The Continuum and Leading Twist Limits of pseudo-PDFs

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(Columbia University)

As part of the
HadStruc Collaboration
Along with

K. Orginos (W&M / JLab)
A. Radyushkin (Old Dominion U / JLab)
S. Zafeiropoulos (Aix-Marseille)

COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Based on JK, K. Orginos, A. Radyushkin, S. Zafeiropoulos, (2021) 2105.13313

LaMET 2021

HadStruc Collaboration

- Carl Carlson, Chris Chamness, Tanjib Khan, Dan Kovner, Chris Monahan, Kostas Orginos, Raza Sufian (W&M)
- Patrick Barry, Robert Edwards, Colin Egerer, Nikhil Karthik, Jian-Wei Qiu, David Richards, Eloy Romero, Frank Winter (JLab)
- Wayne Morris, Anatoly Radyushkin (ODU)
- Bálint Joó (ORNL)
- Savvas Zafeiropoulos (Aix-Marseille)
- Joe Karpie (Columbia U)

Other HadStruc talks!
Today: Colin Egerer, Patrick Barry
Wednesday: Nikhil Karthik, Wayne Morris, Raza Suffian

LaMET and SDF

- Two related methods to analyze the space-like separated fields with Large Momentum Effective Theory (quasi-PDF) or Short Distance Factorization (pseudo-PDF) to obtain PDFs
- LaMET/SDF and the PDF
  - LaMET: factorization relation and power expansion with respect to large momentum scale p_z^{-2}

    X. Ji (2013) 1305.1539
  - SDF: factorization relation and power expansion with respect to short distance scale z^2

V. Braun and D. Müller (2007) 0709.1348
A. Radyushkin (2017) 1705.01488
Y. Q. Ma and J. W. Qiu (2017) 1709.03018

- Wilson Line Operator matrix element

$$M^\alpha(p,z) = \langle p|\bar{\psi}(z)\gamma^\alpha W(z;0)\psi(0)|p\rangle$$

  - Lorentz Composition

$$M^\alpha(p,z) = 2p^\alpha \mathcal{M}(\nu, z^2) + 2z^\alpha \mathcal{N}(\nu, z^2)$$

B. Musch et al (2010) 1011.1213

LaMET and SDF

- Two related methods to analyze the space-like separated fields with Large Momentum Effective Theory (quasi-PDF) or Short Distance Factorization (pseudo-PDF) to obtain PDFs
- LaMET and SDF
  - LaMET: factorization and power expansion with respect to large momentum scale p_z^{-2}
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V. Braun and D. Müller (2007) 0709.1348
A.Radyushkin (2017) 1705.01488
Y. Q. Ma and J. W. Qiu (2017) 1709.03018

- SDF begins with the OPE with a short distance scale
  - Euclidean space matrix elements with power corrections can be ordered by twist
  - The SDF's leading twist kernel is related to LaMET's kernel by integral formula.

$$\mathfrak{M}(\nu, z^2) = \int_0^1 du\, K(u, \mu^2 z^2) Q(u\nu, \mu^2) + O(z^2)$$

T. Izubuchi et al (2018) 1801.03917

- Known to $O(\alpha_s)$

A. Radyushkin (2017) 1710.08813
J.-H. Zhang et. al. (2018) 1801.03023
T. Izubuchi et. al. (2018) 1801.03917

$O(\alpha_s^2)$ Z-Y Li, Y-Q Ma, J-Q Qiu (2020) 2006.12370

The Reduced distribution and renormalization

A.Radyushkin (2017) 1705.01488
T. Izubuchi et al (2020) 2007.06590

- The pseudo-ITD is subject to many systematic errors
  - Lattice spacing, higher twist, incorrect pion mass, finite volume
- A ratio can remove renormalization constants and the low Ioffe time systematic errors
  - Avoids additional gauge fixed scheme dependent RI-Mom calculations
  - Is a renormalization group invariant quantity, guaranteeing finite continuum limit (no power divergences)

$$\mathfrak{M}(\nu, z^2) = \frac{M^0(p,z)/M^0(p,0)}{M^0(0,z)/M^0(0,0)}$$

$$M^\alpha(p,z) = 2p^\alpha \mathcal{M}(\nu, z^2) + 2z^\alpha \mathcal{N}(\nu, z^2)$$

- New ratio method with non-zero momentum could remove different HT errors

The Reduced distribution and renormalization

A.Radyushkin (2017) 1705.01488
T. Izubuchi et al (2020) 2007.06590

- Ratios of hadronic matrix elements can cancel the non-perturbative higher twist contributions
  - What region of Ioffe time depend on the denominator used

$$\mathcal{M}(\nu, z^2) = \mathcal{M}^{\mathrm{lt}}(\nu, z^2) + (z^2\Lambda^2_{\mathrm{QCD}})B(\nu)$$

$$\mathfrak{M}(\nu, \nu_0, z^2) = \frac{\mathcal{M}(\nu, z^2)}{\mathcal{M}(\nu_0, z^2)} = \mathfrak{M}^{\mathrm{lt}}(\nu, \nu_0, z^2) + (z^2\Lambda^2_{\mathrm{QCD}})(B(\nu) - B(\nu_0))$$

- Residual higher twist effects are significantly smaller than would be in other ratios, such as quark matrix elements RI-Mom or vacuum matrix elements
- Similar improvement for all systematic errors!

