Solving numerically the JIMWLK equation with the running coupling constant

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Outline

Plan of the talk:

- Short motivation: origins of the JIMWLK equation
- Initial condition: MV and gaussian models
- Evolution in momentum and position space
- Running coupling effects
- Finite volume effects
- Conclusions

References

- P. Korcyl, arXiv:2009.02045, nearly accepted in Elsevier's SoftwareX
- S. Cali *et al.*, Eur. Phys. J. C81 (2021) 663
- new insights (with K. Cichy and L. Motyka)
- H. Weigert, Nucl. Phys. A 703 (2002) 823
- K. Rummukainen, H. Weigert, Nucl. Phys. A 739 (2004) 183
- T. Lappi, H. Mäntysaari, Eur. Phys. J. C73 (2013) 2307

Motivation

Physical situation: Deep Inelastic Scattering

- $\gamma^* A$ collision - scattering of a virtual photon off a nuclear target, proton or large nuclei

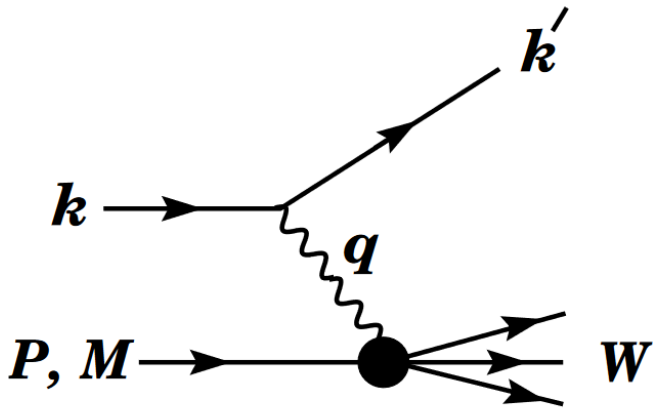


Figure 19.1: Kinematic quantities for the description of deep inelastic scattering. The quantities k and k' are the four-momenta of the incoming and outgoing leptons, P is the four-momentum of a nucleon with mass M, and W is the mass of the recoiling system X. The exchanged particle is a γ, $W^{\pm}$, or Z; it transfers four-momentum $q = k - k'$ to the nucleon.

[Particle Data Group Review, 2015]

Evolution equations

BFKL

- The DIS cross-section is expressed in terms of parton distributions f_i
- using DGLAP equations it is possible to calculate the Q^2 dependence of parton distributions f_i as long as Q^2 is large
- going to larger Q^2, the number of quark and gluons increases, their size decreases as $1/Q$
- at small x, the growth is particularly pronounced, moving towards small x at fixed Q^2 increases the number of gluons of fixed size $1/Q$

The BFKL equation takes the form

$$\frac{\partial f_a(x, k_T)}{\partial \ln x_0/x} \approx \sum_b K_{ab} \otimes f_b(x, k_T)$$

where this time $\otimes$ involves an integration over the transverse plane $\int_0^\infty dl_T^2$.

Evolution equations

BK and JIMWLK

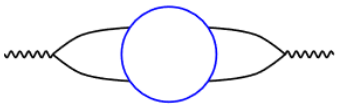
- both DGLAP and BFKL are linear evolution equations
- at very small x gluon recombination terms become equally important leading to a non-linear equation

BK-JIMWLK equations:

$$\frac{\partial f(x,k_T)}{\partial \ln x_0/x} \approx K \otimes f(x,k_T) - f^2(x,k_T)$$

- saturation scale: transverse momentum scale that determines the onset of non-linear corrections in QCD evolution equations
- $Q_S \approx A^{\frac{1}{3}} \left(\frac{1}{x}\right))^{0.2-0.3}$

Origins of the JIMWLK equation

$$\sigma_{\mathrm{DIS}}(x_{\mathrm{Bj}}, Q^2) = \mathrm{Im}$$  $$= \int d^2r\, |\psi^2|(r^2Q^2) \int d^2b\, \langle \frac{\mathrm{tr}(1 - U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}})}{N_c} \rangle$$

$$\sigma_{\mathrm{dipole}}(\tau = \ln(1/x), r^2) := \int d^2b\, \langle \frac{\mathrm{tr}(1 - U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}})}{N_c} \rangle_{\tau = \ln(1/x)}$$

[K. Rummukainen, H. Weigert, Nucl. Phys. A, 2004]

JIMWLK as Langevin equation

$$U(\boldsymbol{x}; s+\delta s) = U(\boldsymbol{x}; s)\exp[it^a\omega^a(\boldsymbol{x}; s)]$$

$$\begin{aligned}\omega^a(\boldsymbol{x}, s) &= \sqrt{\delta s}\sum_{\boldsymbol{y}} K_i(\boldsymbol{x}-\boldsymbol{y})[1-\tilde{U}^\dagger(\boldsymbol{x}; s)\tilde{U}(\boldsymbol{y}; s)]^{ab}\xi_i^b(\boldsymbol{y}) \\ &- \delta s\sum_{\boldsymbol{y}} S(\boldsymbol{x}-\boldsymbol{y})\frac{1}{2}\tilde{\mathrm{tr}}[i\tilde{t}^a\tilde{U}^\dagger(\boldsymbol{x}, s)\tilde{U}(\boldsymbol{y}, s)].\end{aligned}$$

$$K_i(\boldsymbol{r}) = \frac{\boldsymbol{r}_i}{\boldsymbol{r}^2}\,, \qquad S(\boldsymbol{r}) = \frac{1}{\boldsymbol{r}^2},$$

[K. Rummukainen, H. Weigert, Nucl. Phys. A, 2004]

JIMWLK as Langevin equation

A more symmetric formulation was proposed by T. Lappi and H. Mäntysaari where the evolution is applied from the left and right side of U_x,

$$U(\mathrm{x}, s+\delta s) = \exp\left(-\sqrt{\delta s}\sum_{\mathrm{y}} U(\mathrm{y}, s)(\mathrm{K}(\mathrm{x}-\mathrm{y})\cdot\xi(\mathrm{y}))U^{\dagger}(\mathrm{y}, s)\right)\times$$
$$\times\, U(\mathrm{x}, s)\times$$
$$\times \exp\left(\sqrt{\delta s}\sum_{\mathrm{y}} \mathrm{K}(\mathrm{x}-\mathrm{y})\cdot\xi(\mathrm{y})\right).$$

Initial condition

BFKL, BK and JIMWLK equations predict the dependence on x of the unintegrated distribition $f(x, k_T)$. We need the shape of the distribution at some initial value of x_0.

Possible inputs:

- Measurement: extract the distribution from fits to experimental data,
- Lattice QCD: calculate the object non-perturbatively,
- Model: use some approximate description.

Initial condition

Color Glass Condensate

- CGC is an effective theory for the description of high-energy scattering in QCD
- valence degrees of freedom at large-x - static sources
- small-x degrees of freedom - dynamical fields

Solve classical Yang-Mills equations

$$[D_\mu, F^{\mu\nu}] = J^\nu$$

$$\mathcal{L} = -\frac{1}{4} F^2 + J \cdot A$$

$$J^\nu = \rho(x_T)\delta(x_-)\delta^{+\nu}$$

assuming eikonal coupling to dynamical fields and random distribution of color charges $\rho(x_T)$ from some probability weight $W[\rho(x_t)]$.

Initial condition

McLerran-Venugopalan model

- generate random distributions $\rho(x_T)$ with a Gaussian $W[\rho(x_t)]$
- solve Y-M equations
- calculate the distribution

Numerical formulation

- work with the 2D lattice in the transverse plane
- $\rho(\mathrm{x})_n = \rho(\mathrm{x})^a_n \lambda^a$ and $\langle \rho(\mathrm{x})^a_n \rho(\mathrm{y})^b_l \rangle = \delta^{ab}\delta^{nl}\delta(\mathrm{x}-\mathrm{y})\frac{g^2\mu^2}{N_y}$
- we repeat this N_y times to model multiple color scatterings

Initial condition

McLerran-Venugopalan model

- solve Poisson equation to get the color potential

$$U_n^{ab}(\mathrm{x}) = \exp\left(-igA_n^{ab}(\mathrm{x})\right) = \exp\left(-i\frac{g\rho_n^{ab}(\mathrm{x})}{\nabla^2 - m^2}\right),$$

or more explicitly

$$\frac{\rho_n^{ab}(\mathrm{x})}{\nabla^2 - m^2} = \frac{1}{L_x L_y}\sum_{\mathrm{z}\in\tilde{\Lambda}}\sum_{\mathrm{k}\in\Lambda}\frac{e^{i\mathrm{k}(\mathrm{x}-\mathrm{z})}\rho_n^{ab}(\mathrm{z})}{-\frac{4}{a^2}\left[\sin^2\left(\frac{k_x a}{2}\right)+\sin^2\left(\frac{k_y a}{2}\right)\right]-m^2},$$

- calculate correlation function

$$C(\mathrm{x}-\mathrm{y}) = \langle \mathrm{tr}\, U^\dagger(\mathrm{x})U(\mathrm{y})\rangle.$$

- physical parameters: $g^2\mu$, L, m, a

Initial condition

Gaussian model

- work with the 2D lattice in the transverse plane
- $\rho(\mathrm{x})_n = \rho(\mathrm{x})^a_n \lambda^a$ and $\langle \rho(\mathrm{x})^a_n \rho(\mathrm{y})^b_l \rangle = \delta^{ab}\delta^{kl} \exp(-(x-y)^2/(2R^2))$
- we repeat this N_y times to model multiple color scatterings
- physical parameters: R, L, a

MV model: initial condition

Fixed volume: $g^2\mu L = 30.7$.
Different lattice spacings: $L/a = 512, 1024, \ldots$

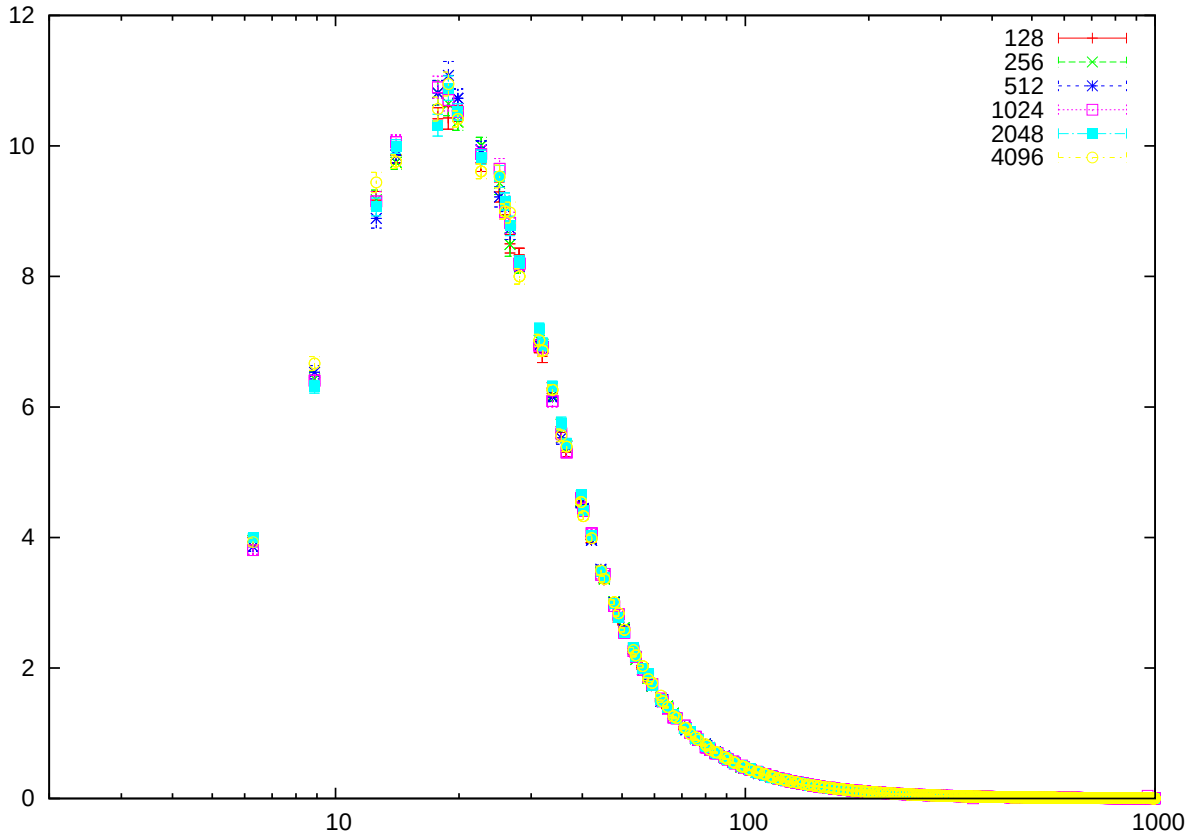


Figure: Lattice spacing dependence of the initial dipole distribution as a function of Lk_T.

