f \delta^{ab} \left\langle (\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\nu\psi)\right\rangle_{0,c} \right] \langle AA\rangle_0$$

$$\Longrightarrow \qquad \Pi(P) = g^2 T_f \delta^{ab} \left\langle (\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\nu\psi)\right\rangle_{0,c} \quad = -g^2 T_f \delta^{ab}\mathrm{tr}[\langle\psi\bar\psi\rangle_0 \gamma^\mu \langle\psi\bar\psi\rangle_0 \gamma^\nu]$$

(only connected contraction; reordered the fermions)

$$=$$ 

$$\Pi(P) = -g^2 T_f \delta^{ab} \int_Q \mathrm{tr}\left\{ \left[\frac{\mathrm{i}\not{Q}}{Q^2}\right] \gamma^\mu \left[\frac{\mathrm{i}(\not{P}+\not{Q})}{(P+Q)^2}\right] \gamma^\nu \right\} = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\left\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\right\}}{Q^2(P+Q)^2}$$

(Remember $Q^0 \to Q^0 + \mathrm{i}\mu$)

Self-energy evaluation; Hard Thermal/Dense Loop limit

$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\left\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\right\}}{Q^2(P+Q)^2}$$

(Remember $Q^0 \to Q^0 + i\mu$)

Self-energy evaluation; Hard Thermal/Dense Loop limit

$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\left\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\right\}}{Q^2(P+Q)^2}$$

(Remember $Q^0 \to Q^0 + i\mu$)

When $P \ll Q$, then we are looking at the UV of this integral

Self-energy evaluation; Hard Thermal/Dense Loop limit

$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\left\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\right\}}{Q^2(P+Q)^2}$$

(Remember $Q^0 \to Q^0 + i\mu$)

When $P \ll Q$, then we are looking at the UV of this integral

$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\left\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\right\}}{Q^2(P+Q)^2} \longrightarrow g^2 \int \mathrm{d}Q Q^3 \frac{Q}{Q^2}\frac{Q}{Q^2} = g^2 \int \mathrm{d}Q\, Q$$

UV dominated

Self-energy evaluation; Hard Thermal/Dense Loop limit

$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\left\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\right\}}{Q^2(P+Q)^2}$$

(Remember $Q^0 \to Q^0 + i\mu$)

When $P \ll Q$, then we are looking at the UV of this integral

$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\left\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\right\}}{Q^2(P+Q)^2} \longrightarrow g^2 \int^\mu \mathrm{d}Q Q^3 \frac{Q}{Q^2}\frac{Q}{Q^2} = g^2 \int^\mu \mathrm{d}Q\, Q$$

UV dominated

Doing residues cuts of the integral at μ:

$$\boxed{\Pi(P) \simeq g^2\mu^2 \text{ for small P}}$$

Self-energy evaluation; Hard Thermal/Dense Loop limit

$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\left\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\right\}}{Q^2(P+Q)^2}$$

(Remember $Q^0 \rightarrow Q^0 + i\mu$)

When $P \ll Q$, then we are looking at the UV of this integral

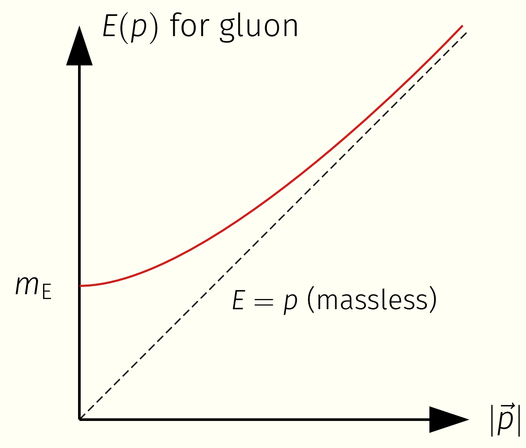
$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\left\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\right\}}{Q^2(P+Q)^2} \longrightarrow g^2 \int^\mu \mathrm{d}Q Q^3 \frac{Q}{Q^2}\frac{Q}{Q^2} = g^2 \int^\mu \mathrm{d}Q\, Q$$

UV dominated

Doing residues cuts of the integral at μ:

$$\boxed{\Pi(P) \simeq g^2\mu^2 \text{ for small P}}$$

gluon has a ***thermal mass***!