Systematic errors of Lattice PDFs

- Pion mass
  - Just use correct values (duh!)
  - Extrapolate PDF to physical pion mass
- Finite Volume
  - Calculate size of effects in a model theory

    R. Briceño et al (2018) 1805.01034 and (2021) 2102.01814
  - Parameterize unknown functional dependence
- Lattice Spacing
  - Parameterizing unknown functional dependence

    X. Gao et al (2020) 2007.06590
  - Interpolate data at fixed hard scale and extrapolate continuum limit

    C. Alexandrou et al (2020) 2011.00964

    H.-W. Lin et al (2020) 2011.14971
- Power Corrections
  - LaMET p_z^{-2}
  - SDF and Lattice Cross Sections z^2
    - Parameterizing unknown functional dependence
  - OPE without OPE and Hadronic Tensors Q^{-2}
- Inverse Problems
  - Get back to these later

C. Alexandrou et al (2018) 1803.02685

J-W Chen et al (2018) 1803.04393

B. Joó, JK, K. Orginos, A. Radyushkin, D. Richards, R. Sufian, S. Zafeiropoulos (2019) 1908.09771

B. Joó, JK, K. Orginos, A. Radyushkin, D. Richards, R. Sufian, S. Zafeiropoulos (2019) 1909.08517

R. Sufian, C. Egerer, JK, R. Edwards, B. Joó, Y-Q Ma, K. Orginos, J-W Qiu, D. Richards (2020) 2001.04960

B. Joó, JK, K. Orginos, A. Radyushkin, D. Richards, S. Zafeiropoulos (2020) 2004.01687

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Unknown functions

- Want to determine a continuous unknown function from the data
- Lattice systematic errors
  - Lattice spacing is the only one used in this study
  - No momentum corrections J-W Chen et al (2017) 1710.01089

$$\mathfrak{M}(p, z, a) = \mathfrak{M}_{\text{cont}}(\nu, z^2) + \sum_{n=1} \left(\frac{a}{|z|}\right)^n P_n(\nu) + (a\Lambda_{\text{QCD}})^n R_n(\nu)$$

- Power Corrections

$$\mathfrak{M}_{\text{cont}}(\nu, z^2) = \mathfrak{M}_{\text{lt}}(\nu, z^2) + \sum_{n=1} (z^2 \Lambda_{\text{QCD}}^2)^n B_n(\nu)$$

- Factorization of the PDF

$$\text{Re}/\text{Im}\, \mathfrak{M}_{\text{lt}}(\nu, z^2) = \int_0^1 dx\, \mathcal{K}_{R/I}(x\nu, \mu^2 z^2) q_{\mp}(x, \mu^2)$$

Inverse Problem Solutions for Lattice PDFs

JK, K. Orginos, A. Rothkopf, S. Zafeiropoulos (2019) 1901.05408

- Parametric
  - Fit a phenomenologically motivated function
    - Method used by most pheno extractions
    - Potentially significant, but controllable model dependence
  - Fit to a neural network S. Forte, L. Garrido, J. Latorre, A. Piccione (2002) 0204232
    - Machine learning is hip
    - Expensive tuning procedure

K. Cichy, L. Del Debbio, T. Giani (2019) 1907.06037
L. Del Debbio, T. Giani, JK, K. Orginos, A. Radyushkin, S. Zafeiropoulos (2020) 2010.03996

- Non-Parametric
  - Backus-Gilbert J. Liang, K-F. Liu, Y-B. Yang (2017) 1710.11145
    - No model dependence, one tunable parameter
  - Bayesian Reconstruction Y. Burnier and A. Rothkopf (2013) 1307.6106, J. Liang et al (2019) 1906.05312
    - Very general, Bayesian statistics has systematics included in meaningful way
  - Bayes-Gauss-Fourier transform

C. Alexandrou, G. Iannelli, K. Jansen, F. Manigrasso (2020) 2007.13800

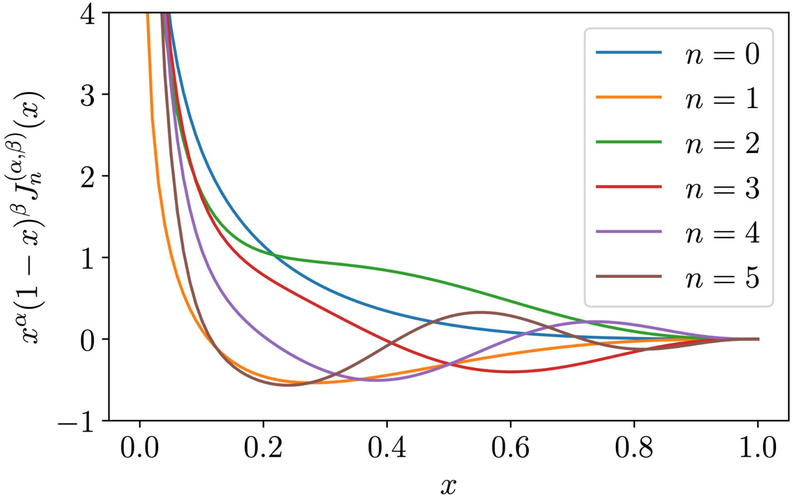
Jacobi Polynomials

- Orthogonal set of Polynomials
  - Textbook orthogonality relationship $\int_{-1}^{1} dz (1-z)^{\alpha}(1+z)^{\beta} j_n^{(\alpha,\beta)}(z) j_m^{(\alpha,\beta)}(z) = \tilde{N}_n^{(\alpha,\beta)} \delta_{n,m}$
- Change variables for more useful metric and integration range: $z = 1 - 2x$

$$\int_0^1 dx\, x^{\alpha}(1-x)^{\beta} J_n^{(\alpha,\beta)}(x) J_m^{(\alpha,\beta)}(x) = N_n^{(\alpha,\beta)} \delta_{n,m}$$

$$J_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \omega_{n,j}^{(\alpha,\beta)} x^j$$

$$\omega_{n,j}^{(\alpha,\beta)} = \binom{n}{j} \frac{(-1)^j}{n!} \frac{\Gamma(\alpha+n+1)\Gamma(\alpha+\beta+n+j+1)}{\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+j+1)}$$