MV model: initial condition

Different volumes: $g^2\mu L = 30.7, 61, 122, \ldots, 1966.$
Different lattice spacings: $L/a = 512, 1024, \ldots$

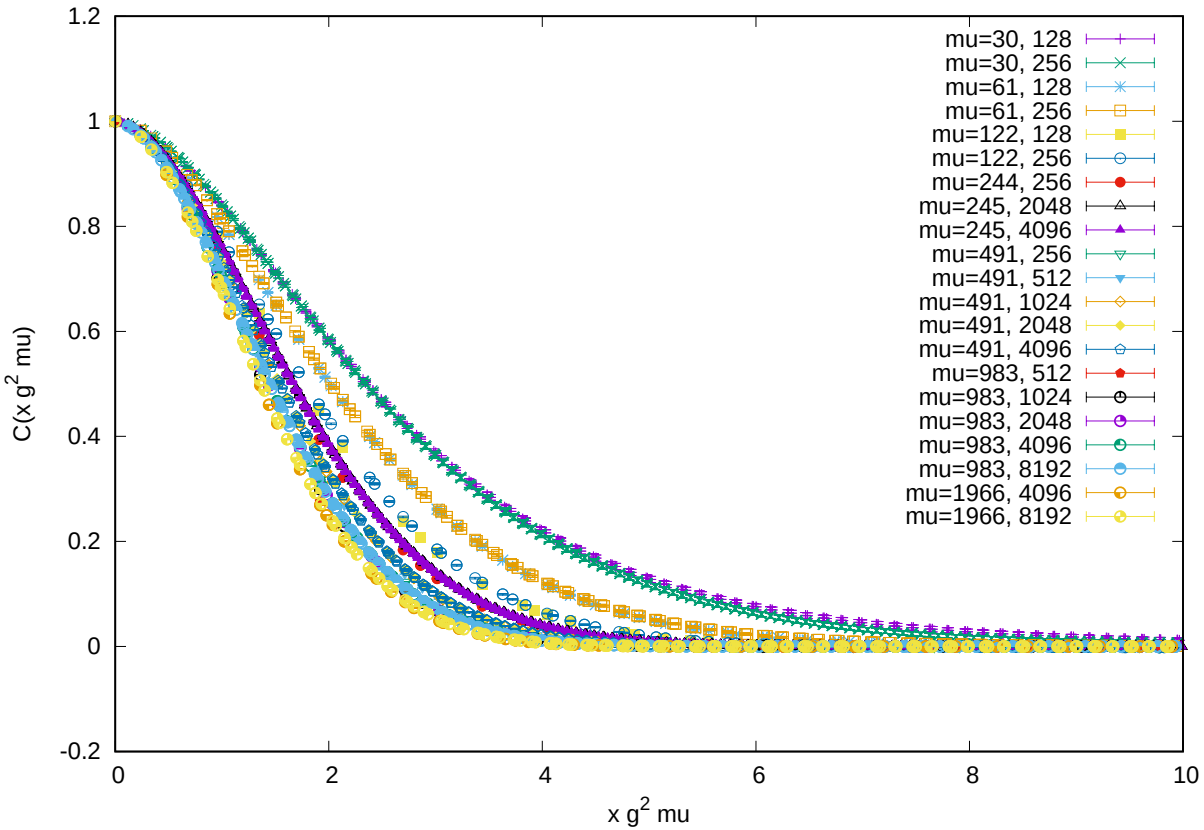


Figure: Volume and lattice spacing dependence of the initial dipole distribution as a function of $g^2\mu x$.

MV model: initial condition

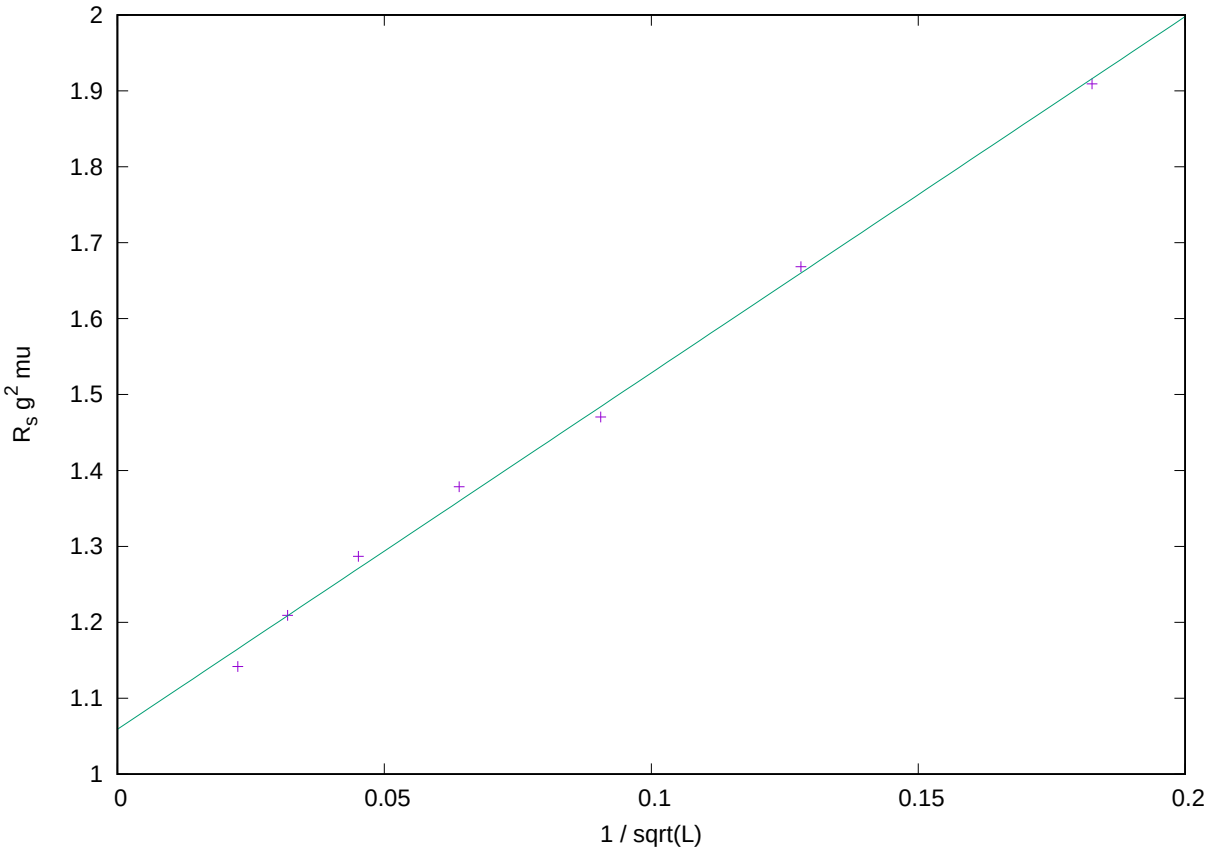


Figure: Volume dependence of the saturation radius of the initial distribution.

Gaussian model: initial condition

Different volumes: $R/L = 1/128, 1/256, \ldots$
Different lattice spacings: $L/a = 1024, 2048, \ldots$

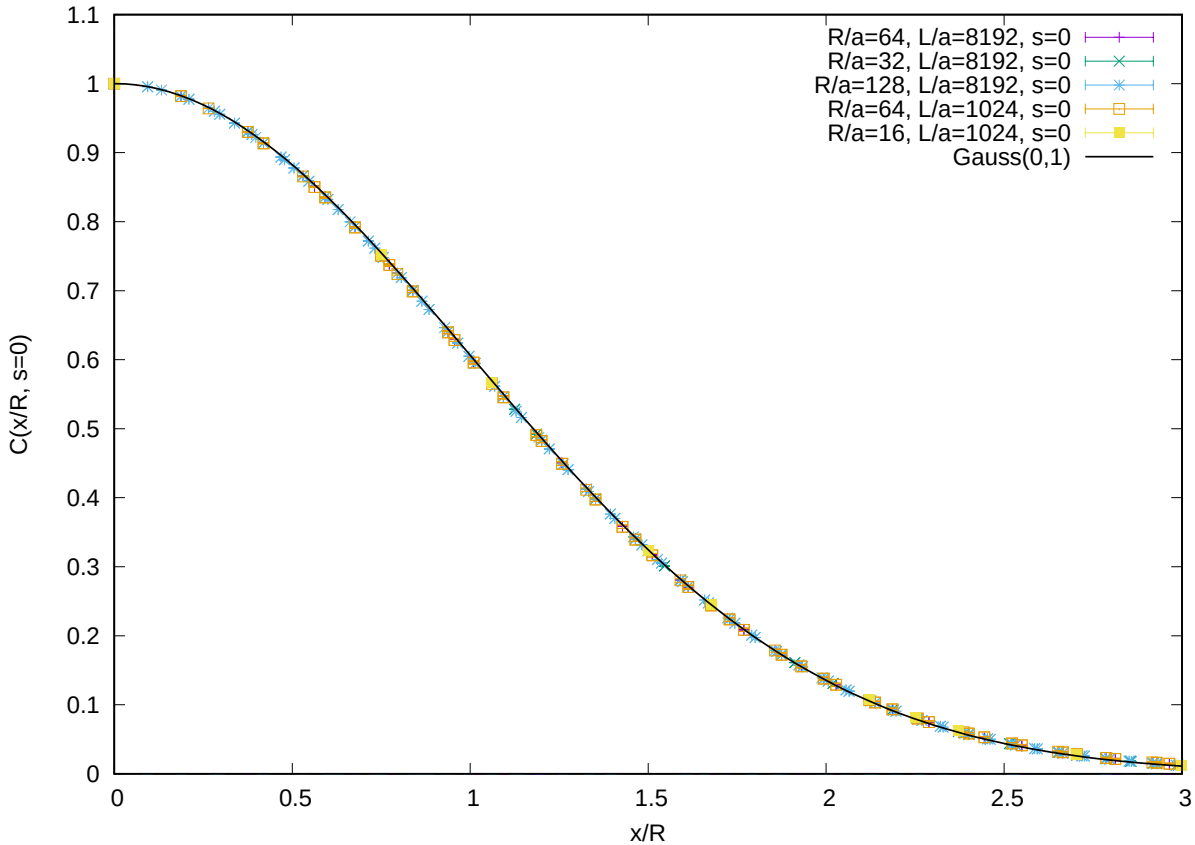


Figure: Volume and lattice spacing dependence of the initial dipole distribution as a function of x/R.