Self-energy evaluation; Hard Thermal/Dense Loop limit

$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\,\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\}}{Q^2(P+Q)^2}$$

(Remember $Q^0 \to Q^0 + i\mu$)

When $P \ll Q$, then we are looking at the UV of this integral

$$\Pi(P) = g^2 T_f \delta^{ab} \int_Q \frac{\mathrm{tr}\,\{\not{Q}\gamma^\mu(\not{P}+\not{Q})\gamma^\nu\}}{Q^2(P+Q)^2} \longrightarrow g^2 \int^\mu \mathrm{d}Q Q^3 \frac{Q}{Q^2}\frac{Q}{Q^2} = g^2 \int^\mu \mathrm{d}Q\, Q$$

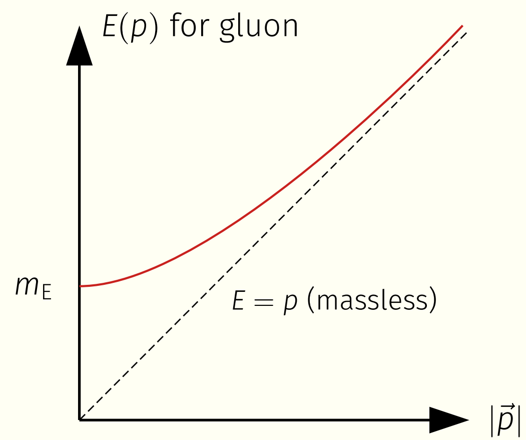
UV dominated

Doing residues cuts of the integral at μ:

$$\boxed{\Pi(P) \simeq g^2\mu^2 \text{ for small P}}$$

gluon has a ***thermal mass***!

"Hard thermal/dense loops"



Braaten & Pisarski, Phys. Rev. D 42 (1990), 46 (1992); in cold QM context: Manuel, Phys. Rev. D 53 (1996)

Self-energy evaluation; Hard Thermal/Dense Loop limit

Nontrivial dependence on $P^0/|\vec{p}|$ in the HTL result (so more than just a thermal mass):

$$\Pi^{\mu\nu}_{ab}(P) = m_{\mathrm{E}}^2 \int_{\hat{v}} \left(\delta^{\mu 0}\delta^{\nu 0} - \frac{\mathrm{i}P^0}{P\cdot V} V^\mu V^\nu \right)$$

$$m_{\mathrm{E}} \equiv \sum_f \frac{g^2\mu_f^2}{2\pi^2}, \quad V^\mu \equiv (-\mathrm{i}, \hat{v}), \quad \hat{v} \in S^2 \text{ (unit vector in } \mathbb{R}^3), \quad \int_{\hat{v}} \text{normalized to 1}$$

Self-energy evaluation; Hard Thermal/Dense Loop limit

Nontrivial dependence on $P^0/|\vec{p}|$ in the HTL result (so more than just a thermal mass):

$$\Pi^{\mu\nu}_{ab}(P) = m_{\mathrm{E}}^2 \int_{\hat{v}} \left(\delta^{\mu 0}\delta^{\nu 0} - \frac{\mathrm{i}P^0}{P \cdot V} V^\mu V^\nu \right)$$

$$m_{\mathrm{E}} \equiv \sum_f \frac{g^2 \mu_f^2}{2\pi^2}, \quad V^\mu \equiv (-\mathrm{i}, \hat{v}), \quad \hat{v} \in S^2 \text{ (unit vector in } \mathbb{R}^3), \quad \int_{\hat{v}} \text{normalized to 1}$$

Nontrivial functional dependence: $-\dfrac{iP^0}{|\vec{p}|} \ln \left(\dfrac{\mathrm{i}P^0 + |\vec{p}|}{\mathrm{i}P^0 - |\vec{p}|} \right)$

Self-energy evaluation; Hard Thermal/Dense Loop limit

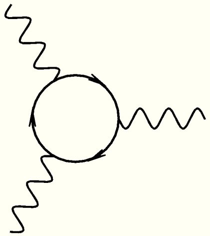
Nontrivial dependence on $P^0/|\vec{p}|$ in the HTL result (so more than just a thermal mass):

$$\Pi^{\mu\nu}_{ab}(P) = m_{\mathrm{E}}^2 \int_{\hat{v}} \left(\delta^{\mu 0}\delta^{\nu 0} - \frac{\mathrm{i}P^0}{P \cdot V} V^\mu V^\nu \right)$$

$$m_{\mathrm{E}} \equiv \sum_f \frac{g^2 \mu_f^2}{2\pi^2}, \quad V^\mu \equiv (-\mathrm{i}, \hat{v}), \quad \hat{v} \in S^2 \text{ (unit vector in } \mathbb{R}^3\text{)}, \quad \int_{\hat{v}} \text{normalized to 1}$$