$\alpha = -0.5\,, \quad \beta = 3$

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Jacobi Polynomial parameterizations

- Parameterize unknown functions
  - Example: PDFs $$q_\pm(x) = x^\alpha (1-x)^\beta \sum_{n=0} {}_\pm d_n^{(\alpha,\beta)} J_n^{(\alpha,\beta)}(x)$$
- How to Fourier transform of this parameterization

$$\sigma_{0,n}^{(\alpha,\beta)}(\nu) = \int_0^1 dx\, x^\alpha (1-x)^\beta \cos(\nu x) J_n^{(\alpha,\beta)}(x)$$

$$\eta_{0,n}^{(\alpha,\beta)}(\nu) = \int_0^1 dx\, x^\alpha (1-x)^\beta \sin(\nu x) J_n^{(\alpha,\beta)}(x)$$

$$\mathrm{Re}\, Q(\nu) = \sum_{n=0} \sigma_{0,n}^{(\alpha,\beta)}(\nu)_- d_n \qquad \mathrm{Im}\, Q(\nu) = \sum_{n=0} \eta_{0,n}^{(\alpha,\beta)}(\nu)_+ d_n$$

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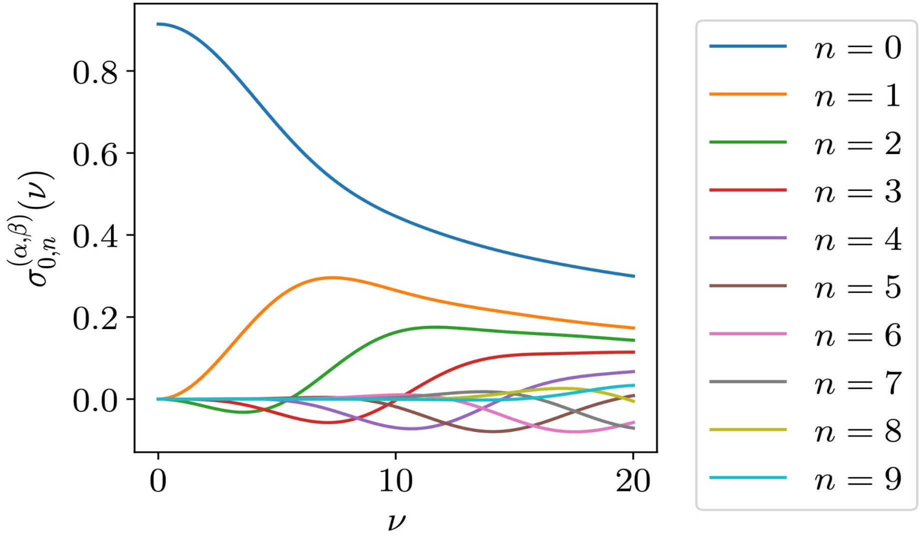
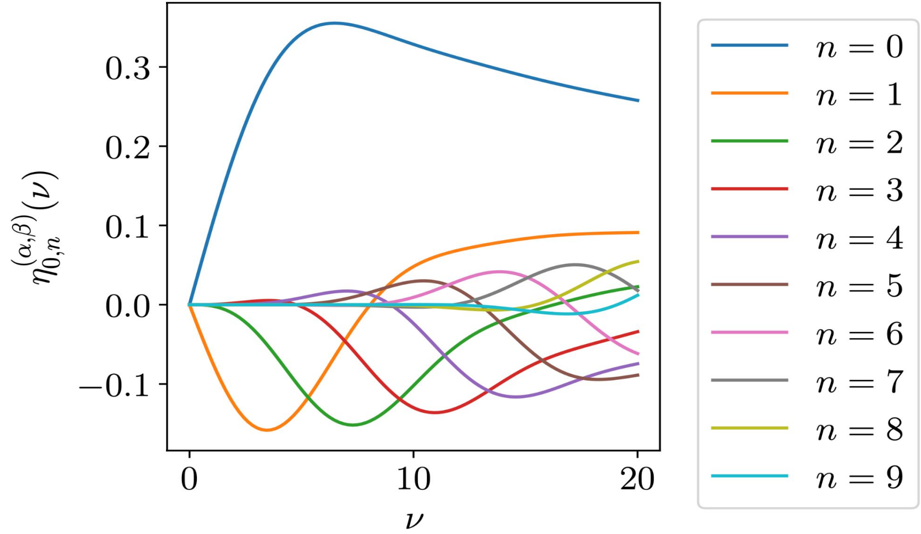
Jacobi Polynomial parameterizations





$$\alpha = -0.5\,, \quad \beta = 3$$

- Decays to 0 with Ioffe time
- Large n only at large Ioffe time if coefficients are small

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Jacobi Polynomial parameterizations

$O(\alpha_s)$ A. Radyushkin (2017) 1710.08813
J.-H. Zhang et. al. (2018) 1801.03023
T. Izubuchi et. al. (2018) 1801.03917

$O(\alpha_s^2)$ Z-Y Li, Y-Q Ma, J-Q Qiu 2006.12370

- Including the factorization kernel

$$\sigma_n^{(\alpha,\beta)}(\nu,\mu^2 z^2) = \int_0^1 dx\, x^\alpha (1-x)^\beta \mathcal{K}_R(x\nu,\mu^2 z^2) J_n^{(\alpha,\beta)}(x)$$

$$\sigma_n^{(\alpha,\beta)}(\nu,\mu^2 z^2) = \sigma_{0,n}^{(\alpha,\beta)}(\nu) + \sigma_n^{(\mathrm{NLO})}(\nu,\mu^2 z^2) + O(\alpha_S^2)$$

$$\eta_n^{(\alpha,\beta)}(\nu,\mu^2 z^2) = \int_0^1 dx\, x^\alpha (1-x)^\beta \mathcal{K}_I(x\nu,\mu^2 z^2) J_n^{(\alpha,\beta)}(x)$$

$$\eta_n^{(\alpha,\beta)}(\nu,\mu^2 z^2) = \eta_{0,n}^{(\alpha,\beta)}(\nu) + \eta_n^{(\mathrm{NLO})}(\nu,\mu^2 z^2) + O(\alpha_S^2)$$