Kernel

Position space kernel

$$\mathrm{K(x)} = \frac{\mathrm{x}}{x^2} \qquad \rightarrow \qquad \mathrm{K(n)} = \frac{\bar{\mathrm{n}}}{\bar{n}^2},$$

where $\mathrm{n} = (n_x, n_y)$ is a vector of integers ($n_i \in (-L, L)$) and the numbers $\bar{n}_i$ ($\bar{n}^2$) are in the chosen discretizations:

- naive with a discontinuity:

$$\bar{n}_i = \begin{cases} n_i - L & \text{if } n_i \geq L/2, \\ n_i & \text{if } -L/2 \leq n_i < L/2, \\ n_i + L & \text{if } n_i < -L/2, \end{cases} \qquad \bar{n}^2 = \bar{n}_x^2 + \bar{n}_y^2$$

- regularized with the sine function:

$$\bar{n}_i = \frac{L}{2\pi} \sin\left(\frac{2\pi n_i}{L}\right), \qquad \bar{n}^2 = \left(\frac{L}{\pi}\right)^2 \left(\sin^2\left(\frac{\pi n_x}{L}\right) + \sin^2\left(\frac{\pi n_y}{L}\right)\right).$$

Kernel

In momentum space we get

$$\int \frac{d^2x}{2\pi} e^{-ikx} \frac{x_i}{x^2} = -2\pi i \frac{k_i}{k^2} .$$

Again, we may discretize the right hand side following one of the two ways

- as is commonly done using lattice momenta $\bar{k}$

$$\bar{k}_i = \sin\left(\frac{2\pi n_i}{L_i}\right), \qquad \bar{k}^2 = 4.0\left(\sin^2\left(\frac{\pi n_x}{L_x}\right) + \sin^2\left(\frac{\pi n_y}{L_y}\right)\right) .$$

- or alternatively keeping naive lattice momenta:

$$\bar{k}_i = 2\pi \frac{n_i}{L_i}$$

Kernel

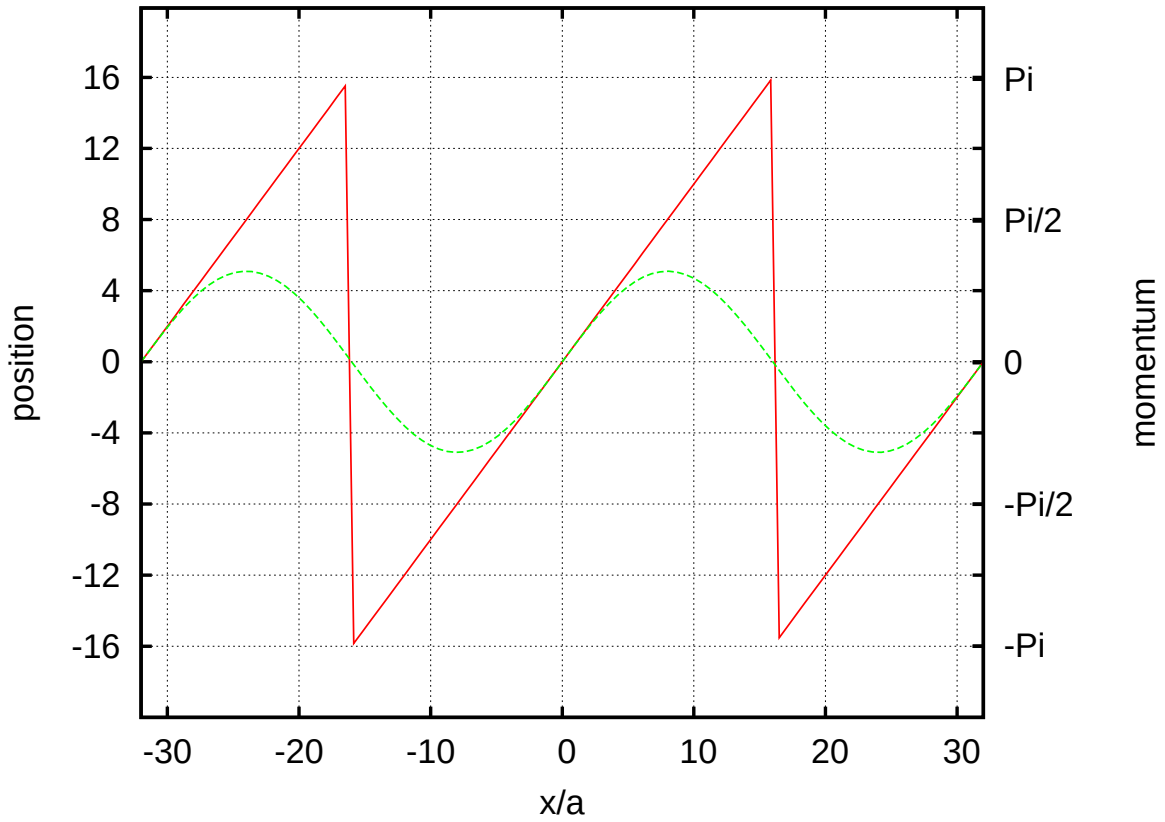


Figure: Comparison of possible discretizations of the numerator of the kernel.

Kernel

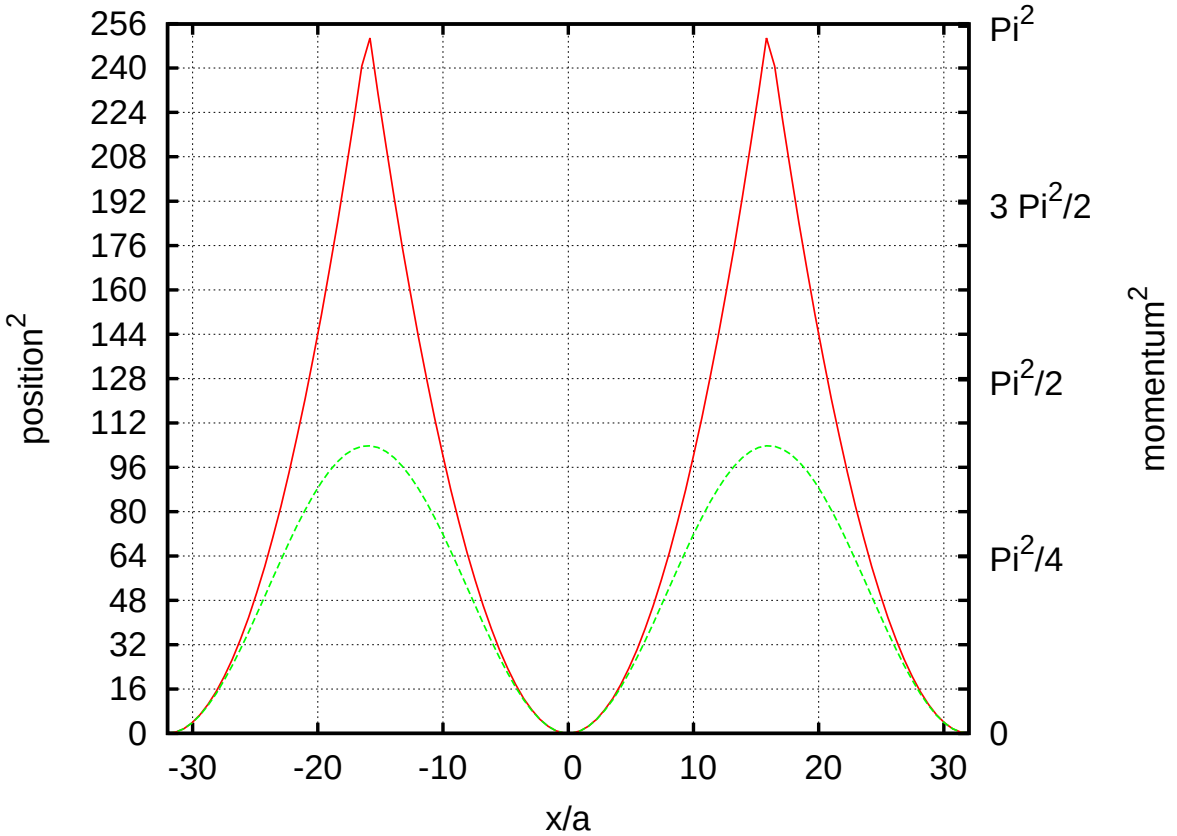


Figure: Comparison of possible discretizations of the denominator of the kernel.

Kernel

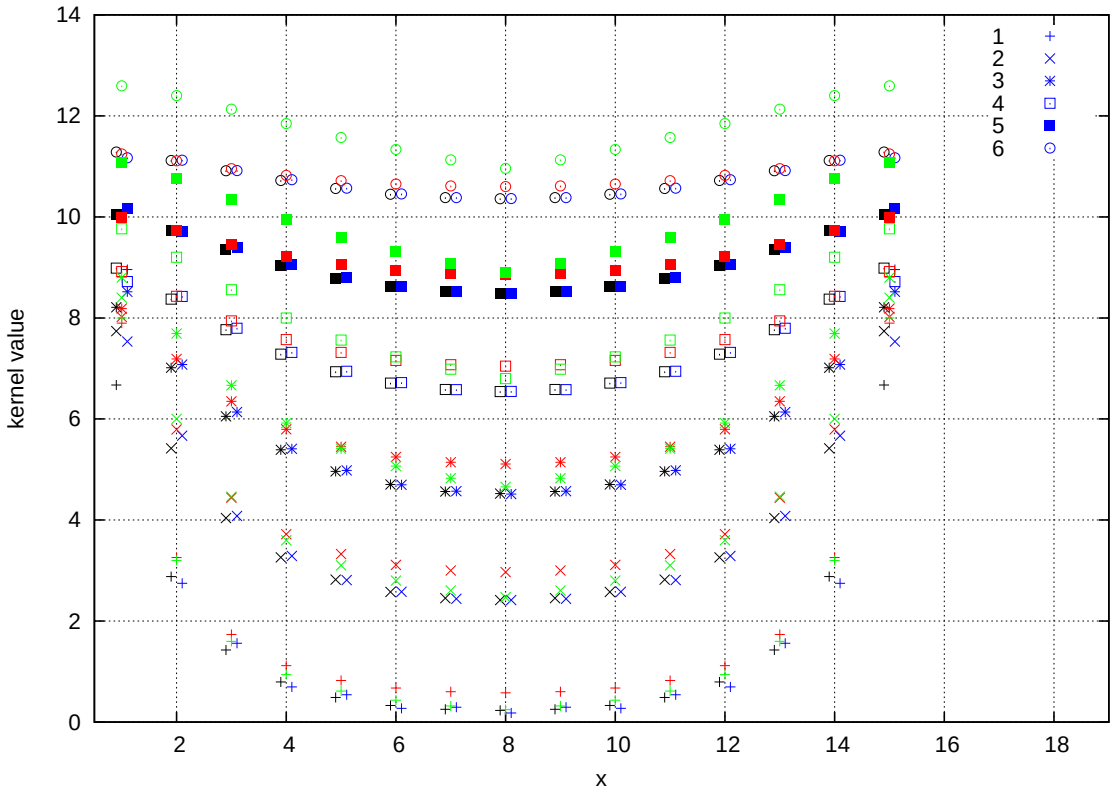


Figure: Comparison of the position and momentum kernels. The blue and black data points show the Fourier transform; the blue correspond to the naive discretization of the position space kernel, whereas the black to the sin regularized kernel. Surprisingly, both data sets agree quite well. The red data points show the corresponding momentum kernel with the usual $\bar{k}/\hat{k}^2$ definition, whereas the green data points are calculated using the simple lattice momenta.

Distribution evolution: MV model and fixed coupling

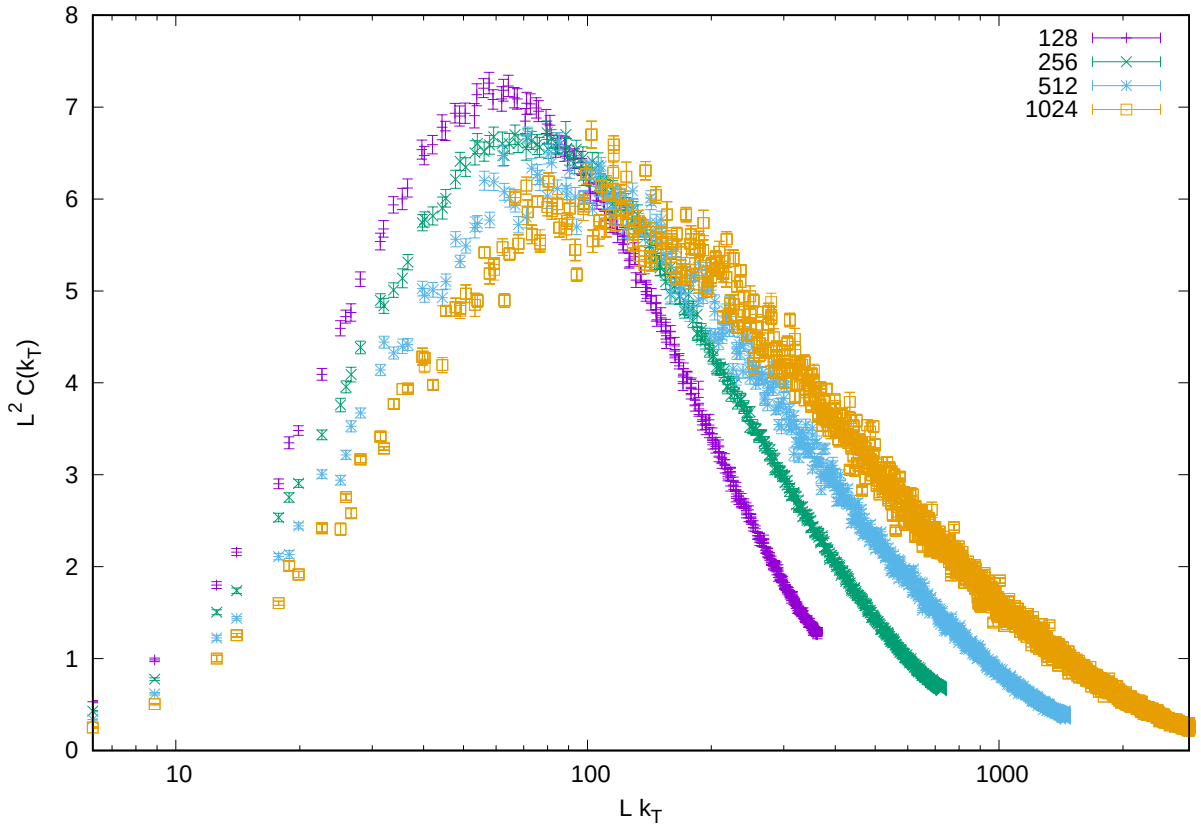


Figure: Comparison of the evolved two-point correlation function as a function of the lattice spacing. Evolution was performed with time step $\delta s = 0.0001$ and the total evolution time was $s = 0.04$.