Nontrivial functional dependence: $-\dfrac{iP^0}{|\vec{p}|} \ln\left(\dfrac{\mathrm{i}P^0 + |\vec{p}|}{\mathrm{i}P^0 - |\vec{p}|}\right)$

Similar HTL contributions for N-point gluon functions:



Corrections to the EoS from different kinematic regions

Current state-of-the-art: contributions from different kinematic regions

$$p = \boxed{\underbrace{p_0}_{\text{free quark gas}} + p_1^h \alpha_s + p_2^h \alpha_s^2 \quad + p_3^h \alpha_s^3} \quad \leftarrow \text{ scale } |P| \gtrsim \mu$$

$$\boxed{+ \underbrace{p_2^s \alpha_s^2}_{\text{free soft pressure (screened)}} \quad + p_3^s \alpha_s^3} \quad \leftarrow \text{ scale } |P| \lesssim m_E$$

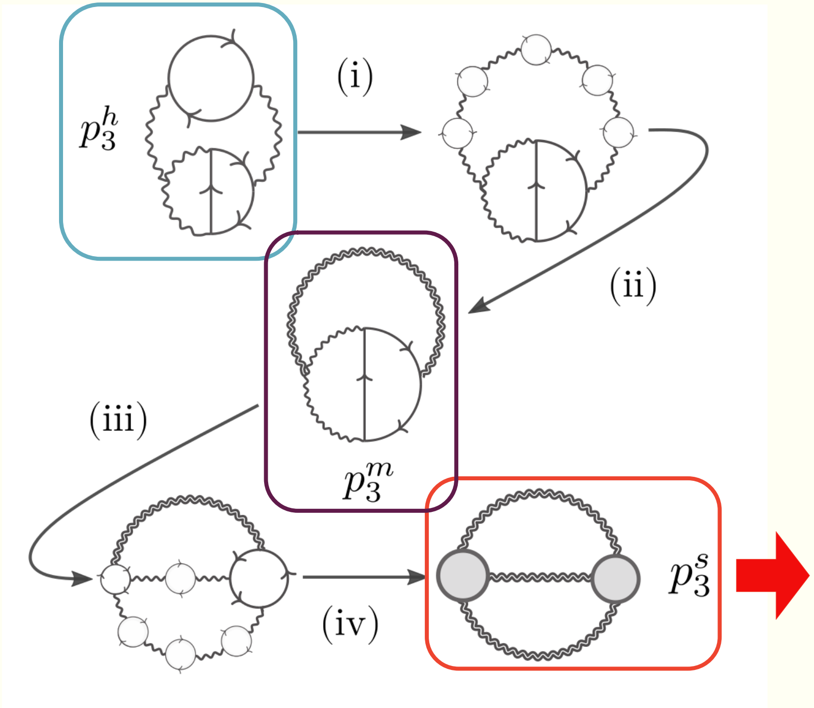
$$\boxed{+ p_3^m \alpha_s^3} \quad \leftarrow \textit{mixed}\text{; both scales}$$

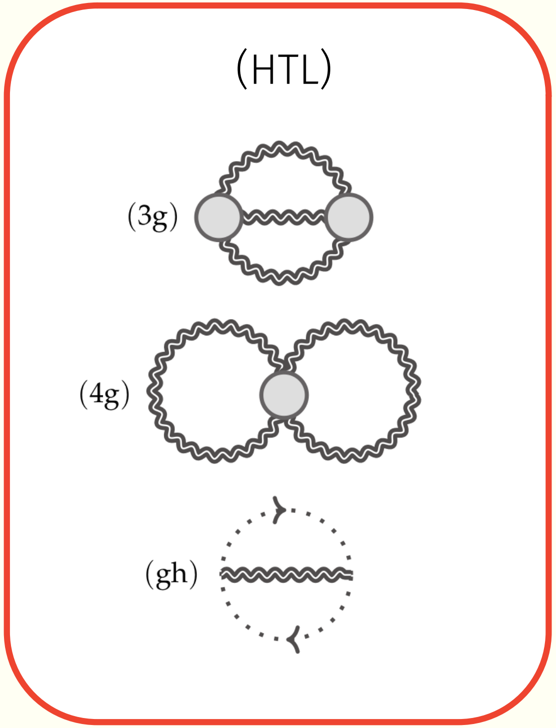
TG+ Phys. Rev. D 104 (2021), Phys. Rev. Lett. 127 (2021);
see also TG+ Phys. Rev. Lett. 121 (2018); $O(\alpha_s^2)$: Freedman & McLerran Phys. Rev. D 16 (1977)