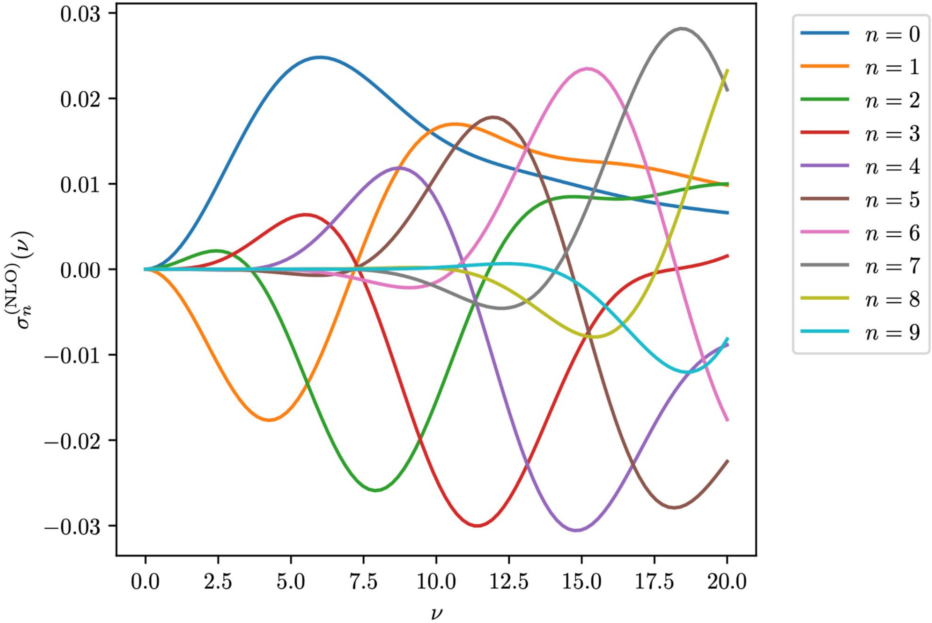
- Parameterize leading twist pseudo-ITD instead of ITD

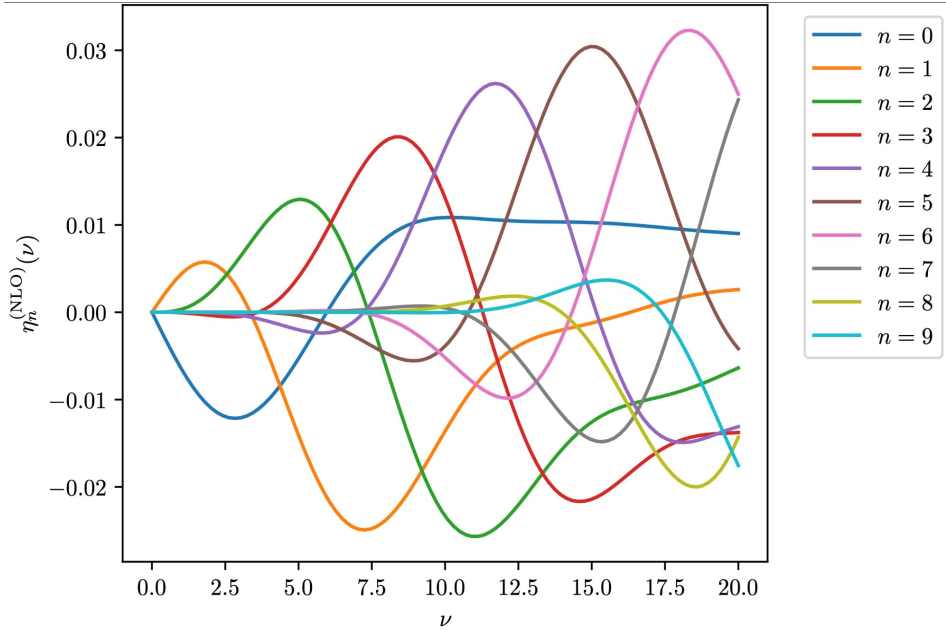
$$\mathrm{Re}\,\mathfrak{M}_{\mathrm{lt}}(\nu,z^2) = \sum_{n=0} \sigma_n^{(\alpha,\beta)}(\nu,\mu^2 z^2)_- d_n$$

$$\mathrm{Im}\,\mathfrak{M}_{\mathrm{lt}}(\nu,z^2) = \sum_{n=0} \eta_n^{(\alpha,\beta)}(\nu,\mu^2 z^2)_+ d_n$$

Jacobi Polynomial parameterizations

$$z = 4a_{\mathrm{E5}}$$





- Remains small function Ioffe time, generating small perturbative corrections at NLO
- Future work will expand to NNLO

$$\alpha = -0.5\,, \quad \beta = 3$$

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Jacobi Polynomial parameterizations