Distribution evolution

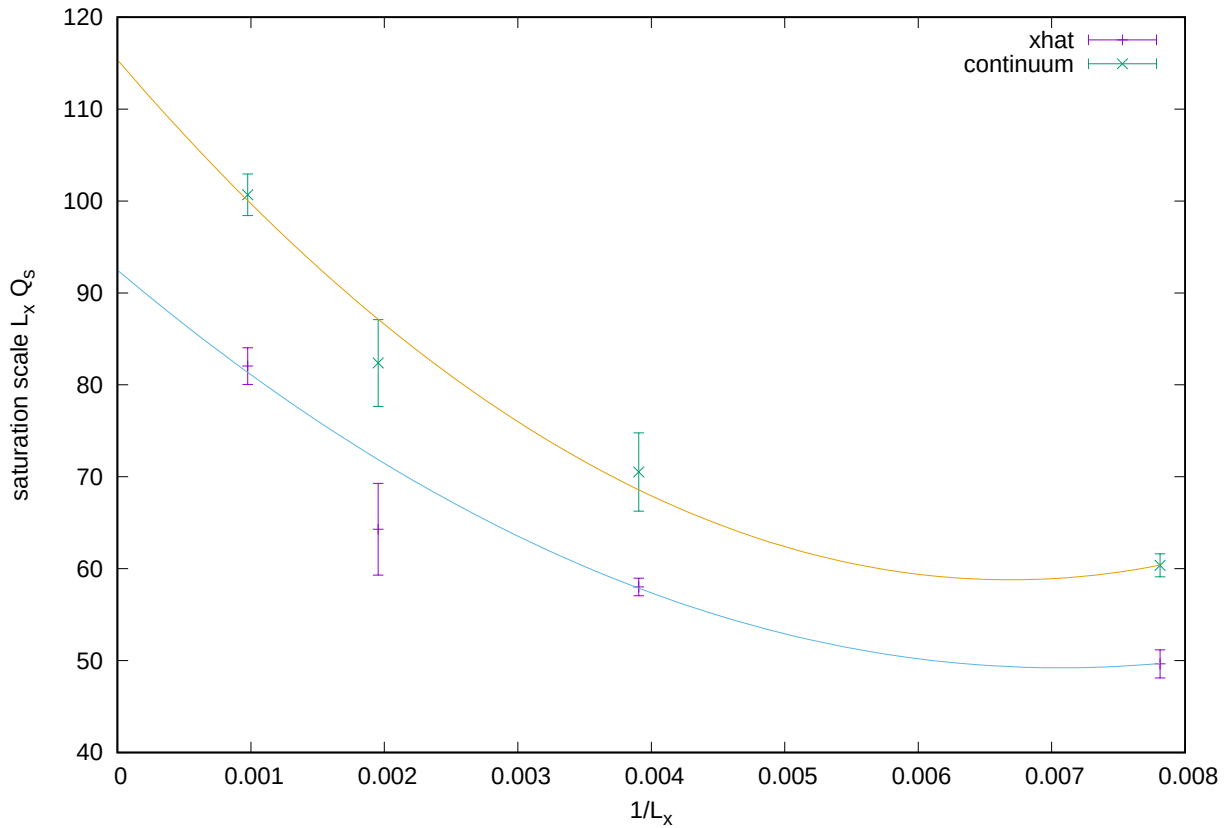


Figure: Lattice spacing dependence of the evolved saturation scale using position space formulation with continuum and lattice JIMWLK kernels. Extrapolation to infinite volume is attempted.

Running coupling effects

Recall the entire JIMWLK formulation

$$U(\mathrm{x}, s+\delta s) = \exp\left(-\sqrt{\delta s}\sum_{\mathrm{y}} U(\mathrm{y}, s)\left(\mathrm{K}(\mathrm{x}-\mathrm{y})\cdot\xi(\mathrm{y})\right)U^{\dagger}(\mathrm{y}, s)\right)\times$$
$$\times\, U(\mathrm{x}, s)\times$$
$$\times\exp\left(\sqrt{\delta s}\sum_{\mathrm{y}}\mathrm{K}(\mathrm{x}-\mathrm{y})\cdot\xi(\mathrm{y})\right),$$

where the coupling constant appears in the Langevin step

$$s = \frac{\alpha_S}{\pi^2}y, \text{ with } y = \ln\frac{x_0}{x}$$

How should we include one-loop running of the coupling constant?

"Square root" prescription

Proposed by Rummukainen and Weingert in 2003. The running coupling effects are accounted for with the coupling at the scale given by the size of the parent dipole $(x-y)^2$, i.e. we introduce a one-loop running

$$\alpha_s \to \alpha_s(1/(x-y)^2) = \frac{4\pi}{\beta_0 \ln \frac{1}{(x-y)^2\Lambda^2}},$$

with $\beta_0 = (11N_c - 2N_f)/3$. In the "square root" prescription becomes

$$U(\mathrm{x}, s+\delta s) =$$

$$= \exp\left[-\frac{\sqrt{\delta y}}{\pi}\sum_{\mathrm{y}} U(\mathrm{y},s)\left(\sqrt{\alpha_s(|\mathrm{x}-\mathrm{y}|)}\mathrm{K}(\mathrm{x}-\mathrm{y})\xi(\mathrm{y})\right)U^\dagger(\mathrm{y},s)\right]\times$$

$$\times U(\mathrm{x},s)\times$$

$$\times \exp\left[\frac{\sqrt{\delta y}}{\pi}\sum_{\mathrm{y}}\sqrt{\alpha_s(|\mathrm{x}-\mathrm{y}|)}\mathrm{K}(\mathrm{x}-\mathrm{y})\xi(\mathrm{y})\right]$$

Noise prescription

An alternative definition of the running coupling was proposed by Lappi and Mantysaari in 2012 as a modification of noise vectors. This has a different physical motivation as the scale of the running coupling will be provided by the momentum of the emitted gluon and this scale is then argued as corresponding to the smallest one of the three relevant dipole sizes (the "parent" and two "daughter" dipoles)

$$
\begin{aligned}
U(\mathrm{x}, s+\delta s) = & \\
& = \exp\left[-\sqrt{\delta s}\sum_{\mathrm{y}} U(\mathrm{y}, s)\Big(\mathrm{K}(\mathrm{x}-\mathrm{y})\eta(\mathrm{y})\Big)U^{\dagger}(\mathrm{y}, s)\right]\times \\
& \qquad \times U(\mathrm{x}, s)\times \\
& \qquad\qquad \times \exp\left[\sqrt{\delta s}\sum_{\mathrm{y}} \mathrm{K}(\mathrm{x}-\mathrm{y})\eta(\mathrm{y})\right],
\end{aligned}
$$

where now

$$
\langle \eta^{a,i}(\mathrm{x})\eta^{b,j}(\mathrm{y})\rangle = \delta^{ab}\delta^{ij}\int \frac{d^2\mathrm{k}}{(2\pi)^2} e^{i\mathrm{k}(\mathrm{x}-\mathrm{y})}\alpha_s(\mathrm{k})
$$

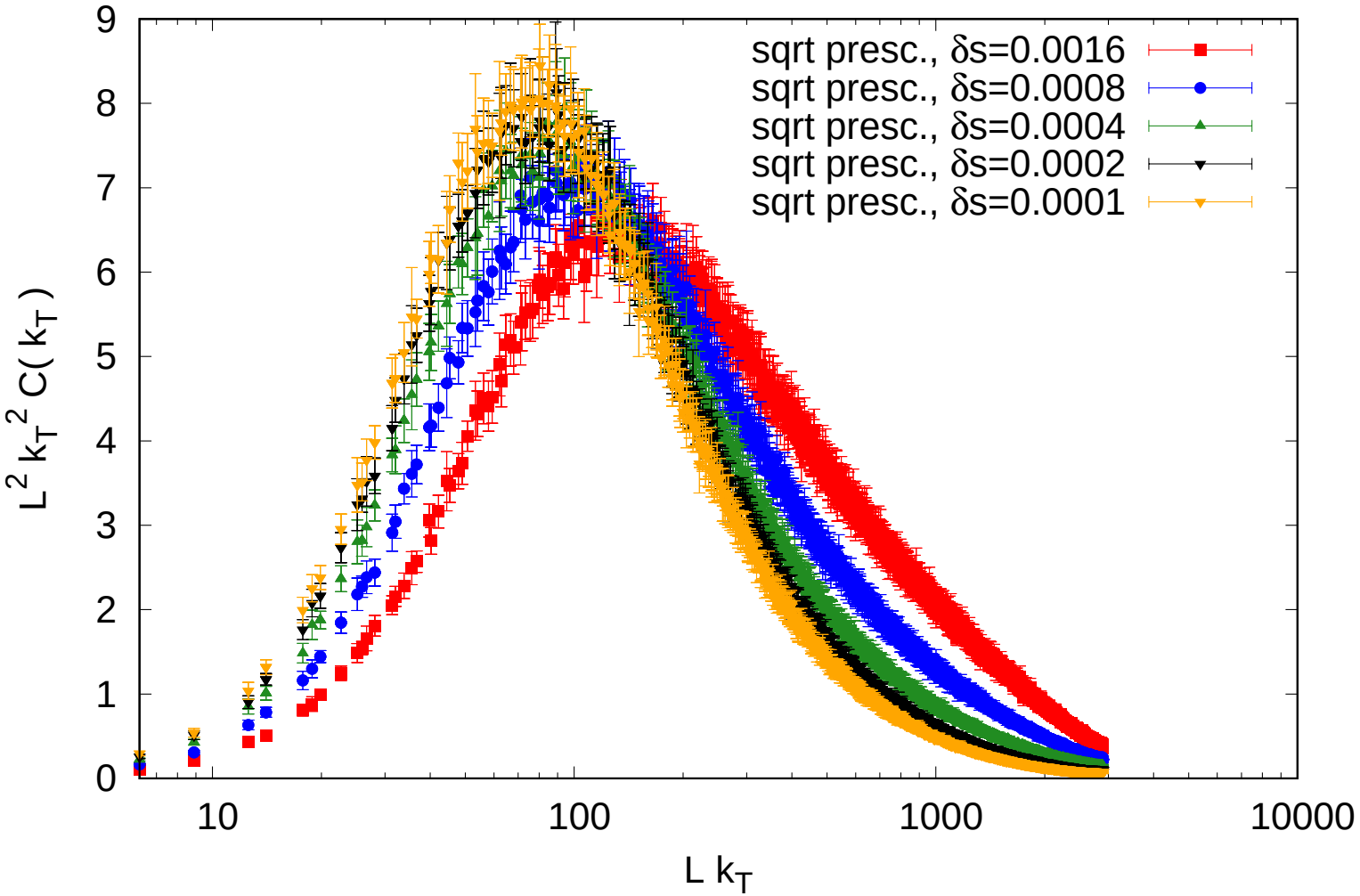
Hatta-Iancu prescription

Hatta and Iancu provided in 2016 a formulation of the JIMWLK equation with collinear resummation which accounts for DGLAP kind of logarithms suppressing collinear pole of the kernel. In particular they re-investigated the relation of momentum space expression for running of α_s to small dipole prescription. According to Hatta-Iancu the smallest dipole prescription corresponds to dependence of α_s on virtuality and not transverse momentum. We proceed with implementation directly in the coordinate space

$$\alpha_s = \alpha_s(\min\{|x-z|, r\}),$$

where r is the size of the projectile.