Corrections to the EoS from different kinematic regions

Current state-of-the-art: have now computed N^{3}LO contributions from ***HTL effective theory***

TG, Kurkela, Paatelainen, Säppi, Vuorinen, Phys. Rev. Lett. 127 (2021), Phys. Rev. D 104 (2021)

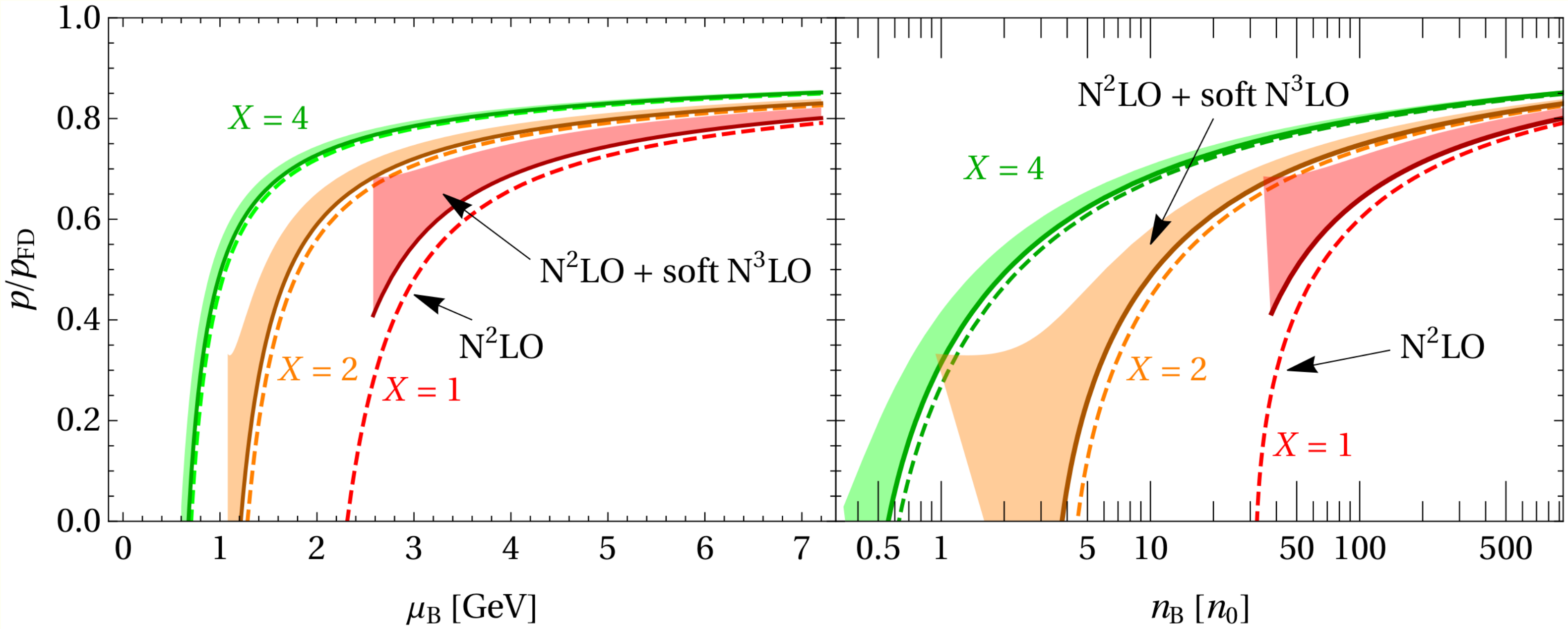




Corrections to the EoS from different kinematic regions

Current state-of-the-art: have now computed N^{3}LO contributions from ***HTL effective theory***

TG, Kurkela, Paatelainen, Säppi, Vuorinen, Phys. Rev. Lett. 127 (2021), Phys. Rev. D 104 (2021)



Decreases renormalization-scale sensitivity

Concluding remarks

- TQFT at high density is systematically improvable framework for calculating corrections to thermodynamic properties
- Rich EFT structure, involving multiple scales
- Current state-of-the-art for cold QM calculations are N3LO, and are ongoing, using approach of kinematic regions
- Cold QM EoS restricts NS-matter EoS at lower densities

References to books:

Two books I referenced, and one I recommend for general QFT:

- A. Altland, B. Simons. Condensed matter field theory. Cambridge, Univ. Press (2006).
- M. Laine, A. Vuorinen. Basics of thermal field theory. Lect. Notes Phys. 925 (2016).
- M. Schwartz. Quantum field theory and the standard model. Cambridge, Univ. Press (2013).