- Final functional form

ITD HT $a\Lambda_{\mathrm{QCD}}$ a/z

$$\mathrm{Re}\,\mathfrak{M}(p,z,a) = \sum_{n=0}^{N_-+1} \sigma_n(\nu,\mu^2 z^2)\,{}_-d_n^{(\alpha,\beta)} + z^2\Lambda_{\mathrm{QCD}} \sum_{n=1}^{N_{R,b}} \sigma_{0,n}(\nu) b_{R,n}^{(\alpha,\beta)} + a\Lambda_{\mathrm{QCD}} \sum_{n=1}^{N_{R,r}} \sigma_{0,n}(\nu) r_{R,n}^{(\alpha,\beta)} + \frac{a}{|z|} \sum_{n=1}^{N_{R,p}} \sigma_{0,n}(\nu) p_{R,n}^{(\alpha,\beta)}$$

$$\mathrm{Im}\,\mathfrak{M}(p,z,a) = \sum_{n=0}^{N_+} \eta_n(\nu,\mu^2 z^2)\,{}_+d_n^{(\alpha,\beta)} + z^2\Lambda_{\mathrm{QCD}} \sum_{n=1}^{N_{I,b}} \eta_{0,n}(\nu) b_{I,n}^{(\alpha,\beta)} + a\Lambda_{\mathrm{QCD}} \sum_{n=1}^{N_{I,r}} \eta_{0,n}(\nu) r_{I,n}^{(\alpha,\beta)} + \frac{a}{|z|} \sum_{n=1}^{N_{I,p}} \eta_{0,n}(\nu) p_{I,n}^{(\alpha,\beta)}$$

- Normalization of PDF

$${}_-d_0^{(\alpha,\beta)} = 1/B(\alpha+1,\beta+1)$$

Bayesian Fits

$$P\left[\theta|\mathfrak{M}_L, I\right] = \frac{P\left[\mathfrak{M}_L|\theta\right] P\left[\theta|I\right]}{P\left[\mathfrak{M}_L|I\right]}$$

- Standard χ^2 minimization, but with modified function

$$P\left[\mathfrak{M}_L|\theta\right] = \frac{\exp[-\frac{\chi^2}{2}]}{Z_\chi} \qquad \chi^2 = \sum_{k,l}(\mathfrak{M}_k^L - \mathfrak{M}_k)C_{kl}^{-1}(\mathfrak{M}_l^L - \mathfrak{M}_l)$$

$$P\left[\theta|\mathfrak{M}_L, I\right] = \frac{\exp[-\frac{L^2}{2}]}{Z} \qquad L^2 = \chi^2 - 2\log\left(P[\theta|I]\right)$$

- Prior Distributions
    - Uniform distribution within bounds
    - Normal distribution
    - Log-Normal distribution

- Additional terms are designed to push weakly push the maximum probability to “reasonable” values

Lattice ensembles

ID	a(fm)	M_π(MeV)	β	c_{SW}	κ	$L^3 \times T$	N_{cfg}
$\widetilde{A}5$	0.0749(8)	446(1)	5.2	2.01715	0.13585	$32^3 \times 64$	1904
E5	0.0652(6)	440(5)	5.3	1.90952	0.13625	$32^3 \times 64$	999
N5	0.0483(4)	443(4)	5.5	1.75150	0.13660	$48^3 \times 96$	477

- E5 and N5 were generated as part of CLS collaboration P. Fritzsch et al (2012) 1205.5380
- Ã5 was generated for this study
- Three lattice spacings for lattice spacing dependence
- Fixed pion mass
- Will ignore the difference between physical volumes until future work

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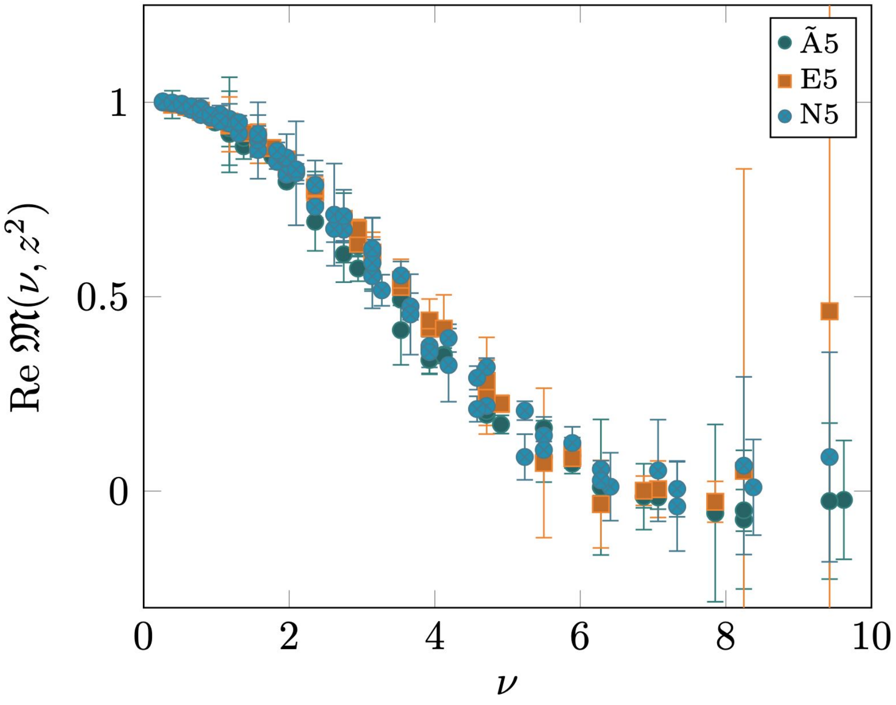
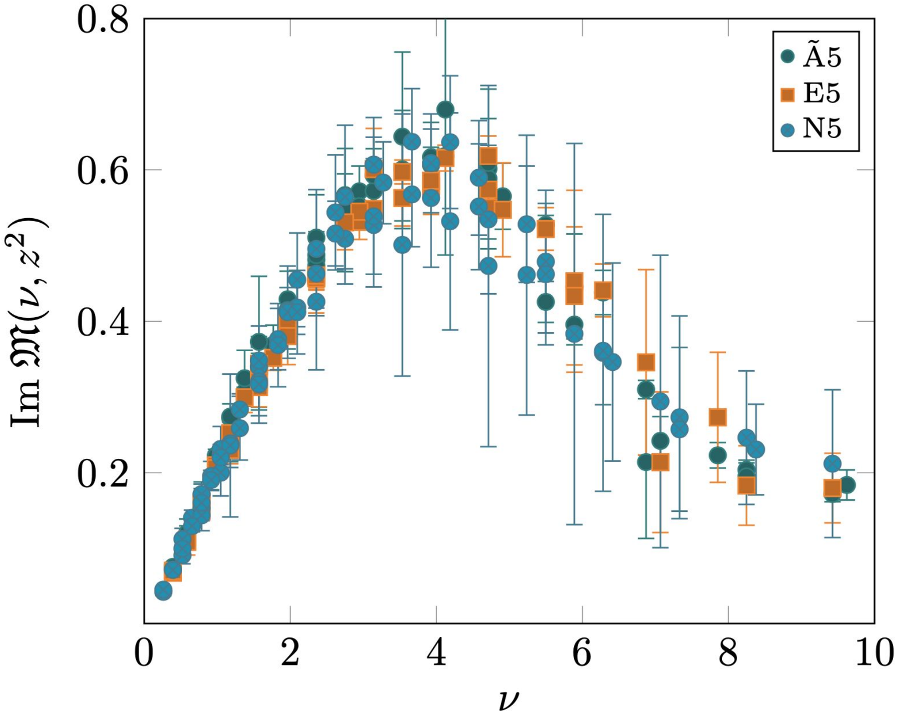
Reduced Matrix Elements