Evolution with running coupling constant: MV model



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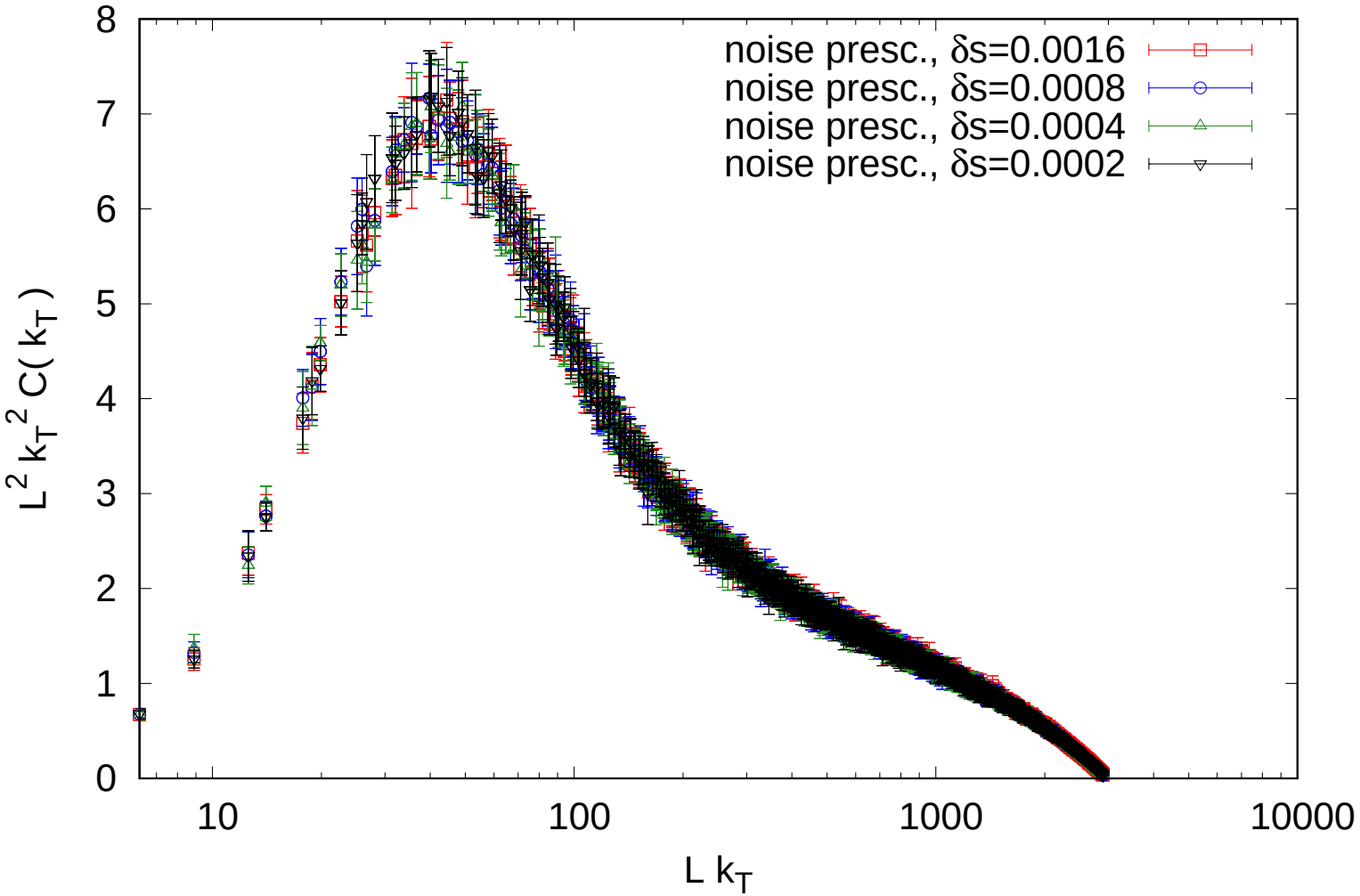

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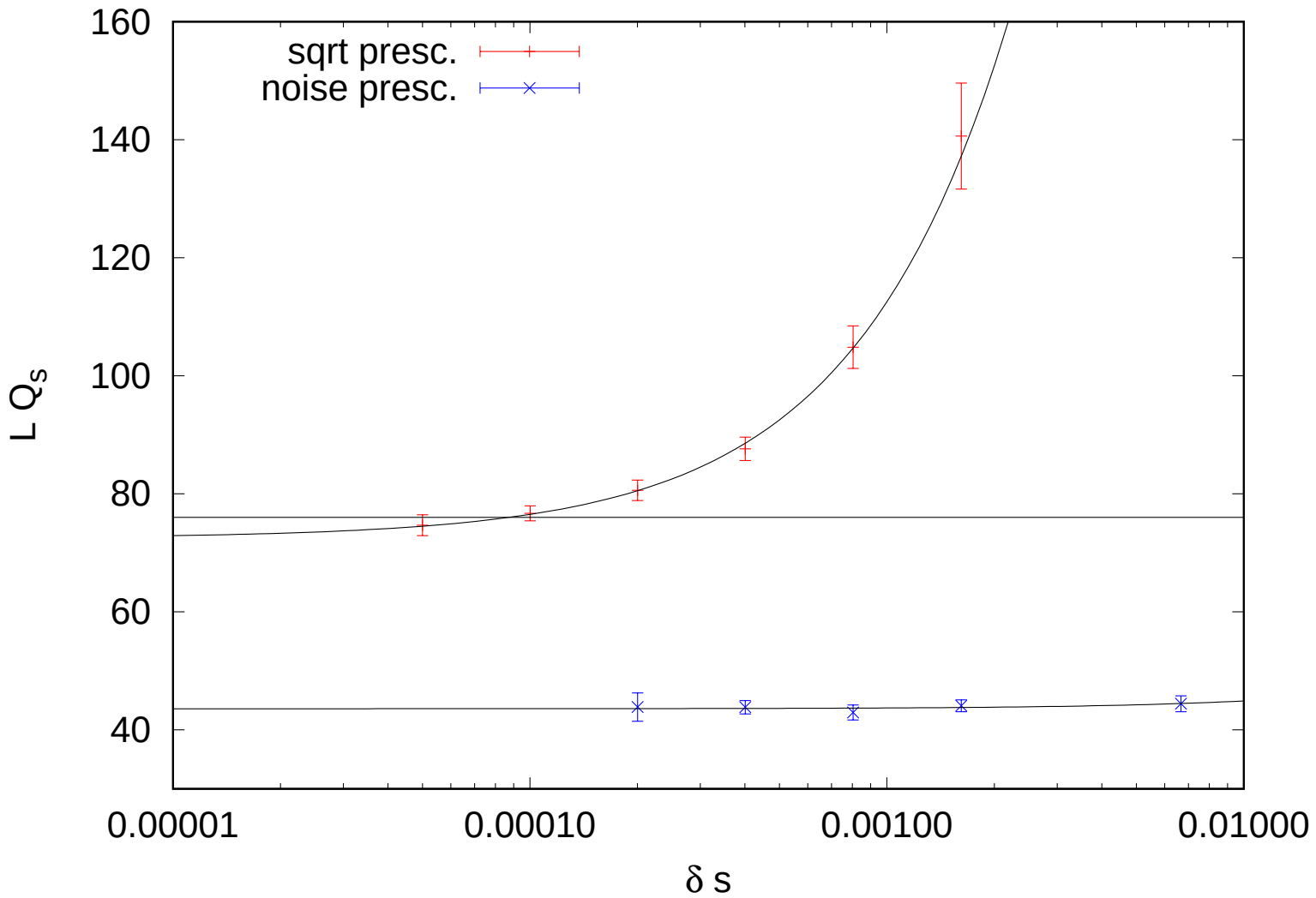

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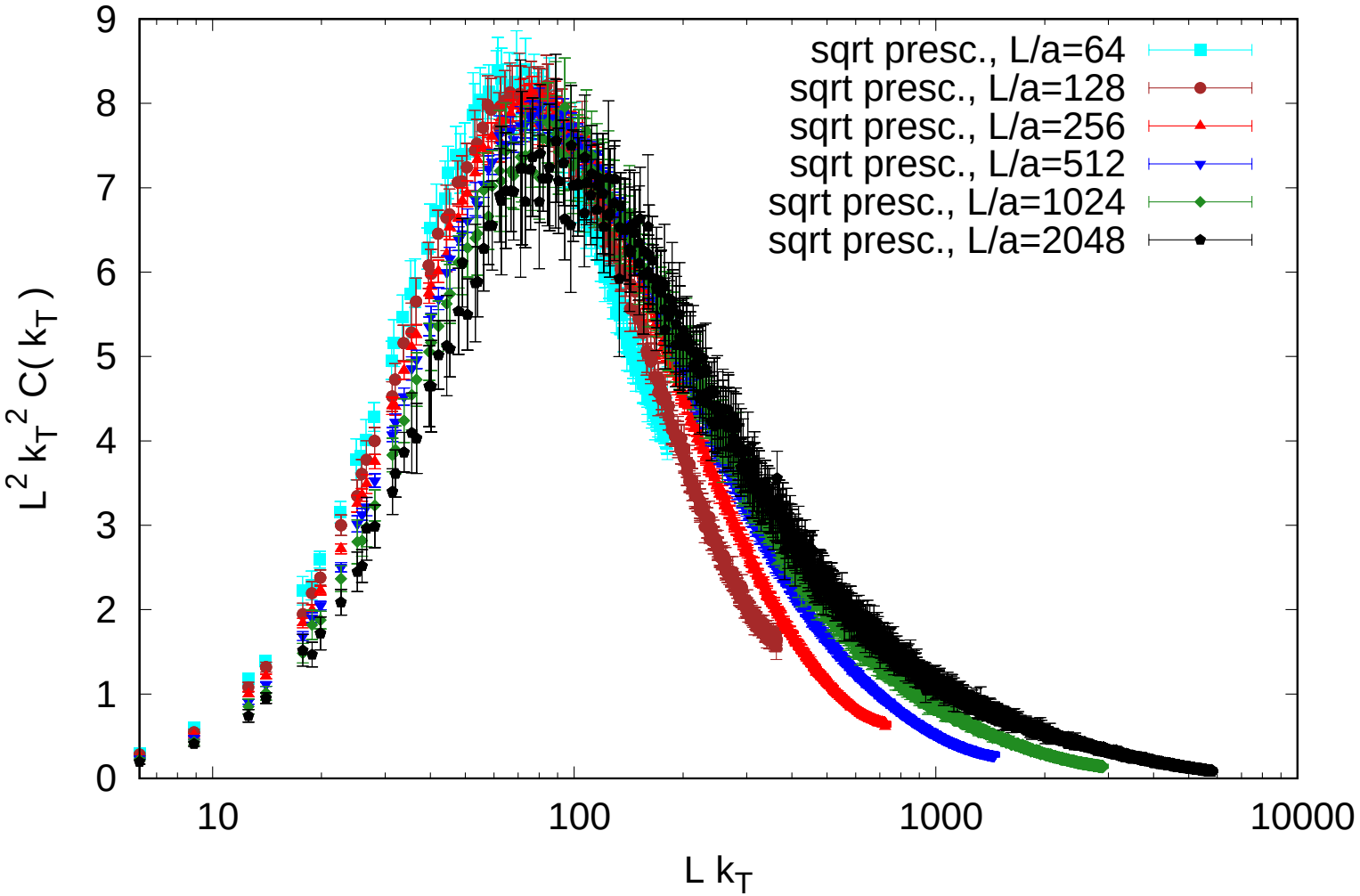

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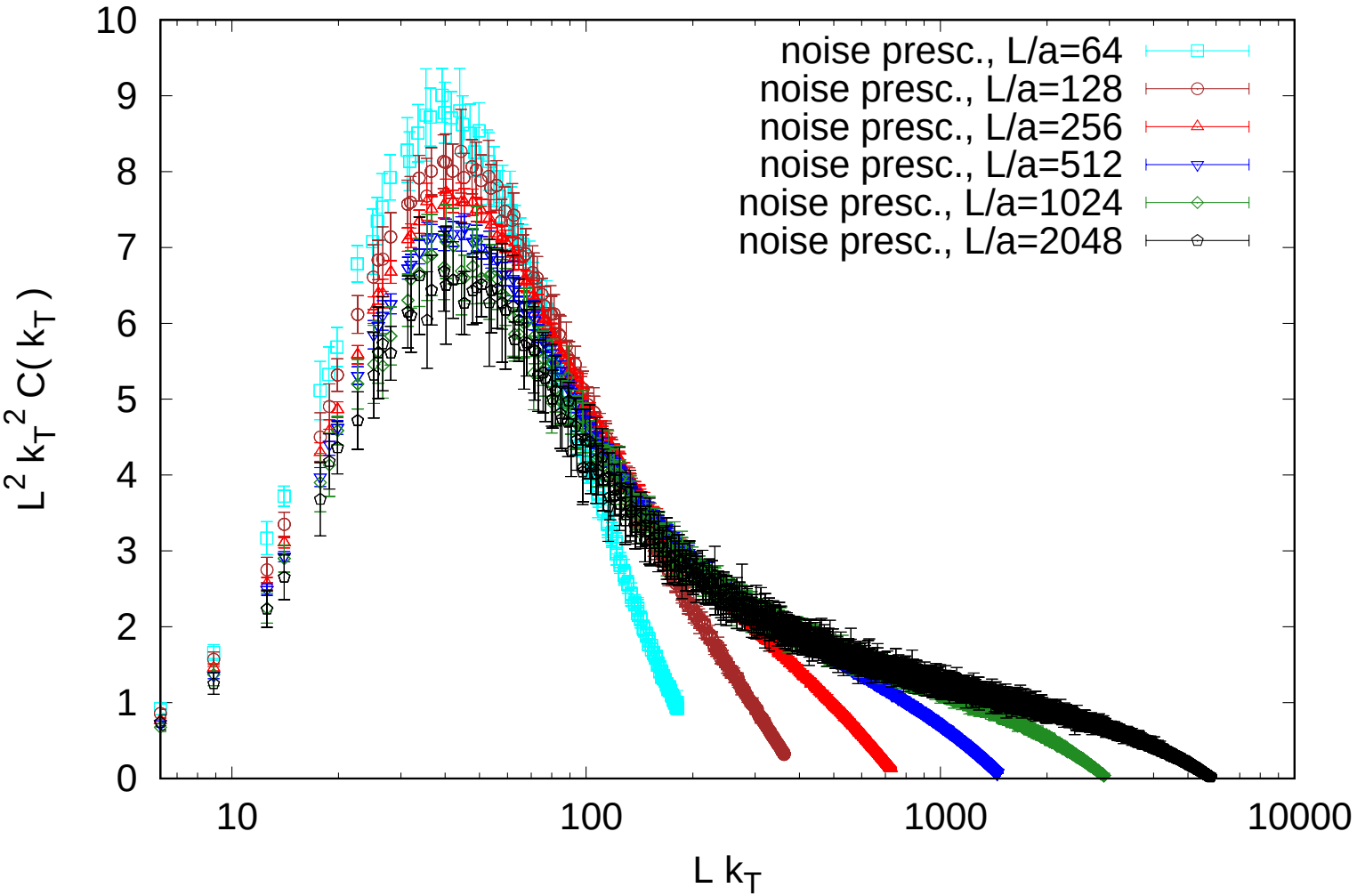

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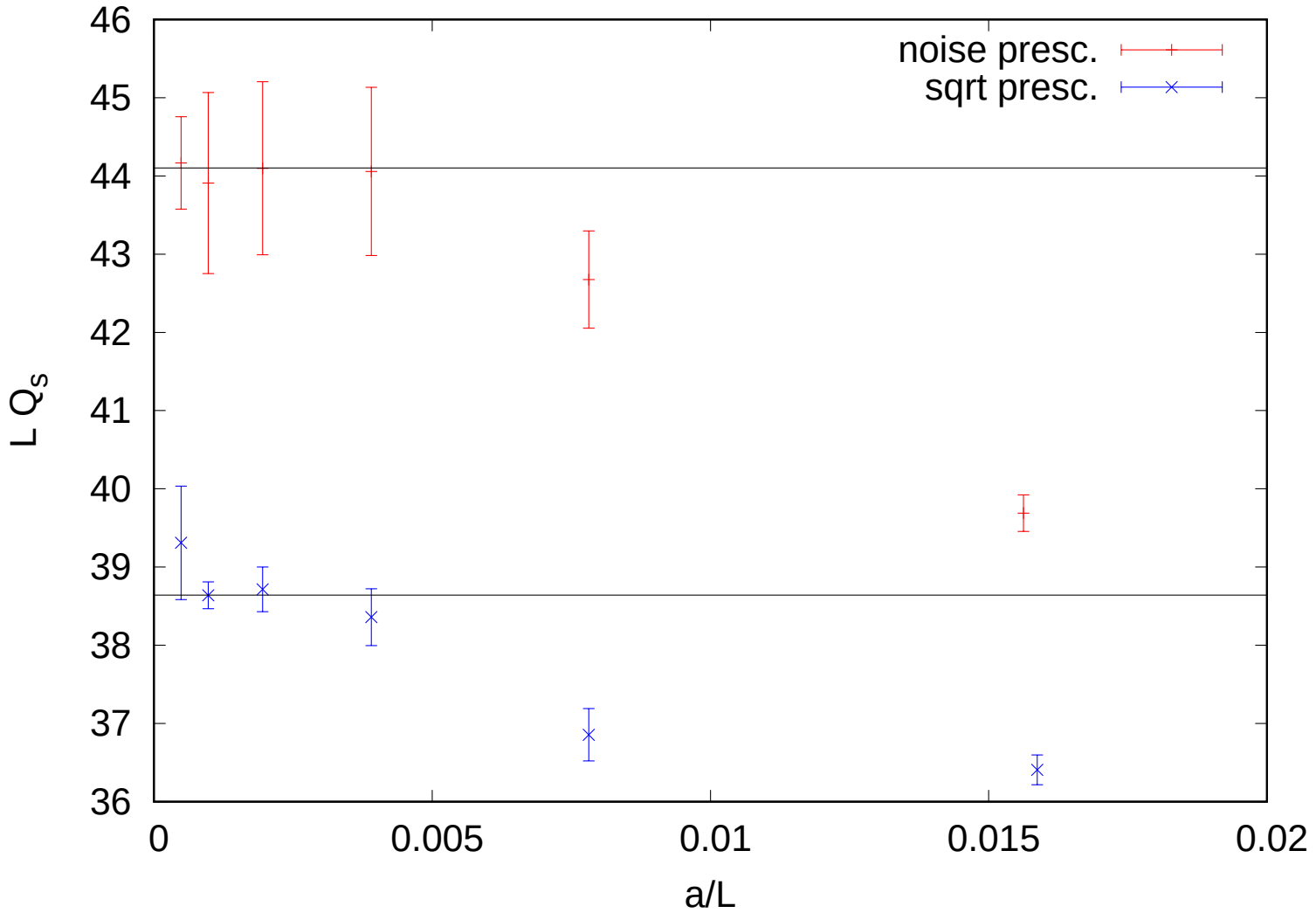

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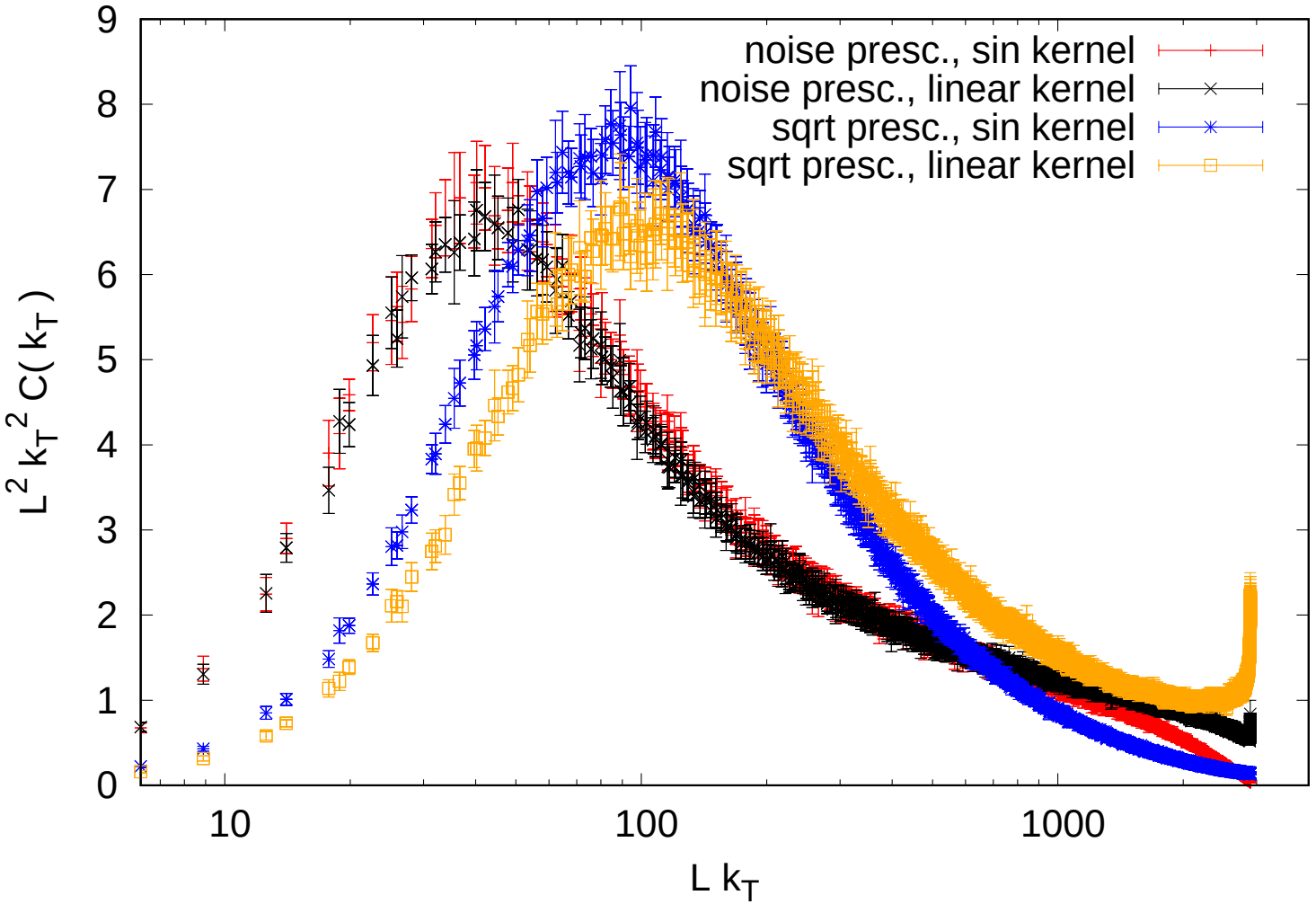
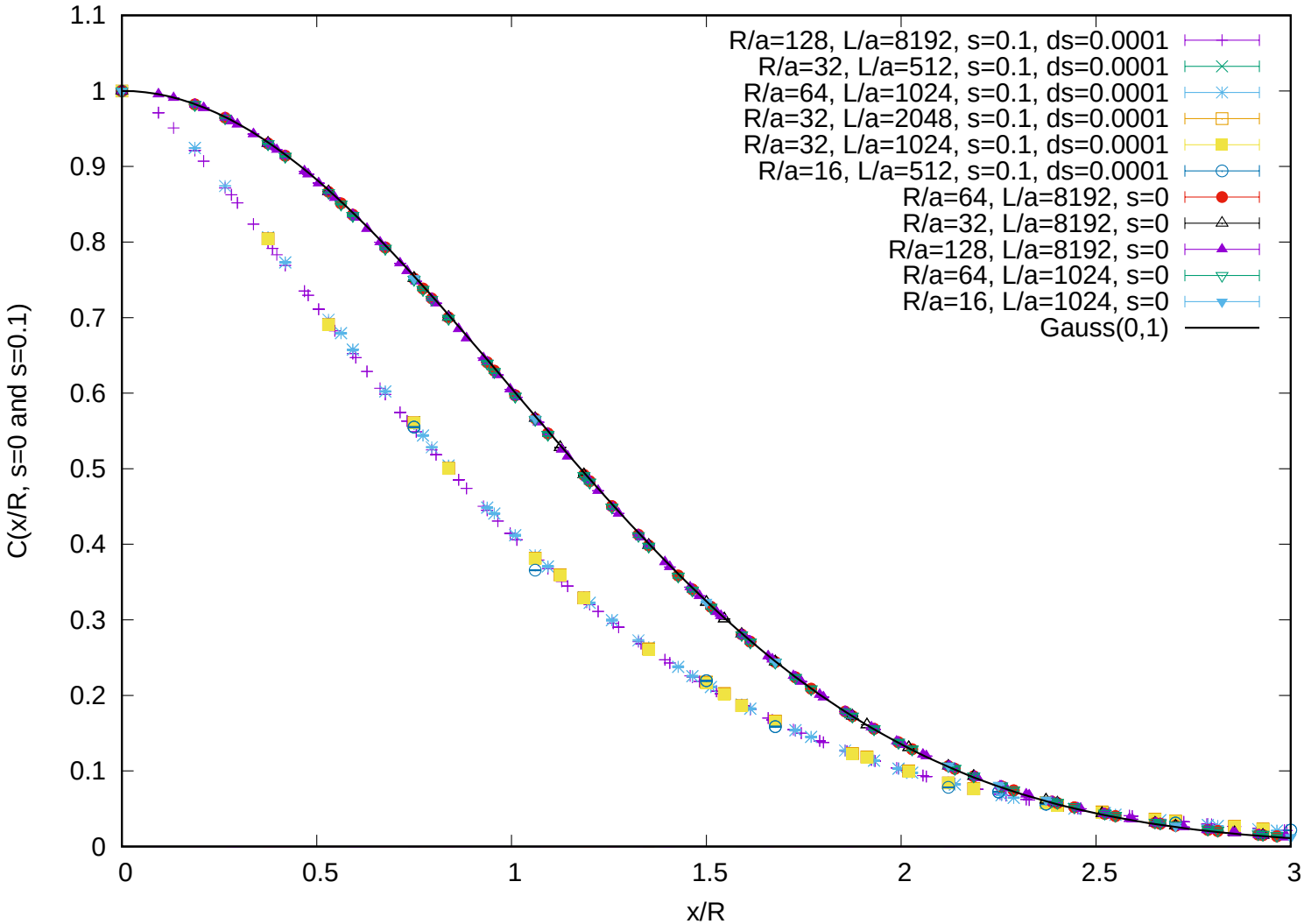
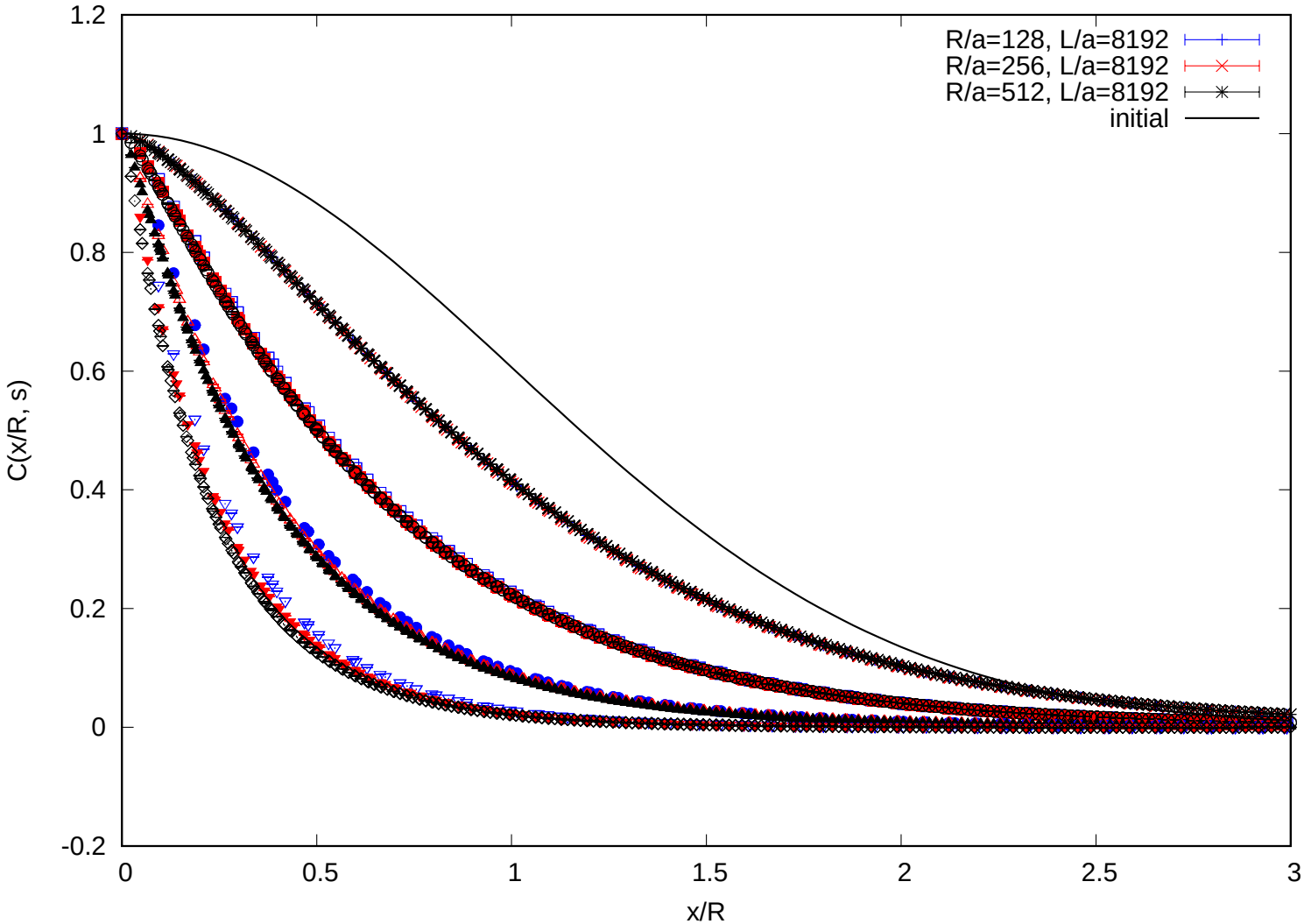