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Chi squared of fits

$$N_{\pm} = 2 \quad N_{R/I,\, b/p/r} = 0,\ 1$$

model	Real L^2/d.o.f.	Real χ^2/d.o.f.	Imag L^2/d.o.f.	Imag χ^2/d.o.f.
Q only	3.173	3.094	3.146	3.095
Q and B_1	2.721	2.479	3.054	2.969
Q and R_1	3.028	2.748	3.068	2.871
Q and P_1	0.876	0.809	1.186	1.088
Q, B_1, and R_1	2.610	2.057	2.917	2.619
Q, B_1, and P_1	0.852	0.723	1.020	0.888
Q, R_1, and P_1	0.881	0.763	1.289	1.063
All terms	0.857	0.727	1.026	0.893

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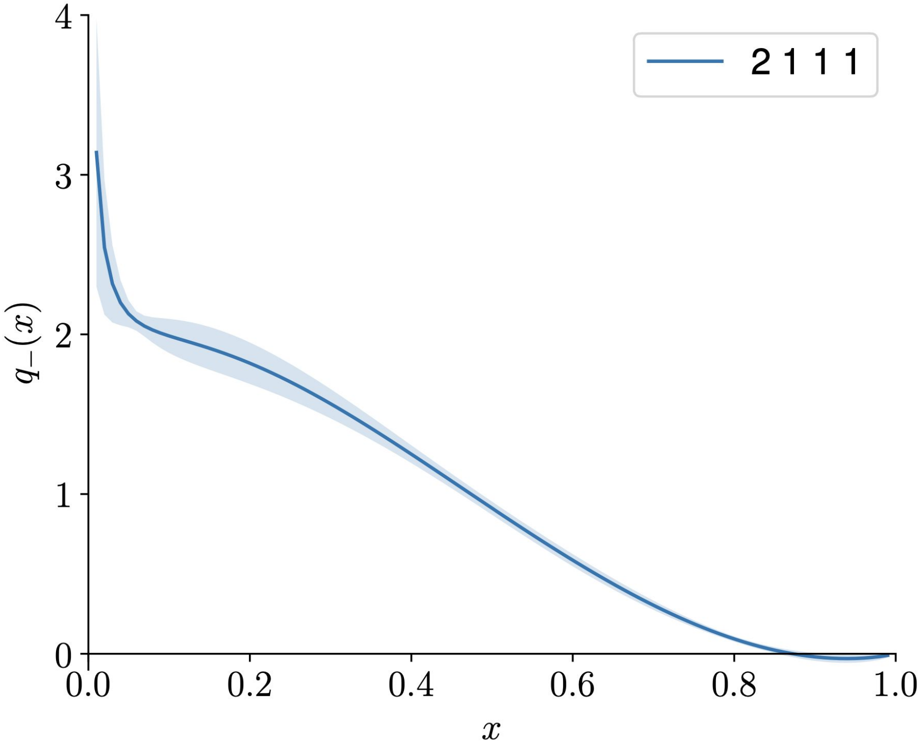
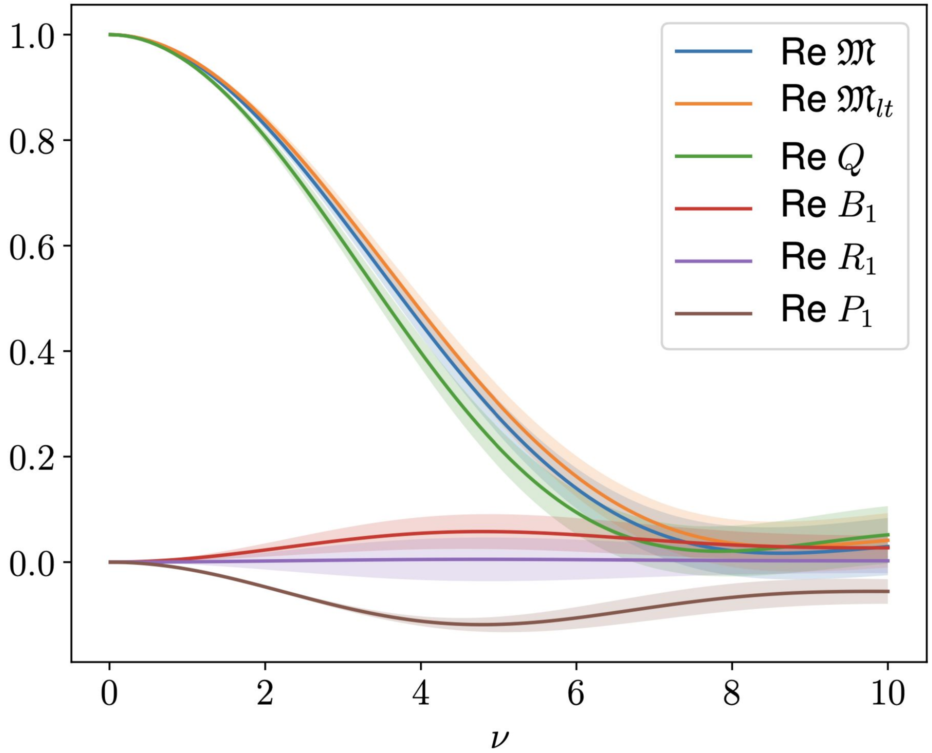
Study nuisance terms