Evolution with running coupling constant: Gaussian model



Evolution with running coupling constant: Gaussian model

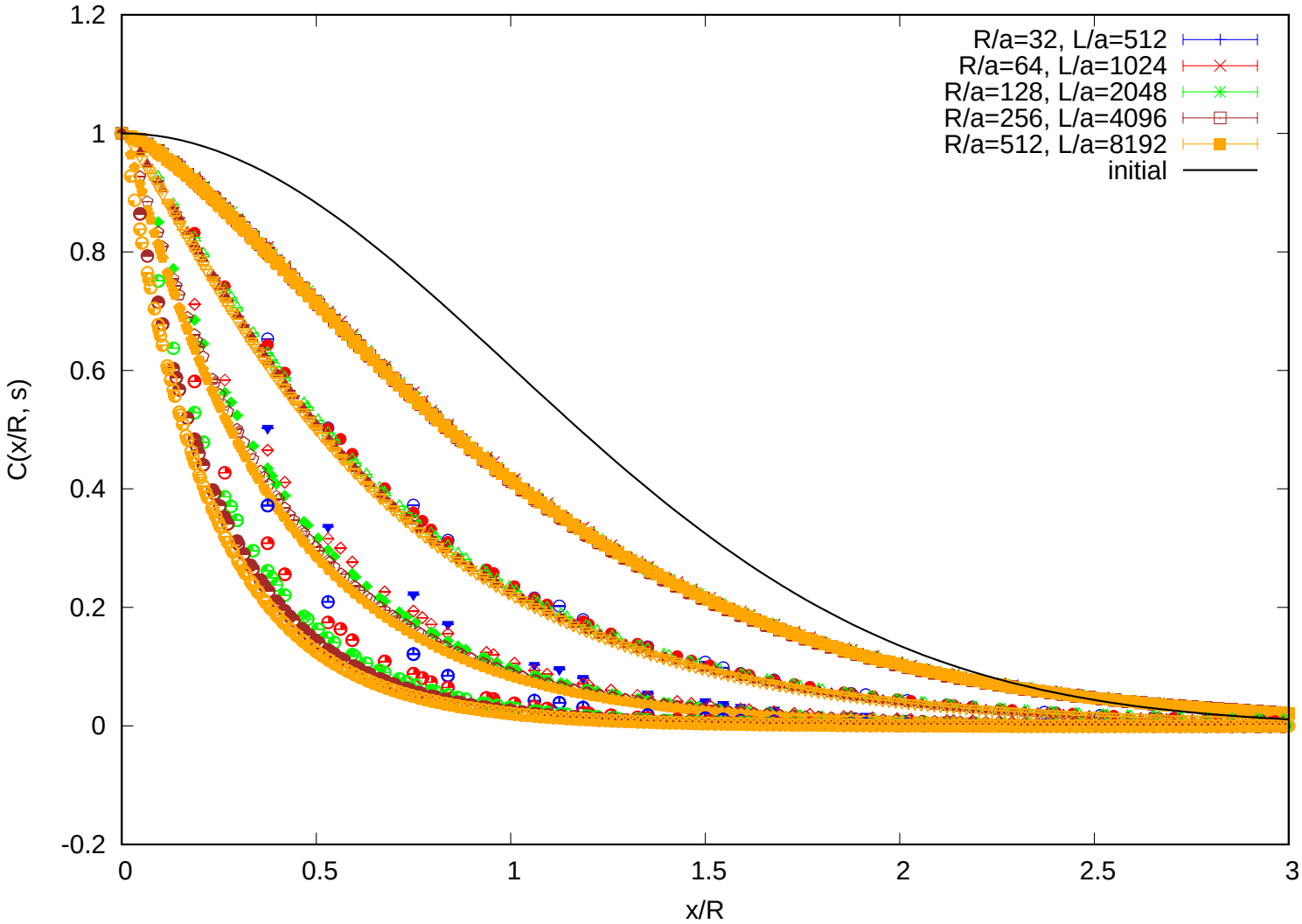
Different volumes: $L/R = 1/64, 1/128, 1/256$
Fixed lattice spacing: $L/a = 8192$



Evolution with running coupling constant: Gaussian model

Fixed volume: $L/R = 1/64$

Different lattice spacings: $L/a = 2048, 4096, 8192$



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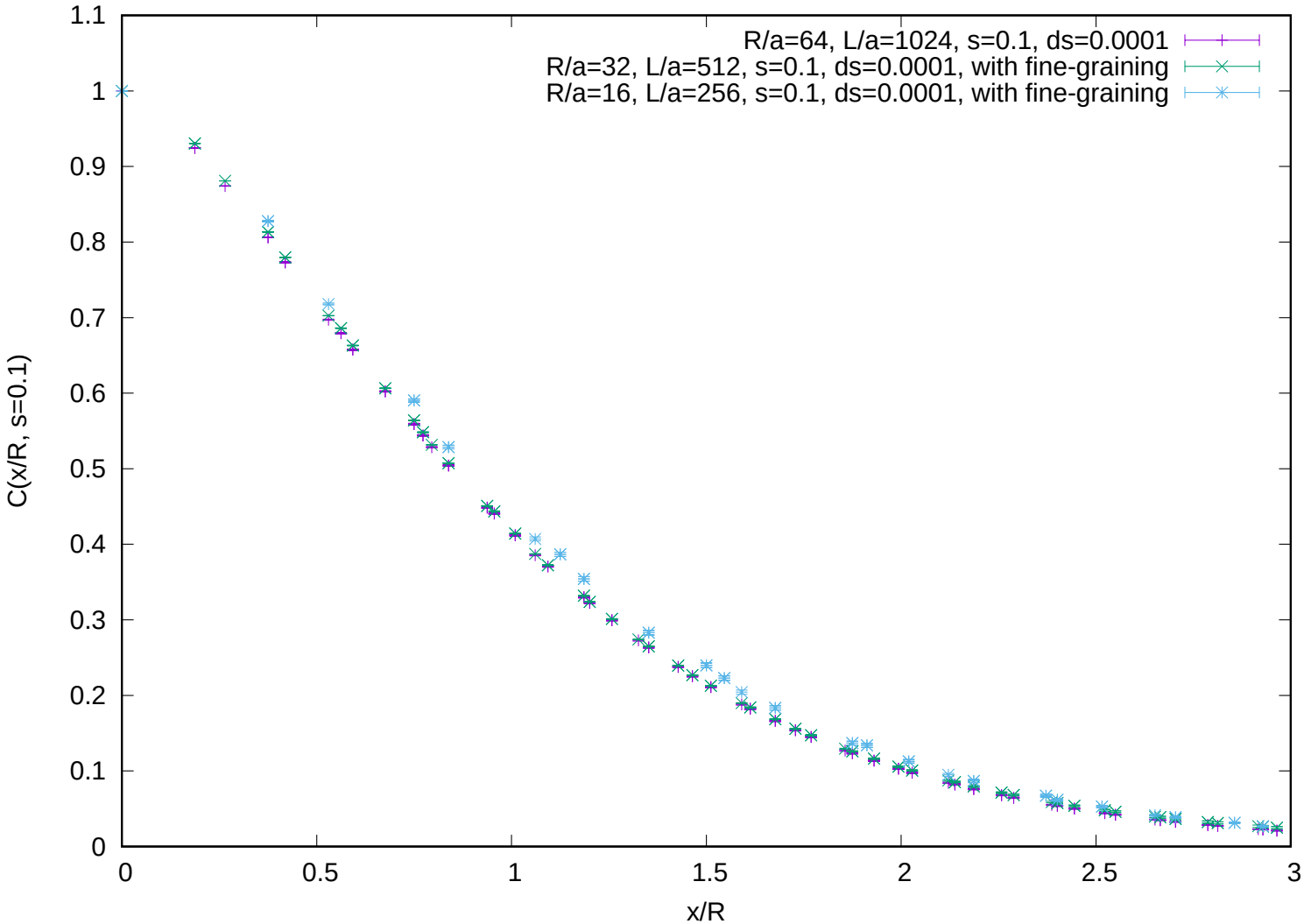
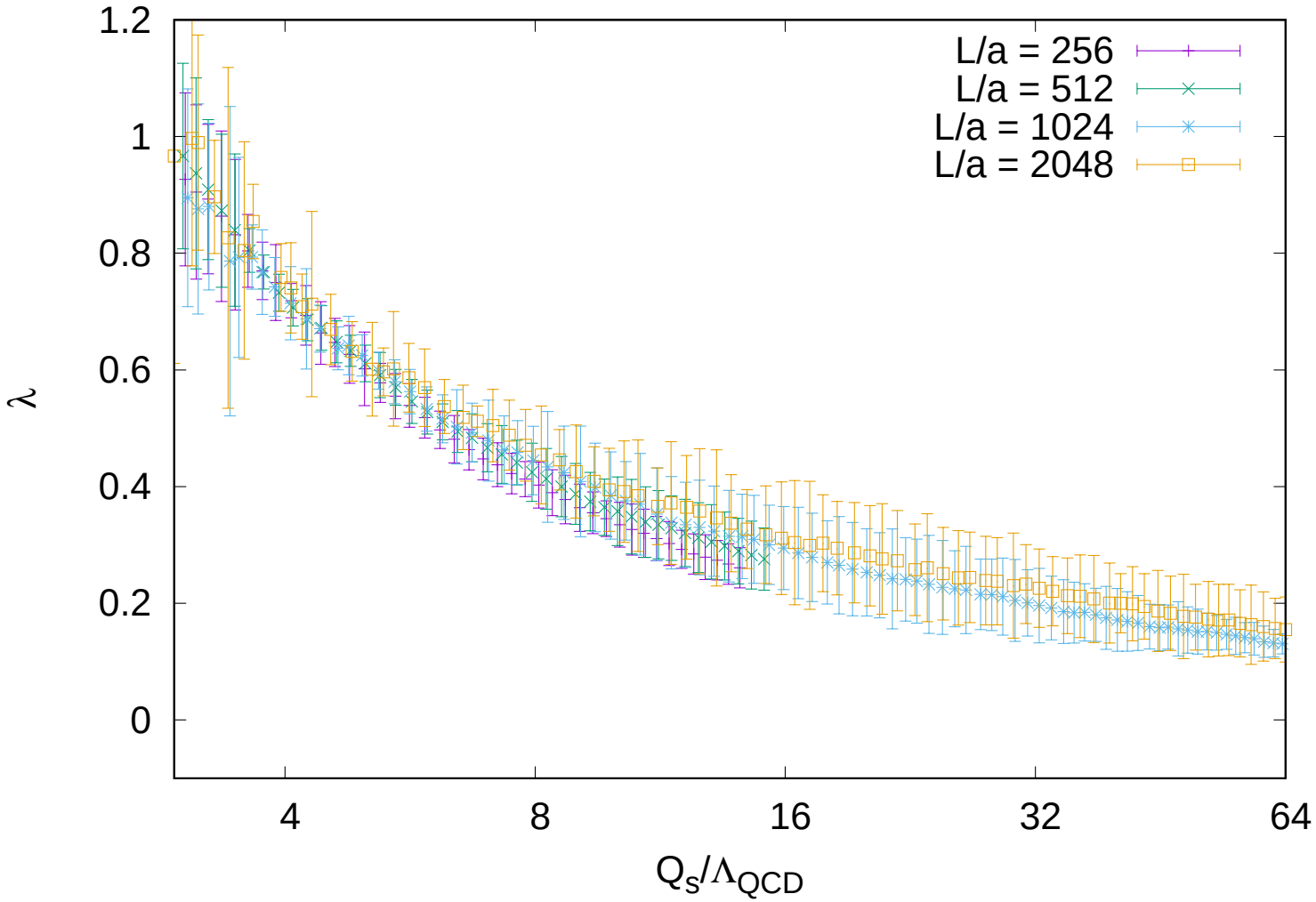


Evolution with running coupling constant: MV model



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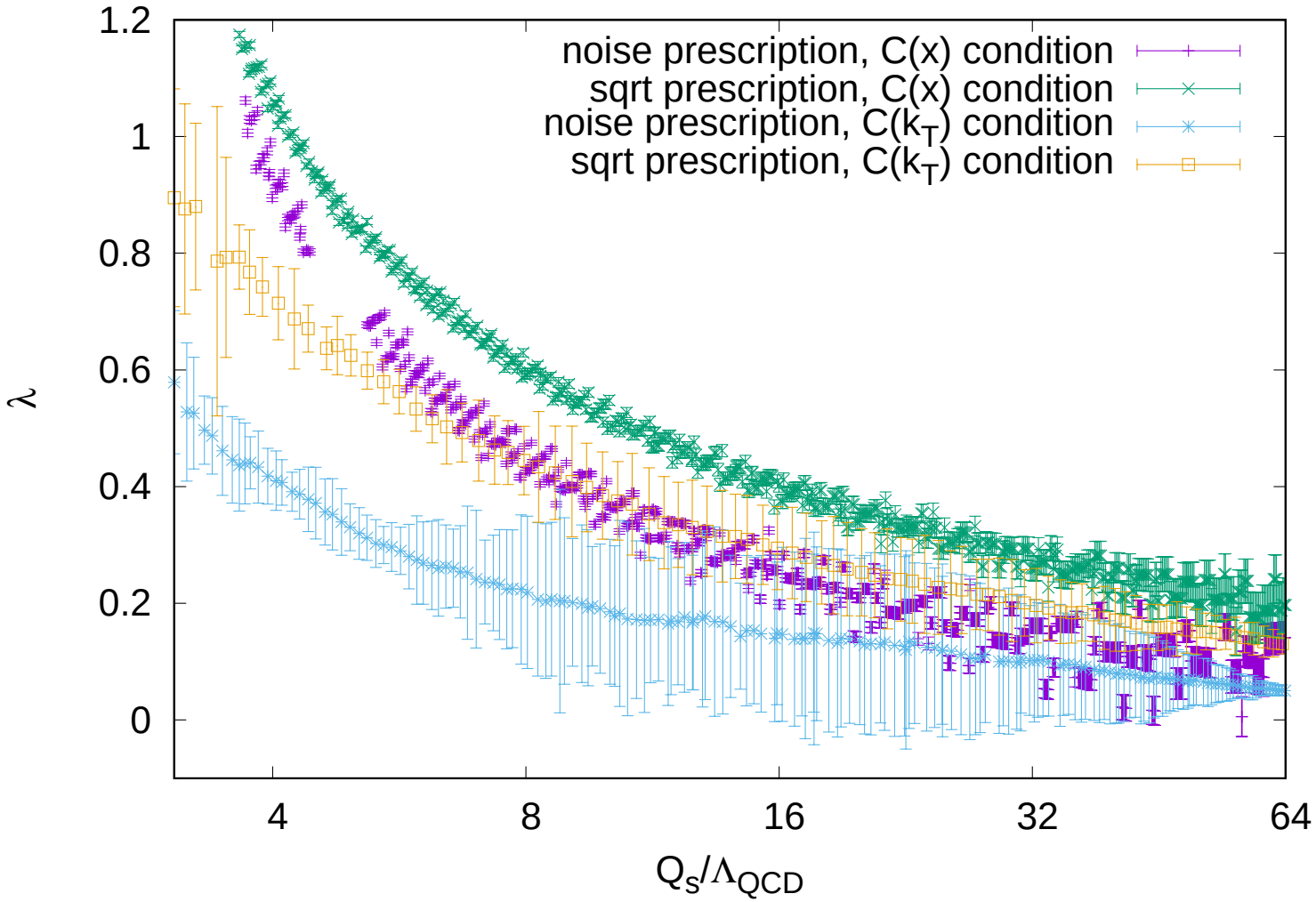
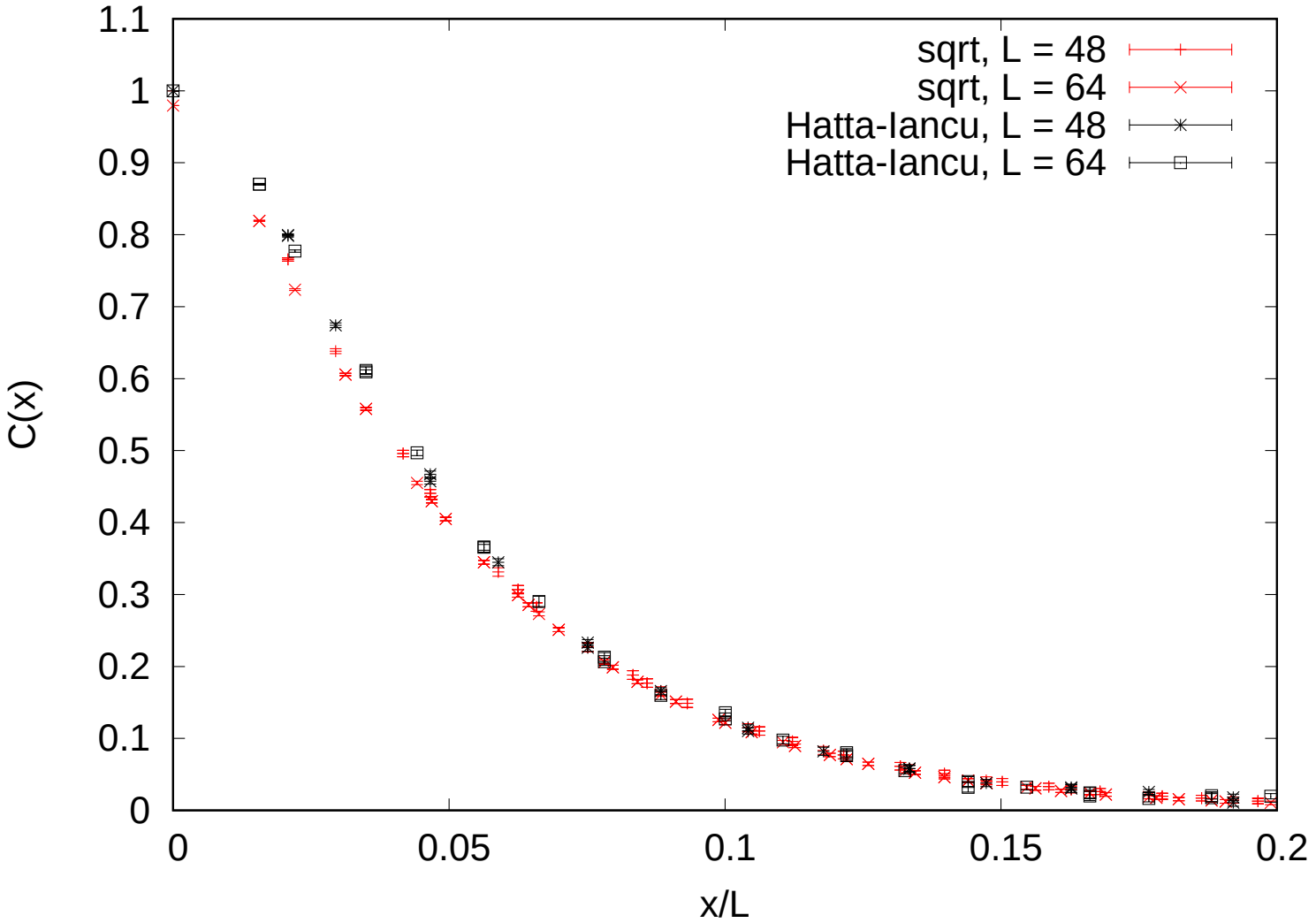


Evolution with running coupling constant: MV model



Conclusions

- JIMWLK equation provides a way to describe DIS data deep in the low-x regime
- numerical implementation and solution possible using the reformulation in terms of Langevin equation
- many systematic effects/ambiguities have to be studied and understood
- efficient implementation provides a useful tool for many further investigations

Thank you very much for your attention