$$N_{\pm} = 2 \quad N_{R/I,b/p/r} = 1$$





$$z = 4a_{\mathrm{E5}}$$

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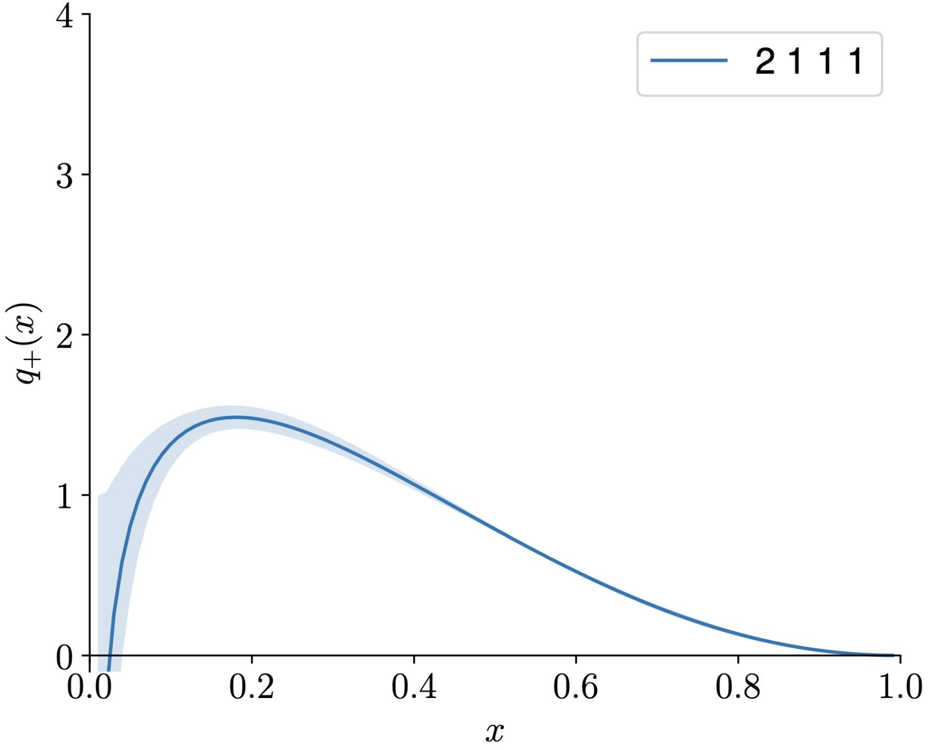

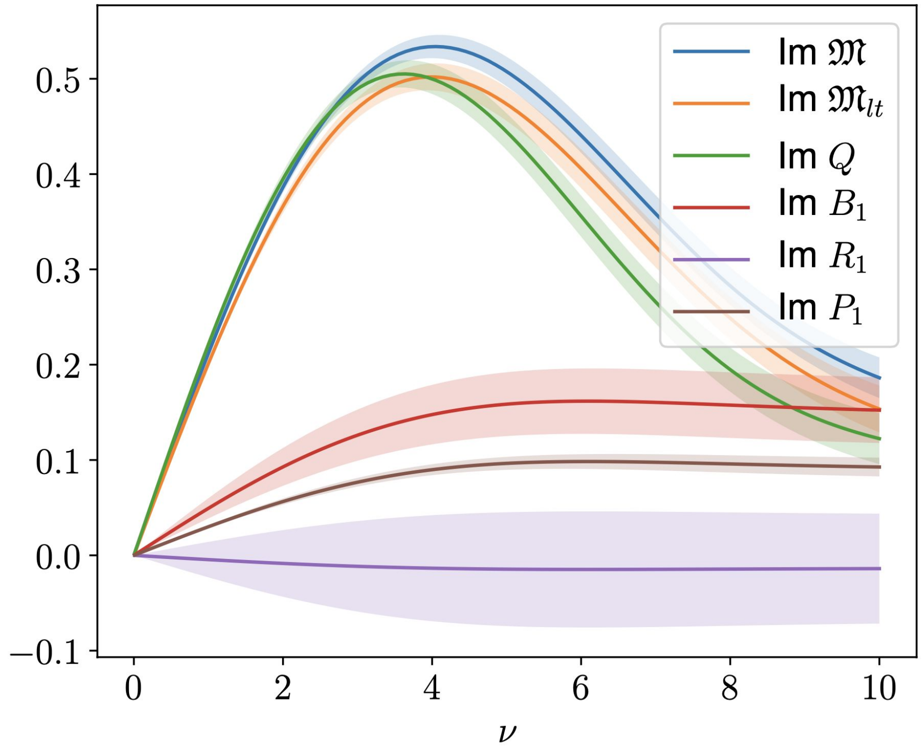

$$z = 4a_{\mathrm{E5}}$$

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AICc averaging

R. Zhang et al (2020) 2005.13955
H.-W. Lin et al (2020) 2011.14971

- Akaike Information Criteria (AIC)
  - Adds weight to disfavor models with too many parameters

$$a_i = 2k_i + 2L_i^2$$

- Corrected AIC (AICc)
  - Used when few number of datapoints compared to number of parameters

$$A_i = a_i + \frac{2k(k+1)}{n-k-1}$$

- Weighted average to determine expectation values of observables
  - Ideally, averages away model biases

$$x = \sum_{i=1}^{N} w_i x_i \, , \qquad w_i = \frac{e^{-\frac{A_i}{2}}}{\sum_{i=1}^{N} e^{-\frac{A_i}{2}}}$$

AICc averaging

- Use a range of models for the AICc weighted average
- To average away model biases, sufficiently many distinct models are required
- Undesirable models could be removed, or their large AICc will exponentially suppress them in the weighted average
- For this study will use models with

$$N_{\pm} = 1,\ 2,\ 3$$
$$N_{R/I,b/p/r} = 0, \ldots, N_{\pm}$$

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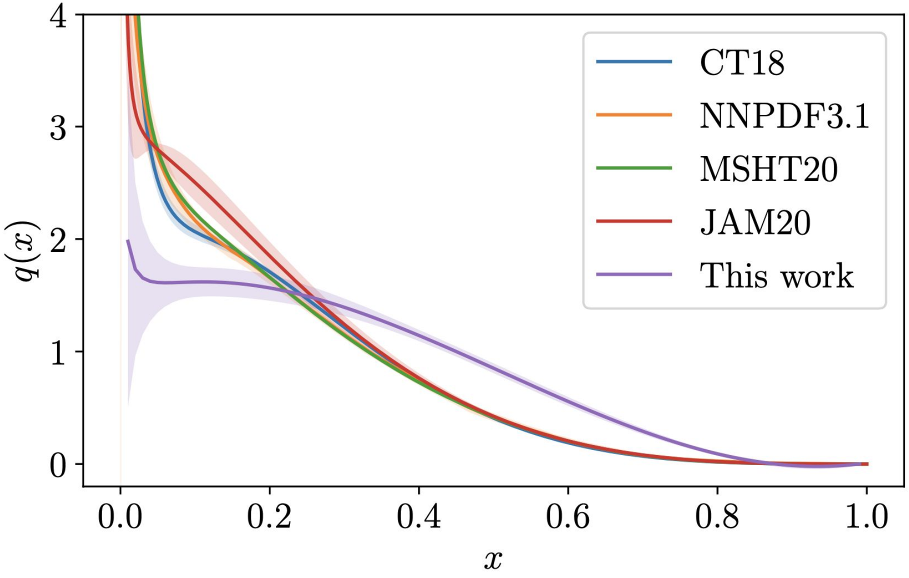
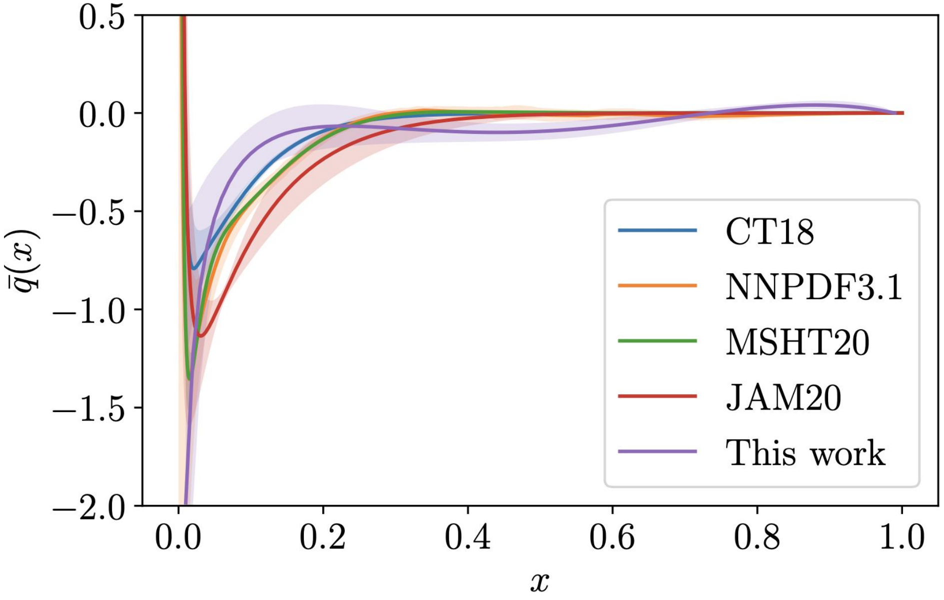
Averaged Results





Conclusions and Outlook

- Jacobi polynomial parameterizations allow for a systematically controlled determination of the PDF
  - With more ensembles, other systematics can be included in the same fashion
  - Pion mass dependence, finite volume, perturbative truncation
  - Can be used with different observables
- Parameterizing in x space allows for Ioffe time functions which decay to 0 at large Ioffe time
  - Also avoids intermediate matching between pseudo-ITD and ITD in our previous works
- We have studied a range of the number of parameters to attempt to handle model dependence with AIC/AICc averaging
  - Truly distinct models, parametric and non-parametric, are required to completely remove remaining model dependent biases